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PERIODIC SOLUTIONS FOR TOTALLY NONLINEAR ITERATIVE DIFFERENTIAL EQUATIONS

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Abstract. This paper studies the existence of periodic solutions of a totally nonlinear iterative differential equation. The equivalent integral equation of the given equation defines a fixed point mapping written as a sum of a large contraction and a compact map. The main results assert the existence of periodic solutions by making use of Krasnoselskii-Burton's fixed point technique.

1. Introduction

Delay or iterative differential equations have attracted considerable attention in mathematics during recent years since these equations have been showed to be valuable tools in the modeling of many phenomena in various fields of science, physics, chemistry and engineering, etc. In particular, periodicity, positivity and stability of solutions for delay or iterative differential equations has been studied extensively by many authors, see the references [**1**]–[**20**]. Motivated by the references [**1**]–[**20**] we consider the following totally nonlinear iterative differential equation

$$\begin{aligned}\frac{d}{dt}x(t) &= -a(t)\,h(x(t)) + \frac{d}{dt}g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right)\\ &\quad + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right),\end{aligned} \tag{1.1}$$

where

$$x^{[1]}(t) = x(t),\ x^{[2]}(t) = x(x(t)),\ ...,\ x^{[n]}(t) = x^{[n-1]}(x(t))$$

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and a is a continuous real-valued function. The functions $h:\mathbb{R}\to\mathbb{R}$, $g,f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are continuous.

Our purpose here is to use Krasnoselskii-Burton's fixed point technique to prove the existence of periodic solutions for (1.1). During the process we use the variation of parameter formula and the integration by parts to transform (1.1) into an equivalent integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use Krasnoselskii-Burton's fixed point theorem, to prove the existence of a periodic solution. The obtained results in this work extend the main results in [**7**].

2. Preliminaries

For $T>0$, define

$$P_T=\{x\in C(\mathbb{R},\mathbb{R}):x(t+T)=x(t)\ \text{ for all } t\in\mathbb{R}\},$$

where $C(\mathbb{R},\mathbb{R})$ denoted the set of all real valued continuous functions map $\mathbb{R}$ into $\mathbb{R}$. Then P_T is a Banach space with the norm

$$\|x\|=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|.$$

For $L,K>0$, define the set

$$P_T(L,K)=\{x\in P_T,\ \|x\|\leqslant L,\ \ |x(t_2)-x(t_1)|\leqslant K|t_2-t_1|\ \text{ for all } t_1,t_2\in\mathbb{R}\},$$

which is a closed convex and bounded subset of P_T.

We assume that

$$a(t+T)=a(t),\ \ \int_0^T a(t)\,dt>0. \tag{2.1}$$

The functions $f(t,x_1,x_2,...,x_n)$ and $g(t,x_1,x_2,...,x_n)$ are supposed periodic in t with period T and globally Lipschitz in x_1, x_2, ..., x_n, i.e,

$$\begin{aligned} f(t+T,x_1,...,x_n) &= f(t,x_1,...,x_n),\\ g(t+T,x_1,...,x_n) &= g(t,x_1,...,x_n), \end{aligned} \tag{2.2}$$

and there exist n positive constants k_1, k_2, ..., k_n and n positive constants c_1, c_2, ..., c_n such that

$$|f(t,x_1,...,x_n)-f(t,y_1,...,y_n)|\leqslant\sum_{i=1}^n k_i|x_i-y_i|, \tag{2.3}$$

and

$$|g(t,x_1,...,x_n)-g(t,y_1,...,y_n)|\leqslant\sum_{i=1}^n c_i|x_i-y_i|. \tag{2.4}$$

The function $g(t,x_1,...,x_n)$ is also supposed globally Lipschitz in t, i.e, there exists a positive constant K_g such that

$$|g(t_2,x_1,...,x_n)-g(t_1,x_1,...,x_n)|\leqslant K_g|t_2-t_1|. \tag{2.5}$$

The following lemma is essential for our results.

