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POSITIVE PERIODIC SOLUTIONS OF ITERATIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this study, we focus on a specific type of functional differential equation

$$x'(t) = a(t)x(t) - \lambda f(t, x(t), x(x(t))),\ t \in \mathbb{R},$$

called iterative differential equation, which may depend on a future state. By employing a cone theoretical fixed point theorem, we propose some sufficient conditions for the existence of positive periodic solutions of the iterative functional differential equation. Our approach enables us to obtain some results that are not covered by the existing literature.

Qualitative properties of differential equations have taken great interest due to the importance of understanding the continuous phenomena in applications of mathematics. Existence, uniqueness, stability, periodicity, and oscillation of solutions of differential equations became topic of many studies in the existing literature. Advanced (or iterative) differential equations that belong to the class of functional equations are visited by several researchers because of their complicated dynamics. We refer to [**1, 2, 3**] and the recent studies [**4, 5, 6**] as inspiring studies on this topic. For example, in [**1**] the equation

$$x'(t) = x(x(t)) \tag{0.1}$$

is considered and existence uniqueness and analyticity of solutions are studied. Afterwards, the iterative differential equation of the form

$$x'(t) = f(x(x(t))) \tag{0.2}$$

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is handled by [**2, 3**]. In recent years, the study of iterative differential equations has become more popular, and we refer the papers [**7, 8, 9, 10, 11, 12, 13, 14, 15**] to readers as rapidly growing literature on the differential equations with iterative terms.

In this paper, we consider a specific type of functional differential equation, called advanced functional differential equation, given in the following form:

$$x'(t) = a(t)x(t) - \lambda f(t, x(t), x(x(t))), \quad t \in \mathbb{R}. \tag{0.3}$$

By using Krasnoselskii's fixed point theorem on cones, we propose some sufficient conditions for the existence of positive periodic solutions of (0.3). A quick literature review can lead to vast literature using Schauder‘s fixed point theorem to prove existence of periodic solutions to advanced functional differential equations. However, we shall emphasize that we present some existence results for positive periodic solutions of a very general form of advanced functional differential equations which may cover some special equations already handled, and the utilization of a cone theoretical fixed point theorem rather than the Schauder's theorem distinguishes our work from the already established literature, since our outcomes also involve the existence of multiple periodic solutions in particular cases. It is well known that obtaining existence results for multiple solutions of differential equations is a grueling task in the qualitative theory of differential equations, and consequentially we contribute the ongoing theory with the outcomes of the manuscript.

1. Setup

This section is devoted to setup of our problem and to obtain some preliminary results that will be used in our further analysis. Consider the following functional equation

$$x'(t) = a(t)x(t) - \lambda f(t, x(t), x(x(t))), \ t \in \mathbb{R}, \tag{1.1}$$

where the functions $a : \mathbb{R} \to (0, \infty)$, $f : \mathbb{R} \times (0, \infty) \times (0, \infty) \to (0, \infty)$ are continuous, and $\lambda > 0$.

For $T > 0$, we define the set P_T as a set of T-periodic functions x i.e., $x(t + T) = x(t)$ for all $t \in \mathbb{R}$. Observe that P_T is a Banach space when it is endowed by the supremum norm $\|x\| := \sup_{t\in[0,T]} |x(t)|$. In preparation for the next result we make the following assumption:

A1: For a positive integer T, the functions a and f are T-periodic in t, that is $a(t + T) = a(t)$, and $f(t + T, \xi, \vartheta) = f(t, \xi, \vartheta)$ for all $t \in \mathbb{R}$, and $(\xi, \vartheta) \in (0, \infty) \times (0, \infty)$.

