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BRANCHES AND OBSTINATE SBE-FILTERS OF SHEFFER STROKE BE-ALGEBRAS

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Abstract. The aim of the study is to introduce an obstinate SBE-filter, a tile, a branch and a chain of Sheffer stroke BE-algebras (briefly, SBE-algebras). An obstinate SBE-filter is defined and some properties are investigated. Also, we determine a tile of a SBE-algebra and state the case which a SBE-subalgebra of a SBE-algebra is its SBE-filter. It is shown that the set of all tiles of a SBE-algebra is a SBE-subalgebra of this algebra but it is not a SBE-filter of this algebra. Finally, we describe a branch of a SBE-algebra by means of a tile of the algebraic structure and branchwise commutative and branchwise self-distibutive branches of SBE-algebras.

1. Section title

Sheffer operation (or Sheffer stroke) is introduced by H. M. Sheffer [**23**]. This operation is known NAND operator in logic and is one of the two operators that can be used by itself, without any other logical operators, to built a logical formal system. The well-known example is Boolean algebras whose axioms can be written in a single axiom using the Sheffer stroke [**7**]. Also, the most important application of Sheffer stroke is to have a single diod on the chip forming processor in a computer, and so, it is simpler and cheaper than to produce different diods for other Boolean operations. Since this operation can be used to reduce the number of axioms in a system, it provides new and easily applicable axiom systems for many algebraic structures. Therefore, Sheffer stroke has many applications in algebraic structures such as ortholattices [**2**], orthoimplication algebras [**1**], Sheffer stroke Hilbert algebras [**8**], their fuzzy filters [**9**] and neutrosophic N-structures [**14**], filters of strong Sheffer stroke non-associative MV-algebras [**10**] and their

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neutrosophic N-structures [**13**], (fuzzy) filters of Sheffer stroke BL-algebras [**11**] and their neutrosophic N-structures [**5**], Sheffer stroke UP-algebras [**12**], Sheffer stroke BG-algebras [**15**] and their fuzzy implivative ideals [**16**] and Sheffer stroke BCK-algebras [**17**].

On the other side, H S. Kim and Y. H. Kim introduced BE-algebras as a generalization of a dual BCK-algebra and defined a filter and an upper set on this algebraic structure [**6**]. A. Rezaei and A. B. Saeid described a regular congruence relation to construct quotient BE-algebras from self-distributive BE-algebras [**19**] and introduced commutative ideals in BE-algebras with several properties [**22**]. Recently, Rezaei et al. stated relations between generalized Hilbert (in short, g-Hilbert) algebras, CI/BE-algebras, implication algebras and other algebraic structures [**18**], [**20**], [**21**]. Recently, T. Katican et al. studied on BE-algebras with Sheffer stroke and various properties [**4**].

In this study, basic definitions and notions of Sheffer stroke BE-algebras (in short, SBE-algebras) are presented. An obstinate SBE-filter of a SBE-algebra are defined and some properties are given. We show that every obstinate SBE-filter of a SBE-algebra is its SBE-filter but the inverse does not generally hold. By describing a tile of a SBE-algebra, we state that every element of a SBE-algebra is its tile if and only if every SBE-subalgebra of this algebraic structure is its SBE-filter and that the set of all tiles of a SBE-algebra is its SBE-subalgebra. Moreover, a (improper and proper) branch of a SBE-algebra is introduced by means of a tile of this algebraic structure and the tile is said to be ultimate element for the branch. Also, it is shown that every branch of a SBE-algebra has an element $1 \circ 1$ of the algebra and that a SBE-algebra equals to an union of its branches, for all ultimate elements. We indicate that the set of all ultimate elements of proper branches of a SBE-algebra have the element 1 of this algebraic structure. After defining a chain of a SBE-algebra, it is demonstrated that a chain initiated by a tile of a SBE-algebra is its SBE-filter. Finally, a branchwise commutative and a branchwise self-distributive branch of a SBE-algebra are determined.

2. Preliminaries

In this section, basic definitions and notions about Sheffer stroke and Sheffer stroke BE-algebras are given.

Definition 2.1. [**2**] Let $\mathcal{S} = \langle S, \circ \rangle$ be a groupoid. The operation $|$ on S is said to be a *Sheffer operation (Sheffer stroke)* if it satisfies the following conditions for all $x, y, z \in S$:
$(S1)$ $x \circ y = y \circ x$,
$(S2)$ $(x \circ x) \circ (x \circ y) = x$,
$(S3)$ $x \circ ((y \circ z) \circ (y \circ z)) = ((x \circ y) \circ (x \circ y)) \circ z$,
$(S4)$ $(x \circ ((x \circ x) \circ (y \circ y))) \circ (x \circ ((x \circ x) \circ (y \circ y))) = x$.