LEMMA 2.1. *Suppose (2.1) and (2.2) hold. If $x \in P_T(L,K)$, then x is a solution of (1.1) if and only if*

$$\begin{aligned} x(t) &= \int_t^{t+T} G(t,s)\, a(s)\, H(x(s))\, ds \\ &+ \int_t^{t+T} \left\{ f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right. \\ &\left. - a(s)\, g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right\} G(t,s)\, ds \\ &+ g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \end{aligned} \tag{2.6}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)\, du\right)}{\exp\left(\int_0^T a(u)\, du\right) - 1}, \tag{2.7}$$

and

$$H(x) = x - h(x). \tag{2.8}$$

PROOF. Let $x \in P_T(L,K)$ be a solution of (1.1). Rewrite (1.1) as

$$\begin{aligned} &\frac{d}{dt} x(t) + a(t)\, x(t) - \frac{d}{dt} g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \\ &= a(t)\, H(x(t)) + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{d}{dt}\left\{ \left[x(t) - g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \right] \exp\left(\int_0^t a(u)\, du \right) \right\} \\ &= \left\{ a(t)\, H(x(t)) - a(t)\, g\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \right. \\ &\left. + f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \right\} \exp\left(\int_0^t a(u)\, du \right). \end{aligned}$$

The integration from t to $t+T$ gives

$$\begin{aligned} &\int_t^{t+T} \frac{d}{ds}\left\{ \left[x(s) - g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right] \exp\left(\int_0^s a(u)\, du \right) \right\} ds \\ &= \int_t^{t+T} \left\{ a(s)\, H(x(s)) - a(s)\, g\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right. \\ &\left. + f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \right\} \exp\left(\int_0^s a(u)\, du \right) ds. \end{aligned}$$

Since

$$\int_t^{t+T} \frac{d}{ds}\left\{\left[x(s)-g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]\exp\left(\int_0^s a(u)\,du\right)\right\}ds$$
$$=\left\{x(t)-g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\right\}$$
$$\times\exp\left(\int_0^t a(u)\,du\right)\left[\exp\left(\int_t^{t+T} a(u)\,du\right)-1\right],$$

then

$$x(t)=g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)$$
$$+\int_t^{t+T}\left\{a(s)H(x(s))-a(s)g\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right.$$
$$\left.+f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right\}\frac{\exp\left(\int_t^s a(u)\,du\right)}{\exp\left(\int_t^{t+T} a(u)\,du\right)-1}ds.$$

The proof is completed. □

Lemma 2.2. *Green function G satisfies the following properties*

$$G(t+T,s+T)=G(t,s),$$

and

$$\alpha=\frac{\exp\left(-\int_0^T a(u)\,du\right)}{\left|\exp\left(\int_0^T a(u)\,du\right)-1\right|}\leqslant|G(t,s)|\leqslant\frac{\exp\left(\int_0^T a(u)\,du\right)}{\left|\exp\left(\int_0^T a(u)\,du\right)-1\right|}=\beta.$$

Lemma 2.3 ([**20**]). *For any $\varphi,\psi\in P_T(L,K)$, we have*

$$\left\|\varphi^{[m]}-\psi^{[m]}\right\|\leqslant\sum_{j=0}^{m-1}K^j\|\varphi-\psi\|,\ m=1,2,....$$

Lemma 2.4 ([**19**]). *It holds*

$$P_T(L,K)$$
$$=\{x\in P_T,\ \|x\|\leqslant L,\ \ |x(t_2)-x(t_1)|\leqslant K|t_2-t_1|\ \text{for all } t_1,t_2\in[0,T]\}.$$

Definition 2.1 (Large contraction [**10**]). Let $(\mathbb{M},d)$ be a metric space and consider $B:\mathbb{M}\to\mathbb{M}$. Then B is said to be a large contraction if given $\phi,\varphi\in\mathbb{M}$ with $\phi\neq\varphi$ then $d(B\phi,B\varphi)\leqslant d(\phi,\varphi)$ and if for all $\varepsilon>0$, there exists a $\delta\in(0,1)$ such that

$$[\phi,\varphi\in\mathbb{M},\ d(\phi,\varphi)\geqslant\varepsilon]\Rightarrow d(B\phi,B\varphi)\leqslant\delta d(\phi,\varphi).$$