Lemma 1.1. *Assume (A1) and consider the mapping H defined on P_T by*

$$(Hx)(t) := \lambda \int_t^{t+T} h(t, u) f(u, x(u), x(x(u)))du, \tag{1.2}$$

where

$$h\left(t,u\right):=\frac{e^{\int_{u}^{t}a(s)ds}}{1-e^{-\int_{0}^{T}a(s)ds}}. \tag{1.3}$$

Then $x\in P_T$ satisfies $\left(Hx\right)\left(t\right)=x\left(t\right)$ for all $t\in\mathbb{R}$ if and only if x is a T-periodic solution of (1.1).

Proof. Suppose (A1) holds and let $x\in P_T$ be a solution of (1.1). Multiplying both sides of (1.1) by $\exp\left(-\int_0^t a\left(s\right)ds\right)$ and taking integral from t to $t+T$, we get

$$x\left(t+T\right)e^{-\int_0^{t+T}a(s)ds}-x\left(t\right)e^{-\int_0^{t}a(s)ds}=\lambda\int_t^{t+T}e^{-\int_0^{u}a(s)ds}f(u,x(u),x\left(x\left(u\right)\right))du,$$

and by employing T-periodicity of x and a we deduce

$$x\left(t\right)e^{-\int_0^{t}a(s)ds}=\frac{\lambda\int_t^{t+T}e^{-\int_0^{u}a(s)ds}f(u,x(u),x\left(x\left(u\right)\right))du}{1-e^{-\int_0^{T}a(s)ds}} \tag{1.4}$$

which gives

$$x\left(t\right)=\lambda\int_t^{t+T}h\left(t,u\right)f(u,x(u),x\left(x\left(u\right)\right))du=\left(Hx\right)\left(t\right)$$

where h is as in (1.3). Conversely, one may easily verify that

$$x\left(t\right)=\lambda\int_t^{t+T}h\left(t,u\right)f(u,x(u),x\left(x\left(u\right)\right))du$$

solves (1.1). □

It is straightforward to verify that $h(t+T,u+T)=h(t,u)$ and

$$0<\frac{\theta}{1-\theta}\leqslant h(t,u)<\frac{1}{1-\theta}\text{ for all }u\in\left[t,t+T\right], \tag{1.5}$$

where

$$\theta:=e^{-\int_0^{T}a(s)ds}. \tag{1.6}$$

Let‘s define the following subset of P_T

$$P_{T,M}:=\left\{x\in P_T:\left|x\left(t_2\right)-x\left(t_1\right)\right|\leqslant M\left|t_2-t_1\right|,\ M\geqslant 0\right\}. \tag{1.7}$$

Similar to [**5**, Lemma 2.1] one may obtain the inequality

$$\left\|x\left(x\right)-y\left(y\right)\right\|\leqslant\left(1+M\right)\left\|x-y\right\| \tag{1.8}$$

for any $x,y\in P_{T,M}$.

We state the following cone theoretical fixed point theorem due to Krasnoselskii.

Theorem 1.1. *Let E be a Banach space and let $P \subset E$ be a cone. Assume Ω_1 and Ω_2 are bounded open balls of E such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Suppose that $L : P \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to P$ is a completely continuous operator such that one of the following conditions holds:*

i. $\|Lu\| \leqslant u$ for $u \in P \cap \delta\Omega_1$ and $\|Lu\| \geqslant u$ for $P \cap \delta\Omega_2$,
ii. $\|Lu\| \geqslant u$ for $P \cap \delta\Omega_1$ and $\|Lu\| \leqslant u$ for $u \in P \cap \delta\Omega_2$.

Then L has a fixed point in $P \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

In preparation for our further analysis, we make the following assumption:

A2: For positive constants $\alpha_{1,2}$, the function f satisfies the Lipschitz condition

$$\left\| f(t, \xi, \vartheta) - f\left(t, \hat{\xi}, \hat{\vartheta}\right) \right\| \leqslant \alpha_1 \left\| \xi - \hat{\xi} \right\| + \alpha_2 \left\| \vartheta - \hat{\vartheta} \right\| \tag{1.9}$$

Next, we define a cone K as a subset of P_T by

$$K := \{x \in P_{T,M} : x(t) \geqslant \theta \|x\| \text{ for all } t \in [0, T]\} \tag{1.10}$$

where θ is as in (1.6).

Now, we present the second auxiliary result.

Lemma 1.2. *Assume (A1), (A2), and that the inequality*

$$\frac{\lambda T}{1-\theta} \left(\alpha_1 + \alpha_2 (1 + M)\right) \leqslant M \tag{1.11}$$

holds, where M is as in (1.7) and θ is as in (1.6). Then the mapping H defined by (1.2) is continuous on K and $H(K) \subset K$.

Proof. Let $x \in K$, one may prove that $(Hx)(t + T) = (Hx)(t)$ for all $t \in \mathbb{R}$. Moreover, we get by (1.5) that

$$\begin{aligned}
(Hx)(t) &= \lambda \int_t^{t+T} h(t, u) f(u, x(u), x(x(u))) du \\
&\geqslant \lambda \int_t^{t+T} \frac{\theta}{1-\theta} f(u, x(u), x(x(u))) du \\
&= \theta \lambda \int_0^T \frac{1}{1-\theta} f(u, x(u), x(x(u))) du \\
&\geqslant \theta \|Hx\|,
\end{aligned}$$

where θ is as in (1.6). From (A2) and (1.11) we obtain $H(K) \subset K$.
Now we prove that the mapping H is continuous. Let the sequence $\{x_n\} \in K$ such

that $x_n \to x$ in K. Since the function f satisfies (1.9), we consider

$$\begin{aligned}
&\|Hx_n - Hx\| \\
&= \max_{t\in[0,T]} \left| \lambda \int_t^{t+T} h(t,u)\left(f(u,x_n(u),x_n(x_n(u))) - f(u,x(u),x(x(u)))\right) du \right| \\
&\leqslant \frac{\lambda}{1-\theta} \max_{t\in[0,T]} \int_t^{t+T} \left|f(u,x_n(u),x_n(x_n(u))) - f(u,x(u),x(x(u)))\right| du
\end{aligned}$$

which converges to 0 as $n \to \infty$ by Lebesgue convergence theorem. This proves the assertion. □

2. Existence results

In this section, we prove our existence results. Throughout this section, we assume the conditions (A1), (A2), and (1.11) hold. In preparation for the existence theorems, we introduce the following limits:

- $\underline{f_0} := \lim_{\xi\to 0^+} \min_{t\in[0,T]} \frac{f(t,\xi,\vartheta)}{\xi}$
- $\overline{f_0} := \lim_{\xi\to 0^+} \max_{t\in[0,T]} \frac{f(t,\xi,\vartheta)}{\xi}$
- $\underline{f_\infty} := \lim_{\xi\to \infty} \min_{t\in[0,T]} \frac{f(t,\xi,\vartheta)}{\xi}$
- $\overline{f_\infty} := \lim_{\xi\to \infty} \max_{t\in[0,T]} \frac{f(t,\xi,\vartheta)}{\xi}$.

THEOREM 2.1. *If $\underline{f_\infty} = \infty$ and $\overline{f_0} = 0$ (or $\underline{f_0} = \infty$ and $\overline{f_\infty} = 0$) for each $\vartheta \in (0,\infty)$, then the equation (1.1) has a positive T-periodic solution for any λ satisfying*

$$\frac{1-\theta}{T} \leqslant \lambda \leqslant \frac{1-\theta}{\theta T} \tag{2.1}$$

where θ is as in (1.6).