Definition 2.2. [**4**] A Sheffer stroke BE-algebra (shortly, SBE-algebra) is a structure $\langle S; \circ, 1\rangle$ of type $(2, 0)$ such that 1 is the constant in S and the following axioms are satisfied for all $sx, y, z \in S$:

$(SBE-1)$ $x \circ (x \circ x) = 1$,
$(SBE-2)$ $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = y \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$.

LEMMA 2.1. [**4**] *Let* $\langle S; \circ, 1\rangle$ *be a SBE-algebra. Then the following hold for all* $x, y \in S$*:*
(i) $x \circ (1|1) = 1$,
(ii) $1 \circ (x \circ x) = x$,
(iii) $x \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x))) = 1$,
(iv) $x \circ (((x \circ (y \circ y)) \circ (y \circ y)) \circ ((x \circ (y \circ y)) \circ (y \circ y))) = 1$,
(v) $(x \circ 1) \circ (x \circ 1) = x$,
(vi) $((x \circ y) \circ (x \circ y)) \circ (x \circ x) = 1$ *and* $((x \circ y) \circ (x \circ y)) \circ (y \circ y) = 1$,
(vii) $x \circ ((x \circ y) \circ (x \circ y)) = x \circ y = ((x \circ y) \circ (x \circ y)) \circ y$.

DEFINITION 2.3. [**4**] A SBE-algebra $\langle S; \circ, 1\rangle$ is called commutative if

$$(x \circ (y \circ y)) \circ (y \circ y) = (y \circ (x \circ x)) \circ (x \circ x),$$

for any $x, y \in S$.

DEFINITION 2.4. [**4**] A SBE-algebra $\langle S; \circ, 1\rangle$ is called self-distributive if

$$x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = (x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z))),$$

for any $x, y, z \in S$.

DEFINITION 2.5. [**4**] Let $\langle S; \circ, 1\rangle$ be a SBE-algebra. Define a relation $\preceq$ on S by

$$x \preceq y \text{ if and only if } x \circ (y \circ y) = 1,$$

for all $x, y \in S$.

The relation is not a partial order on S, since it is only reflexive by $(SBE-1)$.

LEMMA 2.2. [**4**] *Let* $\langle S; \circ, 1\rangle$ *be a SBE-algebra. Then*
(1) *If* $x \preceq y$*, then* $y \circ y \preceq x \circ x$,
(2) $x \preceq y \circ (x \circ x)$,
(3) $y \preceq (y \circ (x \circ x)) \circ (x \circ x)$,
(4) *If* $\langle S; \circ, 1\rangle$ *is self-distributive, then* $x \preceq y$ *implies* $y \circ z \preceq x \circ z$,
(5) *If* $\langle S; \circ, 1\rangle$ *is self-distributive, then* $y \circ (z \circ z) \preceq (z \circ (x \circ x)) \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x)))$.

DEFINITION 2.6. [**4**] A nonempty subset $F \subseteq S$ is called a SBE-filter of a SBE-algebra $\langle S; \circ, 1\rangle$ if it satisfies the following properties:
$(SBEf-1)$ $1 \in F$,
$(SBEf-2)$ For all $x, y \in S$, $x \circ (y \circ y) \in F$ and $x \in F$ imply $y \in F$.

LEMMA 2.3. [**4**] *Let* $\langle S; \circ, 1\rangle$ *be a SBE-algebra. Then a nonempty subset* $F \subseteq S$ *is a SBE-filter of* S *if and only if for all* $x, y \in S$
(i) $x \in F$ *and* $y \in F$ *imply* $(x \circ y) \circ (x \circ y) \in F$,
(ii) $x \in F$ *and* $x \preceq y$ *imply* $y \in F$.

DEFINITION 2.7. [**4**] A subset T of a SBE-algebra $\langle S; \circ, 1\rangle$ is called a SBE-subalgebra of S if $x \circ (y \circ y) \in T$, for any $x, y \in T$. Clearly, S itself and $\{1\}$ are SBE-subalgebras of S.

LEMMA 2.4. [**4**] *Any SBE-filter of a SBE-algebra $\langle S;\circ,1\rangle$ is a SBE-subalgebra of S.*

DEFINITION 2.8. [**3**] A weak BCC-algebra X is an abstract algebra $(X,*,0)$ of type (2, 0) satisfying the following axioms:

(i) $((x*y)*(z*y))*(x*z)=0$,
(ii) $x*x=0$,
(iii) $x*0=x$,
(iv) $x*y=y*x=0\Rightarrow x=y$,

for all $x,y,z\in X$.