Theorem 2.1 (Krasnoselskii-Burton [**10**]). *Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B},\|.\|)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) $x,y\in\mathbb{M}$, *implies* $Ax+By\in\mathbb{M}$,
(ii) *A is compact and continuous,*
(iii) *B is a large contraction mapping.*

Then there exists $z \in \mathbb{M}$ *with* $z = Az + Bz$.

We will use this theorem to show the existence of periodic solutions for (1.1).

THEOREM 2.2 ([**1**]). *Let* $\|.\|$ *be the supremum norm,* $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leqslant L\}$ *where* L *is a positive constant. Suppose that* h *is satisfying the following conditions*

(H1) $h : \mathbb{R} \to \mathbb{R}$ *is continuous on* $[-L, L]$ *and differentiable on* $(-L, L)$,

(H2) *the function* h *is strictly increasing on* $[-L, L]$,

(H3) $\sup_{t \in (-L,L)} h'(t) \leqslant 1$.

Then the mapping H *define by (2.8) is a large contraction on the set* M.

3. Existence of periodic solutions

To apply the Theorem 2.1 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a completely continuous and the other is a large contraction. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and

$$\begin{aligned} \mathbb{M} &= P_T(L, K) \\ &= \left\{\varphi \in P_T,\ \|\varphi\| \leqslant L,\ |\varphi(t_2) - \varphi(t_1)| \leqslant K|t_2 - t_1| \text{ for all } t_1, t_2 \in [0, T]\right\}, \end{aligned} \tag{3.1}$$

with $L, K > 0$. Define a mapping $\mathcal{S} : \mathbb{M} \to P_T$ by

$$\begin{aligned} (\mathcal{S}\varphi)(t) = &\int_t^{t+T} G(t,s)\, a(s)\, H(\varphi(s))\, ds + \int_t^{t+T} \Big\{ f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \\ &-a(s)\, g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right)\Big\} G(t,s)\, ds \\ &+ g\left(t, \varphi(t), \varphi^{[2]}(t), ..., \varphi^{[n]}(t)\right). \end{aligned} \tag{3.2}$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi = A\varphi + B\varphi,$$

where $A, B : \mathbb{M} \to P_T$ are given by

$$\begin{aligned} (A\varphi)(t) = &\int_t^{t+T} \Big\{ f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \\ &-a(s)\, g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right)\Big\} G(t,s)\, ds \\ &+ g\left(t, \varphi(t), \varphi^{[2]}(t), ..., \varphi^{[n]}(t)\right), \end{aligned} \tag{3.3}$$

and

$$(B\varphi)(t) = \int_t^{t+T} G(t,s)\, a(s)\, H(\varphi(s))\, ds. \tag{3.4}$$

To simplify notations, we introduce the following constants

$$\sigma = \max_{t \in [0,T]} |a(t)|,\ \rho_1 = \max_{t \in [0,T]} |f(t, 0, 0, ..., 0)|,\ \rho_2 = \max_{t \in [0,T]} |g(t, 0, 0, ..., 0)|. \tag{3.5}$$

We need the following assumptions

$$J\left[\beta T(\rho_1 + \sigma\rho_2) + \rho_2 + L\sum_{i=1}^{n}\left[c_i + \beta T(k_i + \sigma c_i)\right]\sum_{j=0}^{i-1} K^j\right] \leqslant L, \tag{3.6}$$

and

$$J\left((2\beta+T\alpha\|a\|)(\rho_1+\sigma\rho_2)+K_g\right.$$
$$\left.+\sum_{i=1}^{n}\left[(2\beta+T\alpha\|a\|)L(k_i+\sigma c_i)+Kc_i\right]\sum_{j=0}^{i-1}K^j\right)\leqslant K, \tag{3.7}$$

where J is a positive constant with $J\geqslant 3$.

Lemma 3.1. *For A defined in (3.3), suppose that (2.1)–(2.5) and (3.5)–(3.7) hold. Then $A:\mathbb{M}\to\mathbb{M}$.*

Proof. Let $\varphi\in\mathbb{M}$. For having $A\varphi\in\mathbb{M}$ we will show that $A\varphi\in P_T$, $\|A\varphi\|\leqslant L$ and $|(A\varphi)(t_2)-(A\varphi)(t_1)|\leqslant K|t_2-t_1|$ for all $t_1,t_2\in[0,T]$. First, it is easy to prove that $(A\varphi)(t+T)=(A\varphi)(t)$. That is, if $\varphi\in P_T$ then $A\varphi\in P_T$. By (3.5), we get

$$\begin{aligned}|(A\varphi)(t)|\leqslant\beta&\int_t^{t+T}\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\&+\beta\sigma\int_t^{t+T}\left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\&+\left|g\left(t,\varphi(t),\varphi^{[2]}(t),...,\varphi^{[n]}(t)\right)\right|,\end{aligned}$$

and in view of conditions (2.4), (2.5) and Lemma 2.3, we obtain

$$\begin{aligned}&\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|\\&\leqslant\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)-f(s,0,0,...,0)\right|+|f(s,0,0,...,0)|\\&\leqslant\rho_1+\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\|\varphi\|\\&\leqslant\rho_1+L\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j,\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&\left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|\\&\leqslant\left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)-g(s,0,0,...,0)\right|+|g(s,0,0,...,0)|\\&\leqslant\rho_2+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^j\|\varphi\|\\&\leqslant\rho_2+L\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^j.\end{aligned}\tag{3.9}$$

Thus, it follows from (3.8) and (3.9) that

$$\begin{aligned}|(A\varphi)(t)| &\leqslant \beta T\left(\rho_1 + L\sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} K^j\right) \\ &\quad + (\beta\sigma T + 1)\left(\rho_2 + L\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} K^j\right) \\ &= \beta T(\rho_1 + \sigma\rho_2) + \rho_2 + L\sum_{i=1}^{n}\left[c_i + \beta T(k_i + \sigma c_i)\right]\sum_{j=0}^{i-1} K^j.\end{aligned}$$

Therefore, from (3.6), we get

$$\|A\varphi\| \leqslant \frac{L}{J} \leqslant L.$$

Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we obtain

$$\begin{aligned}&|(A\varphi)(t_2) - (A\varphi)(t_1)| \\ &\leqslant \left|\int_{t_2}^{t_2+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2, s)\, ds \right. \\ &\left. - \int_{t_1}^{t_1+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_1, s)\, ds\right| \\ &+ \left|\int_{t_2}^{t_2+T} a(s)\, g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2, s)\, ds \right. \\ &\left. - \int_{t_1}^{t_1+T} a(s)\, g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_1, s)\, ds\right| \\ &+ \left|g\left(t_2, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_1), \varphi^{[2]}(t_1), ..., \varphi^{[n]}(t_1)\right)\right|.\end{aligned}$$

But,

$$\begin{aligned}&\left|\int_{t_2}^{t_2+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2, s)\, ds \right. \\ &\left. - \int_{t_1}^{t_1+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_1, s)\, ds\right| \\ &\leqslant \left|\int_{t_2}^{t_1} G(t_2, s)\, f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \right. \\ &\left. + \int_{t_1+T}^{t_2+T} G(t_2, s)\, f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds\right| \\ &+ \left|\int_{t_1}^{t_1+T} \left[G(t_2, s) - G(t_1, s)\right] f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds\right|\end{aligned}$$

$$\leqslant \int_{t_2}^{t_1} |G(t_2,s)| \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right| ds$$
$$+ \int_{t_1+T}^{t_2+T} |G(t_2,s)| \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right| ds$$
$$+ \frac{1}{\left|\exp\left(\int_0^T a(u)\,du\right) - 1\right|} \int_{t_1}^{t_1+T} \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right|$$
$$\times \left| \exp\left(\int_{t_2}^{s} a(u)\,du\right) - \exp\left(\int_{t_1}^{s} a(u)\,du\right) \right| ds,$$