PROOF. Assume $\underline{f_\infty} = \infty$. Then there exists constants $R > 0$ and $c_1 \geqslant \frac{1}{(1-\theta)\theta}$ such that $f(t,\xi,\vartheta) \geqslant c_1\xi$ for each $(t,\vartheta) \in [0,T] \times (0,\infty)$ and all $\xi \geqslant R$. We define

$$\Omega_1 := \left\{ x \in P_{T,M} : x < \frac{R}{1-\theta} \right\}$$

as an open, bounded subset of the cone K. Obviously, if $x \in \partial\Omega_1 \cap K$, then $\|x\| = \frac{R}{1-\theta}$ and

$$R \leqslant x(t) \leqslant \frac{R}{1-\theta} \text{ for all } t \in [0,T]. \tag{2.2}$$

Then, combining (1.5), (2.1) and (2.2), we have

$$\|Hx\| \geqslant \frac{\lambda\theta}{1-\theta}\int_0^T f\left(u, x\left(u\right), x\left(x\left(u\right)\right)\right)du \geqslant \frac{\lambda\theta}{1-\theta}\int_0^T c_1 x\left(u\right)du \geqslant \frac{\theta}{T}\int_0^T c_1 x\left(u\right)du \geqslant \frac{R}{1-\theta} = \|x\|.$$

This means $\|Hx\| \geqslant \|x\|$ for all $x \in \partial\Omega_1 \cap K$. On the other hand, suppose $\overline{f_0} = 0$ and choose $\widetilde{R} > \frac{R}{1-\theta} > 0$ large enough so that $f(t,\xi,\vartheta) \leqslant c_2\xi$ for all $0 \leqslant \xi \leqslant \widetilde{R}$, $(t,\vartheta) \in [0,T] \times (0,\infty)$, and $0 < c_2 \leqslant \theta$. We introduce the ball

$$\Omega_2 := \left\{x \in P_{T,M} : \|x\| < \widetilde{R}\right\} \subset K,$$

and if $x \in \partial\Omega_2 \cap K$, then one may get

$$\|Hx\| \leqslant \frac{\lambda}{1-\theta}\int_0^T f\left(u, x\left(u\right), x\left(x\left(u\right)\right)\right)du \leqslant \frac{\lambda}{1-\theta}\int_0^T c_2 x\left(u\right)du \leqslant \widetilde{R} = \|x\|$$

by using (1.5) and (2.1). Thus, $\|Hx\| \leqslant \|x\|$ for all $x \in \partial\Omega_2 \cap K$ and Theorem 1.1 implies the mapping H has a fixed point for $x \in K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right)$. Consequently, the advanced differential equation (1.1) has a positive T-periodic solution.

In the second part of the proof we assume $\underline{f_0} = \infty$. Then there exists $S > 0$ such that $f(t,\xi,\vartheta) \geqslant c_3\xi$ for all $0 < \xi < S$ and $(t,\vartheta) \in [0,T] \times (0,\infty)$, where $c_3 \geqslant \frac{1}{\theta^2}$. We define

$$\Upsilon_1 := \{x \in P_{T,M} : x < S\}$$

and if $x \in \partial\Upsilon_1 \cap K$, then we have $\|x\| = S$. By using the inequalities (1.5), (2.1) and the defined cone (1.10), we deduce $\theta S \leqslant x \leqslant S$ when $x \in \partial\Upsilon_1 \cap K$. In the light of above-given arguments, we obtain

$$\|Hx\| \geqslant \frac{\lambda\theta}{1-\theta}\int_0^T f\left(u, x\left(u\right), x\left(x\left(u\right)\right)\right)du \geqslant \frac{1-\theta}{T}\frac{\theta}{1-\theta}\int_0^T c_3 x\left(u\right)du \geqslant S = \|x\|.$$

This proves $\|Hx\| \geqslant \|x\|$ for $x \in \partial\Upsilon_1 \cap K$. Furthermore, if $\overline{f_\infty} = 0$, then there exists $\widetilde{S} > 0$ such that $f(t,\xi,\vartheta) \leqslant c_4\xi$ for all $\xi \geqslant \widetilde{S}$, $(t,\vartheta) \in [0,T] \times (0,\infty)$, where $0 < c_4 \leqslant \theta$. We set $S_{\max} \geqslant \max\left\{2S, \frac{1}{\theta}\widetilde{S}\right\}$ and define

$$\Upsilon_2 := \{x \in P_{T,M} : x < S_{\max}\}.$$

Obviously, if $x \in \partial\Upsilon_2$, then $\|x\| = S_{\max}$ and $\widetilde{S} \leqslant \theta\|x\| \leqslant x(t)$ for $t \in [0,T]$. Assume $x \in \partial\Upsilon_2 \cap K$, and by inequalities (1.5) and (2.1) consider

$$\|Hx\| \leqslant \frac{\lambda}{1-\theta}\int_0^T f\left(u, x\left(u\right), x\left(x\left(u\right)\right)\right)du \leqslant \frac{\lambda}{1-\theta}\int_0^T c_4 x\left(u\right)du \leqslant \frac{1}{\theta T}\int_0^T c_4 x\left(u\right)du \leqslant S_{\max} = \|x\|.$$

This implies $\|Hx\| \leqslant \|x\|$ for $x \in \partial\Upsilon_2 \cap K$ and from Theorem 1.1, the mapping H has a fixed point in $x \in K \cap \left(\overline{\Upsilon_2} \backslash \Upsilon_1\right)$. As a result, (1.1) has a positive T-periodic solution. The proof is complete. □

THEOREM 2.2. *Assume* $\underline{f_0} = \underline{f_\infty} = \infty$ *for each* $\vartheta \in (0, \infty)$, *and there exists a constant* $W > 0$ *such that*

$$f(t, \xi, \vartheta) \leqslant W\theta \tag{2.3}$$

for all $\xi \in (0, W]$, *and* $(t, \vartheta) \in [0, T] \times (0, \infty)$. *Then (1.1) has two positive periodic solutions for any* λ *satisfying*

$$\frac{1-\theta}{T} \leqslant \lambda \leqslant \frac{1-\theta}{\theta T}, \tag{2.4}$$

where θ *is as in (1.6).*

PROOF. If $\underline{f_0} = \infty$ holds, then there exists $W > \underline{W} > 0$ such that $f(t, \xi, \vartheta) \geqslant c_5\xi$ for $(t, \vartheta) \in [0, T] \times (0, \infty)$ and all $0 < \xi \leqslant \underline{W}$, where $c_5 \geqslant \frac{1}{\theta^2}$. We introduce

$$\Lambda_1 := \{x \in P_{T,M} : \|x\| < \underline{W}\}.$$

Then for $x \in \partial\Lambda_1 \cap K$, we get

$$\begin{aligned}\|Hx\| &\geqslant \frac{\lambda\theta}{1-\theta}\int_0^T f(u, x(u), x(x(u)))\,du \geqslant \frac{\lambda\theta}{1-\theta}\int_0^T c_5 x(u)\,du \\ &\geqslant \frac{\theta}{T}\int_0^T c_5 x(u)\,du \geqslant \frac{\theta}{T}\frac{1}{\theta^2}T\underline{W}\theta = \underline{W} = \|x\|\end{aligned}$$

due to (1.5), (2.4), and the cone (1.10). This means $\|Hx\| \geqslant \|x\|$ for $x \in \partial\Lambda_1 \cap K$. Moreover, if $\underline{f_\infty} = \infty$, then there exists $\overline{W} > W$ such that $f(t, \xi, \vartheta) \geqslant c_6\xi$ for all $\xi \geqslant \overline{W}$, $c_6 \geqslant \frac{1}{\theta^2}$ and $(t, \vartheta) \in [0, T] \times (0, \infty)$. Set

$$\Lambda_2 := \left\{x \in P_{T,M} : \|x\| < \overline{W}\right\}.$$

Then for $x \in \partial\Lambda_2 \cap K$, we obtain the result $\|Hx\| \geqslant \|x\|$ by applying the same procedure given above.