DEFINITION 2.9. [**3**] The set $B(a)=\{x\in X:a\leqslant x\}$, where a is an atom of a weak BCC-algebra X, is called a branch of X. The element a is called initial for $B(a)$. In the case when there exists $b\neq a$ such that $B(a)\subset B(b)$, we say that the branch $B(a)$ is improper. So, $B(a)$ is proper if no $b\in X$ such that $b\neq a$ and $b\leqslant a$. The set of all initial elements of proper branches of X is denoted by $I(X)$.

3. Obstinate SBE-filters

In this section, we introduce obstinate SBE-filters of SBE-algebras. Unless otherwise specified, S denotes a SBE-algebra.

DEFINITION 3.1. Let F be a SBE-filter of a SBE-algebra S. Then F is called obstinate if $x,y\notin F$ implies $x\circ(y\circ y),y\circ(x\circ x)\in F$, for all $x,y\in S$.

EXAMPLE 3.1. Consider the SBE-algebra $\langle S;\circ,1\rangle$ where $S=\{0,u,v,w,t,1\}$ and Sheffer operation $\circ$ with the following Cayley table [**4**]:

TABLE 1. Table of the Sheffer operation $\circ$ on S

$\circ$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	1	t	w	1	1	t
v	1	w	w	1	1	w
w	1	1	1	v	u	v
t	1	1	1	u	u	u
1	1	t	w	v	u	0

Then $\{u,v,1\}$ is an obstinate SBE-filter of S.

LEMMA 3.1. *Let F be a SBE-filter of a SBE-algebra S. Then F is obstinate if and only if $x\in F$ or $x\circ x\in F$, for all $x\in S$.*

PROOF. Let F be an obstinate SBE-filter of S. Assume that $x\circ x\notin F$. Since $x\circ x\notin F$ and $1\circ 1\notin F$, it follows from Lemma 2.1 (i)-(ii), (S1) and (S2) that $x=1\circ(x\circ x)=(x\circ x)\circ((1\circ 1)\circ(1\circ 1))\in F$ and $1=(1\circ 1)\circ((x\circ x)\circ(x\circ x))=x\circ(1\circ 1)\in F$. Suppose that $x\notin F$. Since $x\notin F$ and $1\circ 1\notin F$, it is obtained from Lemma 2.1 (i), (v), (S1) and (S2) that $x\circ x=x\circ 1=x\circ((1\circ 1)\circ(1\circ 1))\in F$ and $1=(x\circ x)\circ(1\circ 1)=(1\circ 1)\circ(x\circ x)\in F$.

Conversely, let F be a SBE-filter of S such that $x \in F$ or $x \circ x \in F$, for any $x \in S$. Assume that $x, y \notin F$. Then $x \circ x \in F$ anf $y \circ y \in F$. Since $x \circ x \preceq (y \circ y) \circ ((x \circ x) \circ (x \circ x)) = x \circ (y \circ y)$ and $y \circ y \preceq (x \circ x) \circ ((y \circ y) \circ (y \circ y)) = y \circ (x \circ x)$ from Lemma 2.2 (2), (S1) and (S2), we have from Lemma 2.3 (ii) that $x \circ (y \circ y) \in F$ and $y \circ (x \circ x) \in F$. Thus, F is obstinate. □

LEMMA 3.2. *Let F be a SBE-filter of a SBE-algebra S. Then F is obstinate if and only if*

$$x \circ (y \circ y) \in F \quad or \quad y \circ (x \circ x) \in F, \tag{3.1}$$

for all $x, y \in S$.

PROOF. Let F be an obstinate SBE-filter of a SBE-algebra S. Assume that $x \circ (y \circ y) \notin F$. Then $(x \circ (y \circ y)) \circ (x \circ (y \circ y)) \in F$ from Lemma 3.1. Since

$$\begin{aligned}
&((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ ((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \\
&= ((((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ y) \circ (((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \circ y)) \circ (x \circ x) \\
&= ((x \circ ((y \circ (y \circ y)) \circ (y \circ (y \circ y)))) \circ (x \circ ((y \circ (y \circ y)) \circ (y \circ (y \circ y))))) \circ (x \circ x) \\
&= ((x \circ (1 \circ 1)) \circ (x \circ (1 \circ 1))) \circ (x \circ x) \\
&= 1 \in F
\end{aligned}$$

from Lemma 2.1 (i), (S1), (S3), (SBE-1) and (SBEf-1), it follows from (SBEf-2) that $y \circ (x \circ x) \in F$. Suppose that $y \circ (x \circ x) \notin F$. Similarly, $(y \circ (x \circ x)) \circ (y \circ (x \circ x)) \in F$ from Lemma 3.1. Since

$$\begin{aligned}
&((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) \\
&= ((((y \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ x) \circ (((y \\
&\quad \circ (x \circ x)) \circ (y \circ (x \circ x))) \circ x)) \circ (y \circ y) \\
&= ((y \circ ((x \circ (x \circ x)) \circ (x \circ (x \circ x)))) \circ (y \circ \\
&\quad ((x \circ (x \circ x)) \circ (x \circ (x \circ x))))) \circ (y \circ y) \\
&= ((y \circ (1 \circ 1)) \circ (y \circ (1 \circ 1))) \circ (y \circ y) \\
&= 1 \in F
\end{aligned}$$

from Lemma 2.1 (i), (S1), (S3), (SBE-1) and (SBEf-1), we get from (SBEf-2) that $x \circ (y \circ y) \in F$.

Conversely, let F be a SBE-filter of S satisfying the statement (3.1). Suppose that $x, x \circ x \notin F$. Then it is obtained from (S2) and the statement (3.1) that $x = (x \circ x) \circ (x \circ x) \in F$ or $x \circ x = x \circ ((x \circ x) \circ (x \circ x)) \in F$, which is a contradiction. Hence, $x \in F$ or $x \circ x \in F$, for all $x \in S$. By Lemma 3.1, F is obstinate. □

REMARK 3.1. Every obstinate SBE-filter of a SBE-algebra S is a SBE-filter of S but the inverse is generally not true.

EXAMPLE 3.2. Consider the SBE-algebra S in Example 3.1. Then $\{1\}$ is a SBE-filter of S but it is not obstinate since $w \circ (u \circ u) = u \notin F$ and $u \circ (w \circ w) = w \notin F$ when $u, w \notin F$.

4. Branches

In this section, we present tiles and branches of SBE-algebras.

DEFINITION 4.1. An element s of a SBE-algebra S is called a tile of S if $s \circ (x \circ x) = 1$ implies $x = s$ or $x = 1$. The set of all tiles of S is denoted by $R(S)$.

EXAMPLE 4.1. Consider the SBE-algebra $\langle S; \circ, 1\rangle$ where $S = \{0, u, v, w, t, 1\}$ and Sheffer stroke $|$ with the following Cayley table [**4**]:

TABLE 2. Table of the Sheffer stroke $\circ$ on S

$\circ$	0	u	v	w	t	1
0	1	1	1	1	1	1
u	1	v	1	1	1	v
v	1	1	u	1	1	u
w	1	1	1	t	1	t
t	1	1	1	1	w	w
1	1	v	u	t	w	0

Then 1 is a tile of S while u is not since $w \neq u$ and $w \neq 1$ when $u \circ (w \circ w) = 1$.

REMARK 4.1. For every SBE-algebra S, it is clear from (SBE-1) that $1 \in R(S)$.

LEMMA 4.1. *Let S be a SBE-algebra. Then $1 \neq s$ is a tile of S if and only if a subset $\{s, 1\}$ of S is a SBE-filter of S.*

PROOF. Let $1 \neq s$ be a tile of S. It is obvious that $1 \in \{s, 1\}$. Assume that $x, x \circ (y \circ y) \in \{s, 1\}$. If $x = s$, then $s \circ (y \circ y) = 1$. Thus, $y = s$ or $y = 1$, and so, $y \in \{s, 1\}$. If $x = 1$, then $y = s$ or $y = 1$ from Lemma 2.1 (ii), and so, $y \in \{s, 1\}$. Hence, $\{s, 1\}$ is a SBE-filter of S.

Conversely, let $\{s, 1\}$ be a SBE-filter of S and $x \in S$ such that $s \circ (x \circ x) = 1$. Since $s \circ (x \circ x) = 1 \in \{s, 1\}$ and $s \in \{s, 1\}$, it is obtained from (SBEf-2) that $x \in \{s, 1\}$. Then $x = s$ or $x = 1$, which means that s is a tile of S. $\square$

LEMMA 4.2. *A SBE-algebra S contains only tiles (i.e., every element of S is a tile of S) if and only if every SBE-subalgebra of S is a SBE-filter of S.*

PROOF. Let every element s of S be a tile of S and T be a SBE-subalgebra of S. Then it is obvious that $1 \in T$. Assume that $x, x \circ (y \circ y) \in T$. Since $y \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) = 1$ from Lemma 2.1 (iii) and y is a tile of S, it follows that $x \circ (y \circ y) = y$ or $x \circ (y \circ y) = 1$. If $x \circ (y \circ y) = y$, then $y \in T$. If $x \circ (y \circ y) = 1$, then $y = x$ or $y = 1$ since x is a tile of S. Thus, $y \in T$. Hence, T is a SBE-filter of S.

Conversely, let every SBE-subalgebra T of S be a SBE-filter of S and $s \in S$ be not a tile of S. