BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **12**(2)(2022), 213-217
DOI: 10.7251/BIMVI2202213S

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON EDGE IRREGULARITY STRENGTH OF SUNLET GRAPH

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Abstract. For a simple graph G, a vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called k-labeling. The weight of an edge xy in G, written $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e., $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f, $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, written $es(G)$. In this paper, we determine the bounds for the edge irregularity strength of line graph of sunlet graph $C_n \bigodot K_1$ for $n \geqslant 3$; and find the exact value of edge irregularity strength of block graph of sunlet graph.

1. Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set (the edge-set), then the labeling is called *vertex labelings* (*edge labelings*). If the domain is $V(G) \cup E(G)$, then the labeling is called *total labeling.* Thus, for an edge k-labeling $\delta : E(G) \to \{1, 2, \ldots, k\}$ the associated weight of a vertex $x \in V(G)$ is $w_\delta(x) = \sum \delta(xy)$, where the sum is over all vertices y adjacent to x.

Chartrand et al. [9] defined irregular labeling for a graph G as an assignment of labels from the set of natural numbers to the edges of G such that the sums of the labels assigned to the edges of each vertex are different.

2010 *Mathematics Subject Classification.* 05C38, 05C78.

Key words and phrases. Irregularity strength, edge irregularity strength, line graph, block graph, sunlet graph.

Communicated by Dusko Bogdanic.

The minimum value of the largest label of an edge over all existing irregular labelings is known as the *irregularity strength* of G and it is denoted by $s(G)$. Finding the irregularity strength of a graph seems to be hard even for simple graphs [9,10].

Motivated by this, Baca et al. [7] investigated two modifications of the irregularity strength of graphs, namely total edge irregularity strength, denoted by $tes(G)$; and total vertex irregularity strength, denoted by $tvs(G)$. Some results on the total edge irregularity strength and the total vertex irregularity strength can be found in [1, 4, 6, 8].

Motivated by the work of Chartrand et al. [9], Ahmad et al. [3] introduced the concept of edge irregular k-labelings of graphs.

A vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called k-labeling. The *weight of an edge* xy *in* G, written $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e., $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling of a graph G is defined to be an *edge irregular* k*-labeling* of the graph G if for every two different edges e and f, $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, written $es(G)$.

2. Preliminary results

The following theorem in [3] establishes the lower bound for the edge irregularity strength of a graph G.

THEOREM 2.1. *Let* $G = (V, E)$ *be a simple graph with maximum degree* $\Delta(G)$*. Then*

$$es(G) \geqslant max\{\lceil \frac{E(G)+1}{2} \rceil, \Delta(G)\}$$

Over the last years, $es(G)$ has been investigated for different families of graphs including trees with the help of algorithmic solutions, see [2, 5, 12−15].

The most complete recent survey of graph labelings is [11].

3. Edge irregularity strength of line graph of sunlet graph

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, the block graphs, and their generalizations.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.

The corona product of two graphs G and H, denoted by $G \bigodot H$, is a graph obtained by taking one copy of G (which has n vertices) and n copies $H_1, H_2, \ldots, H_n$ of H, and then joining the ith vertex of G to every vertex in H_i. The corona product $C_n \bigodot K_1$ is called the *sunlet graph.*

In the next theorem, we determine the bounds for the edge irregularity strength of line graph of sunlet graph.

THEOREM 3.1. *Let* $G = C_n \bigodot K_1$, $n \geqslant 3$, *be a sunlet graph. Then*

$$\lceil \frac{3n+1}{2} \rceil \leqslant es(L(G)) \leqslant 4n - 3.$$

PROOF. Let $G = C_n \bigodot K_1$, $n \geqslant 3$, be a sunlet graph. Let us consider the vertex set and the edge set of $L(G)$:

$$V(L(G)) = \{v_i : 1 \leqslant i \leqslant n\} \cup \{v_j : n+1 \leqslant j \leqslant 2n\},$$
$$E(L(G)) = \{v_iv_{i+1}, v_jv_{j+n}, v_kv_{k+n-1}, v_1v_n, v_1v_{2n} : 1 \leqslant i \leqslant n-1; 1 \leqslant j \leqslant n; 2 \leqslant k \leqslant n\}.$$

Clearly, $|V(L(G))| = 2n$, $|E(L(G))| = 3n$, and the maximum degree $\Delta(L(G)) = 4$. According to the Theorem 2.1, $es(L(G)) \geqslant max\{\lceil \frac{3n+1}{2} \rceil, 4\}$.

Since $\lceil \frac{3n+1}{2} \rceil > 4$ for $n \geqslant 3$, $es(L(G)) \geqslant \lceil \frac{3n+1}{2} \rceil$.

For the upper bound, we consider the following two cases.

Case 1: When $n = 2i+1, i \geqslant 1$. Define a labeling ϕ on vertex set of $L(G)$ as follows:

$$\phi(v_i) = i \text{ for } 1 \leqslant i \leqslant n;\ \phi(v_i) = i+n-2 \text{ for } n+1 \leqslant i \leqslant 2n-1;\ \phi(v_2) = 4n-3.$$

The edge weights are as follows:

$$w_\phi(v_1v_n) = n+1; w_\phi(v_nv_{2n}) = 5n-3; w_\phi(v_1v_{2n}) = 4n-2.$$
$$w_\phi(v_iv_{i+1}) = 2i+1 \ \text{ for } 1 \leqslant i \leqslant n-1.$$
$$w_\phi(v_iv_{n+i}) = 2(i+n-1) \ \text{ for } 1 \leqslant i \leqslant n-1.$$
$$w_\phi(v_iv_{n+i}) = 2(i+n)-1 \ \text{ for } 1 \leqslant i \leqslant n-1.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Thus the vertex labeling ϕ is the desired edge irregular $(4n-3)-$labeling. Thus $es(L(G)) \leqslant 4n-3$ for $n = 2i+1, i \geqslant 1$.