and

$$\int_{t_1}^{t_1+T} \left| \exp\left(\int_{t_2}^{s} a(u)\,du\right) - \exp\left(\int_{t_1}^{s} a(u)\,du\right) \right| ds$$
$$= \int_{t_1}^{t_1+T} \exp\left(\int_{t_2}^{s} a(u)\,du\right) \left| 1 - \exp\left(\int_{t_1}^{t_2} a(u)\,du\right) \right| ds$$
$$\leqslant T\|a\| |t_2 - t_1| \exp\left(-\int_0^T a(u)\,du\right),$$

so,

$$\left| \int_{t_2}^{t_2+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2,s)\,ds \right.$$
$$\left. - \int_{t_1}^{t_1+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_1,s)\,ds \right|$$
$$\leqslant 2\beta |t_2 - t_1| \left(\rho_1 + L\sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} K^j\right) + T\alpha \|a\| |t_2 - t_1| \left(\rho_1 + L\sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} K^j\right)$$
$$\leqslant |t_2 - t_1| \left(\rho_1 + L\sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} K^j\right) (2\beta + T\alpha\|a\|). \tag{3.10}$$

Similarly, we get

$$\left| \int_{t_2}^{t_2+T} a(s)\, g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2,s)\,ds \right.$$
$$\left. - \int_{t_1}^{t_1+T} a(s)\, g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_1,s)\,ds \right|$$

$$\leqslant \left| \int_{t_2}^{t_1} a(s) g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2, s)\, ds \right.$$

$$\left. + \int_{t_1+T}^{t_2+T} a(s) g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2, s)\, ds \right|$$

$$+ \left| \int_{t_1}^{t_1+T} a(s) \left[G(t_2, s) - G(t_1, s)\right] g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds \right|$$

$$\leqslant \int_{t_2}^{t_1} |a(s)| \, |G(t_2, s)| \left| g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right| ds$$

$$+ \int_{t_1+T}^{t_2+T} |a(s)| \, |G(t_2, s)| \left| g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right| ds$$

$$+ \frac{1}{\left| \exp\left(\int_0^T a(u)\, du\right) - 1 \right|} \int_{t_1}^{t_1+T} |a(s)| \left| g\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right|$$

$$\times \left| \exp\left(\int_{t_2}^{s} a(u)\, du\right) - \exp\left(\int_{t_1}^{s} a(u)\, du\right) \right| ds$$

$$\leqslant |t_2 - t_1| \, \sigma \left(\rho_2 + L \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} K^j \right) (2\beta + T\alpha \|a\|). \tag{3.11}$$

Also, we have

$$\left| g\left(t_2, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_1), \varphi^{[2]}(t_1), ..., \varphi^{[n]}(t_1)\right) \right|$$

$$= \left| g\left(t_2, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) \right.$$

$$\left. + g\left(t_1, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_1), \varphi^{[2]}(t_1), ..., \varphi^{[n]}(t_1)\right) \right|$$

$$\leqslant \left| g\left(t_2, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) \right|$$

$$+ \left| g\left(t_1, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_1), \varphi^{[2]}(t_1), ..., \varphi^{[n]}(t_1)\right) \right|.$$

By (2.3)–(2.5) and Lemma 2.3, we get

$$\left| g\left(t_2, \varphi(t_2), \varphi^{[2]}(t_2), ..., \varphi^{[n]}(t_2)\right) - g\left(t_1, \varphi(t_1), \varphi^{[2]}(t_1), ..., \varphi^{[n]}(t_1)\right) \right|$$

$$\leqslant K_g |t_2 - t_1| + \sum_{i=1}^{n} c_i \left\| \varphi^{[i]}(t_2) - \varphi^{[i]}(t_1) \right\|$$

$$\leqslant \left(K_g + \sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} K^{j+1} \right) |t_2 - t_1|. \tag{3.12}$$

Thus, it follows from (3.10)–(3.12) and (3.7) that

$$\begin{aligned}
&\left|(A\varphi)(t_2)-(A\varphi)(t_1)\right| \\
&\leqslant \left((2\beta+T\alpha\|a\|)\left(\rho_1+\sigma\rho_2+L\sum_{i=1}^{n}(k_i+\sigma c_i)\sum_{j=0}^{i-1}K^j\right)\right. \\
&\left.+\left(K_g+\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^{j+1}\right)\right)|t_2-t_1|.
\end{aligned}$$

Therefore,

$$\left|(A\varphi)(t_2)-(A\varphi)(t_1)\right| \leqslant \frac{K}{J}|t_2-t_1| \leqslant K|t_2-t_1|.$$

Consequently, $A:\mathbb{M}\to\mathbb{M}$. □

Lemma 3.2. *Suppose that conditions (2.1)–(2.5) and (3.5)–(3.7) hold. Then the operator $A:\mathbb{M}\to\mathbb{M}$ given by (3.3), is continuous and compact.*

Proof. Since $\mathbb{M}$ is a uniformly bounded and equicontinuous subset of the space of continuous functions on the compact $[0,T]$ we can apply the Ascoli-Arzela theorem to confirm that $\mathbb{M}$ is a compact subset from this space. Also, and since any continuous operator maps compact sets into compact sets, then to prove that A is a compact operator it's suffices to prove that it is continuous. For $\varphi,\psi\in\mathbb{M}$, we have

$$\begin{aligned}
&\left|(A\varphi)(t)-(A\psi)(t)\right| \\
&\leqslant \int_t^{t+T}|G(t,s)|\left|f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right. \\
&\left.-f\left(s,\psi(s),\psi^{[2]}(s),...,\psi^{[n]}(s)\right)\right|ds \\
&+\int_t^{t+T}|a(s)||G(t,s)|\left|g\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right. \\
&\left.-g\left(s,\psi(s),\psi^{[2]}(s),...,\psi^{[n]}(s)\right)\right|ds \\
&+\left|g\left(t,\varphi(t),\varphi^{[2]}(t),...,\varphi^{[n]}(t)\right)-g\left(t,\psi(t),\psi^{[2]}(t),...,\psi^{[n]}(t)\right)\right|.
\end{aligned}$$

In view of conditions (2.4) and (2.5) and notations (3.5), we have

$$\begin{aligned}
&\left|(A\varphi)(t)-(A\psi)(t)\right| \\
&\leqslant \beta T\sum_{i=1}^{n}k_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|+(\beta\sigma T+1)\sum_{i=1}^{n}c_i\left\|\varphi^{[i]}-\psi^{[i]}\right\|.
\end{aligned}$$

From Lemma 2.3, it follows that

$$\begin{aligned}
&\left|(A\varphi)(t)-(A\psi)(t)\right| \\
&\leqslant \beta T\sum_{i=1}^{n}k_i\sum_{j=0}^{i-1}K^j\left\|\varphi-\psi\right\|+(\beta\sigma T+1)\sum_{i=1}^{n}c_i\sum_{j=0}^{i-1}K^j\left\|\varphi-\psi\right\| \\
&=\sum_{i=1}^{n}\left(\beta Tk_i+(\beta\sigma T+1)\,c_i\right)\sum_{j=0}^{i-1}K^j\left\|\varphi-\psi\right\|.
\end{aligned}$$

which proves that the operator A is continuous. Therefore, A is compact and continuous. □

The next result proves the relationship between the mappings H and B in the sense of large contractions. Assume that

$$\beta\sigma T\leqslant 1, \tag{3.13}$$

$$\max\left(\left|H(-L)\right|,\left|H(L)\right|\right)\leqslant\frac{(J-1)}{J}L, \tag{3.14}$$

and

$$\left(2\beta+T\alpha\left\|a\right\|\right)\sigma L\leqslant K. \tag{3.15}$$

Lemma 3.3. *Let B be defined by (3.4), suppose (2.1), (3.13), (3.14), (3.15) and all conditions of Theorem 2.2 hold. Then $B:\mathbb{M}\to\mathbb{M}$ is a large contraction.*