Moreover, we introduce

$$\Lambda_3 := \{x \in P_{T,M} : \|x\| < W\}.$$

In this case, when $x \in \partial\Lambda_3 \cap K$ we have the following

$$\|Hx\| \leqslant \frac{\lambda}{1-\theta}\int_0^T f(u, x(u), x(x(u)))\,du \leqslant \frac{1}{\theta T}\int_0^T W\theta du = W = \|x\|$$

by (1.5), (2.3), and (2.4). Since $\underline{W} < W < \overline{W}$, by Theorem 1.1 the mapping H has two fixed points in $K \cap \left(\overline{\Lambda_3} \backslash \Lambda_1\right)$ and $K \cap \left(\overline{\Lambda_2} \backslash \Lambda_3\right)$. This is equivalent to existence of at least two positive periodic solutions of (1.1). The proof is complete. □

The following result can be proven similar to the proof of Theorem 2.2. Therefore, we omit its proof.

Theorem 2.3. *Suppose* $\overline{f_0} = \overline{f_\infty} = 0$ *hold and there exists a constant* $Y > 0$ *such that*

$$f(t, \xi, \vartheta) \geqslant \frac{Y}{\theta},$$

for all $\xi \in [\theta Y, Y]$ *and* $(t, \vartheta) \in [0, T] \times (0, \infty)$. *Then (1.1) has two positive periodic solutions for*

$$\frac{1-\theta}{T} \leqslant \lambda \leqslant \frac{1-\theta}{\theta T}. \tag{2.5}$$

As an implementation of our results, we provide the following example.

Example 2.1. For the construction of the iterated differential equation in (1.1), we set $a(t) = 2 + \sin(10\pi t)$, $\lambda = 2$, and $f(t, \xi, \vartheta) = \frac{1}{16} e^{-\xi - \vartheta}$. At first, the function a is $\frac{1}{5}$-periodic, and the initial condition (A1) holds. Furthermore, the condition given in (A2) can be easily verified for the function f with constants $\alpha_1 = \frac{1}{16}$ and $\alpha_2 = \frac{1}{16}$. Additionally, the limit results $\underline{f_0} = \infty$ and $\overline{f_\infty} = 0$ hold. Then, the advanced equation in the spotlight is as follows

$$x'(t) = (2 + \sin(10\pi t))\, x(t) - \frac{1}{8} e^{-x(t) - x(x(t))}. \tag{2.6}$$

As the next task, we focus on the crucial inequality (1.11) in Lemma 1.2, and we observe it holds for any $M \geqslant 1$, where $\theta = e^{-2/5}$ (see (1.6)). Besides, it should be highlighted that $\lambda = 2$ satisfies the inequality (2.5). Thus, all conditions of Theorem 2.1 are satisfied, and consequentially the iterated differential equation (2.6) has a positive $\frac{1}{5}$-periodic solution.

3. Concluding comments

In [**7**], the authors generalize the results of [**5**] and study the existence and uniqueness of the periodic solutions of the equation

$$\frac{d}{dt} x(t) = -a(t)x(t) + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) + \frac{d}{dt} g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right)$$

by using Schauder's fixed point theorem over the set

$$P_T(L, M) := \{x \in P_T : \|x\| \leqslant L, |x(t_2) - x(t_1)| \leqslant M |t_2 - t_1|, \ M \geqslant 0\}. \tag{3.1}$$

First of all, we study the existence of positive periodic solutions (not general periodic solutions) of iterative differential equations by using a different methodology. Secondly, our existence results are valid in some particular cases that existing literature does not cover. For instance, existence of periodic solution of the equation (2.6) in $P_{\frac{1}{5}}(L, M)$ for $M \geqslant 13$ cannot be proved by using [**7**, Theorem 3.3] since condition 9 is not satisfied for $M \geqslant 13$. However, Example 2.1 shows that (2.6) has a positive periodic solution for any $M \geqslant 1$. Thus, in this study, we not only provide an alternative technique to discuss the existence of periodic solutions of iterated differential equations but also improve the existing literature.

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