Case 2: When $n = 2(i+1), i \geqslant 1$. Define a labeling ϕ_1 as follows:

$$\phi_1(v_i) = i \text{ for } 1 \leqslant i \leqslant n-1;\ \phi_1(v_n) = n+1;\ \phi_1(v_i) = i+n-1 \text{ for } n+1 \leqslant i \leqslant 2n-1; \text{ and } \phi_1(v_{2n}) = 4n-3.$$

The edge weights are as follows:

$$w_{\phi_1}(v_iv_{i+1}) = 2i+1 \ \text{ for } 1 \leqslant i \leqslant n-2.$$
$$w_{\phi_1}(v_iv_{i+1}) = 2(i+1), i = n-1.$$
$$w_{\phi_1}(v_iv_{i+n}) = 2n+2i-3 \ \text{ for } 1 \leqslant i \leqslant n-1.$$
$$w_{\phi_1}(v_iv_{i+n}) = 2n+2i-3 \ \text{ for } 2 \leqslant i \leqslant n-1.$$
$$w_{\phi_1}(v_1v_n) = n+2; w_{\phi_1}(v_1v_{2n}) = 4n-2.$$
$$w_{\phi_1}(v_nv_{2n-1}) = 4n-1; w_{\phi_1}(v_nv_{2n}) = 5n-2.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Thus the vertex labeling ϕ_1 is the desired edge irregular $(4n-3)-$labeling. Thus $es(L(G)) \leqslant 4n-3$ for $n = 2(i+1), i \geqslant 1$. This completes the proof. □

In Theorem 3.1, we determined the bounds for the edge irregularity strength of line graph of sunlet graph. Finding the exact value of $es(L(C_n \bigodot K_1))$ seems to be very challenging and hence we propose the following open problem.

Open Problem: Determine the exact value of edge irregularity strength of line graph of sunlet graph $C_n \bigodot K_1$, $n \geqslant 3$.

4. Edge irregularity strength of block graph of sunlet graph

A graph G is connected if between any two distinct vertices there is a path. A maximal connected subgraph of G is called a *component* of G. A *cut-vertex* of a graph is one whose removal increases the number of components. A *non-separable graph* is connected, non-trivial, and has no cut-vertices. A *block* of a graph is a maximal non-separable subgraph. If two distinct blocks B_1 and B_2 are incident with a common cut-vertex, then they are called *adjacent blocks*.

The *block graph* of a graph G, written $B(G)$, is the graph whose vertices are the blocks of G and in which two vertices are adjacent whenever the corresponding blocks have a cut-vertex in common.

In the next theorem, we find the exact value of edge irregularity strength of block graph of sunlet graph.

THEOREM 4.1. *Let $G = C_n \bigodot K_1$, $n \geqslant 3$, be a sunlet graph. Then $es(B(G)) = n$.*

PROOF. Let $G = C_n \bigodot K_1$, $n \geqslant 3$, be a sunlet graph. By definition, the block graph of sunlet graph is a star graph $K_{1,n}, n \geqslant 3$, i.e., $B(G) = K_{1,n}$. Let us consider the vertex set and the edge set of $B(G)$:

$$V(B(G)) = \{v\} \cup \{v_i : 1 \leqslant i \leqslant n\},\ E(B(G)) = \{vv_i : 1 \leqslant i \leqslant n\}.$$

Clearly, $|V(B(G))| = n + 1$, $|E(B(G))| = n$, and the maximum degree $\Delta(B(G)) = n$. According to the Theorem 2.1, $es(B(G)) \geqslant max\{\lceil \frac{n+1}{2} \rceil, n\}$.

Since $\lceil \frac{n+1}{2} \rceil < n$ for $n \geqslant 3$, $es(B(G)) \geqslant n$.

To prove the equality, it suffices to prove the existence of $n-$labeling. Define a labeling ϕ_2 as follows:

$$\phi_2(v) = 1;\ \phi_2(v_i) = i \text{ for } 1 \leqslant i \leqslant n.$$

The edge weights are as follows:

$$w_{\phi_2}(vv_i) = i + 1 \ \text{ for } 1 \leqslant i \leqslant n.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Thus the vertex labeling ϕ_2 is the desired edge irregular $n-$labeling. Thus $es(B(G)) = n$. This completes the proof.

□

5. Conclusion

In this paper, we investigated the edge irregularity strength, as a modification of the well-known irregularity strength, total edge irregularity strength and total vertex irregularity strength. We determined the bounds for the edge irregularity strength of line graph of sunlet graph; and found the exact value of edge irregularity strength of block graph of sunlet graph. However, to find the exact values of edge irregularity strength of different graph operators still remains open.

Acknowledgement. We thank the anonymous referee for many useful comments and suggestions that helped us to improve the quality of the paper.

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Received by editors 23.12.2021; Revised version 28.1.2022; Available online 7.2.2022.

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