Proof. Let B be defined by (3.4). For having $B\varphi\in\mathbb{M}$ we will show that $\|B\varphi\|\leqslant L$ and $\left|(B\varphi)(t_2)-(B\varphi)(t_1)\right|\leqslant K\left|t_2-t_1\right|$ for all $t_1,t_2\in[0,T]$. First, it is easy to show that $(B\varphi)(t+T)=(B\varphi)(t)$. That is, if $\varphi\in P_T$ then $B\varphi\in P_T$. Let $\varphi\in\mathbb{M}$, by (3.14), we obtain

$$\begin{aligned}
\left|(B\varphi)(t)\right| &\leqslant \int_t^{t+T}\left|G(t,s)\right|\left|a(s)\right|\left|H(\varphi(s))\right|ds \\
&\leqslant \beta\sigma T\max\left\{\left|H(-L)\right|,\left|H(-L)\right|\right\}\leqslant\frac{(J-1)\,L}{J}\leqslant L.
\end{aligned}$$

Then, for any $\varphi\in\mathbb{M}$, we have

$$\|B\varphi\|\leqslant L.$$

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, by (3.13)–(3.15), we get

$$\begin{aligned}
&\left|(B\varphi)(t_1) - (B\varphi)(t_2)\right| \\
&\leqslant \left|\int_{t_2}^{t_2+T} G(t_2,s)\,a(s)\,H(\varphi(s))\,ds - \int_{t_1}^{t_1+T} G(t_1,s)\,a(s)\,H(\varphi(s))\,ds\right| \\
&\leqslant \left|\int_{t_2}^{t_1} G(t_2,s)\,a(s)\,H(\varphi(s))\,ds + \int_{t_1+T}^{t_2+T} G(t_2,s)\,a(s)\,H(\varphi(s))\,ds\right| \\
&+ \left|\int_{t_1}^{t_1+T} \left[G(t_2,s) - G(t_1,s)\right] a(s)\,H(\varphi(s))\,ds\right| \\
&\leqslant \int_{t_2}^{t_1} |G(t_2,s)|\,|a(s)|\,|H(\varphi(s))|\,ds + \int_{t_1+T}^{t_2+T} |G(t_2,s)|\,|a(s)|\,|H(\varphi(s))|\,ds \\
&+ \frac{1}{\left|\exp\left(\int_0^T a(u)\,du\right) - 1\right|} \int_{t_1}^{t_1+T} |a(s)|\,|H(\varphi(s))| \\
&\times \left|\exp\left(\int_{t_2}^{s} a(u)\,du\right) - \exp\left(\int_{t_1}^{s} a(u)\,du\right)\right| ds \\
&\leqslant 2\beta\sigma\left(\frac{(J-1)\,L}{J}\right)|t_2 - t_1| + \frac{1}{\left|\exp\left(\int_0^T a(u)\,du\right) - 1\right|} \int_{t_1}^{t_1+T} |a(s)|\,|H(\varphi(s))| \\
&\times \exp\left(\int_{t_2}^{s} a(u)\,du\right)\left|1 - \exp\left(\int_{t_1}^{t_2} a(u)\,du\right)\right| ds \\
&\leqslant 2\beta\sigma\frac{(J-1)\,L}{J}\,|t_2 - t_1| + T\alpha\,\|a\|\,\sigma\frac{(J-1)\,L}{J}\,|t_2 - t_1| \\
&= (2\beta + T\alpha\,\|a\|)\,\sigma\frac{(J-1)\,L}{J}\,|t_2 - t_1|.
\end{aligned}$$

Then

$$\left|(B\varphi)(t_1) - (B\varphi)(t_2)\right| \leqslant \frac{(J-1)\,K}{J}\,|t_2 - t_1| \leqslant K\,|t_2 - t_1|.$$

Therefore, $B : \mathbb{M} \to \mathbb{M}$.

It remains to prove that B is a large contraction. By Theorem 2.2, H is a large contraction on $\mathbb{M}$, then for any $\varphi, \psi \in \mathbb{M}$, with $\varphi \neq \psi$ we get

$$\begin{aligned}
&\left|(B\varphi)(t) - (B\psi)(t)\right| \\
&\leqslant \left|\int_{t}^{t+T} G(t,s)\,a(s)\left[H(\varphi(s)) - H(\psi(s))\right] ds\right| \\
&\leqslant \beta\sigma T\,\|\varphi - \psi\| \leqslant \|\varphi - \psi\|.
\end{aligned}$$

Then $\|B\varphi - B\psi\| \leqslant \|\varphi - \psi\|$. Now, let $\varepsilon \in (0,1)$ be given and let $\varphi, \psi \in \mathbb{M}$, with $\|\varphi - \psi\| \geqslant \varepsilon$ from the proof of Theorem 2.2, we have found a $\delta \in (0,1)$, such that

$$\left|(H\varphi)(t) - (H\psi)(t)\right| \leqslant \delta\,\|\varphi - \psi\|.$$

Thus,

$$\begin{aligned}&\left|(B\varphi)(t)-(B\psi)(t)\right|\\&\leqslant\left|\int_t^{t+T}G(t,s)\,a(s)\left[H(\varphi(s))-H(\psi(s))\right]ds\right|\\&\leqslant\beta\sigma T\delta\left\|\varphi-\psi\right\|\leqslant\delta\left\|\varphi-\psi\right\|.\end{aligned}$$

The proof is complete. □

THEOREM 3.1. *Suppose the hypothesis of Lemmas 3.1–3.3 hold. Let* $\mathbb{M}$ *defined by (3.1), then (1.1) has a* T*-periodic solution in* $\mathbb{M}$.

PROOF. By Lemmas 3.1 and 3.2 $A:\mathbb{M}\to\mathbb{M}$ is continuous and $A(\mathbb{M})$ is contained in a compact set. Also, from Lemma 3.3, the mapping $B:\mathbb{M}\to\mathbb{M}$ is a large contraction. Next, we prove that if $\varphi,\psi\in\mathbb{M}$, we have $\|A\varphi+B\psi\|\leqslant L$ and $|(A\varphi+B\psi)(t_2)-(A\varphi+B\psi)(t_1)|\leqslant K|t_2-t_1|$ for all $t_1,t_2\in[0,T]$. Let $\varphi,\psi\in\mathbb{M}$ with $\|\varphi\|,\|\psi\|\leqslant L$. By (3.6) and (3.14), we have

$$\begin{aligned}&\left\|A\varphi+B\phi\right\|\\&\leqslant\beta T(\rho_1+\sigma\rho_2)+\rho_2+L\sum_{i=1}^{n}\left[c_i+\beta T(k_i+\sigma c_i)\right]\sum_{j=0}^{i-1}K^j+\frac{(J-1)L}{J}\\&\leqslant\frac{L}{J}+\frac{(J-1)L}{J}=L.\end{aligned}$$

Now, let $\varphi,\psi\in\mathbb{M}$ and $t_1,t_2\in[0,T]$. By (3.7) and (3.15), we get

$$\begin{aligned}&\left|(A\varphi+B\psi)(t_2)-(A\varphi+B\psi)(t_1)\right|\\&\leqslant\left|(A\varphi)(t_2)-(A\varphi)(t_1)\right|+\left|(B\psi)(t_2)-(B\psi)(t_1)\right|\\&\leqslant\frac{K}{J}\left|t_2-t_1\right|+\frac{(J-1)K}{J}\left|t_2-t_1\right|\\&\leqslant K\left|t_2-t_1\right|.\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z\in\mathbb{M}$ such that $z=Az+Bz$. By Lemma 2.1, this fixed point is a solution of (1.1). Hence (1.1) has a T-periodic solution. □

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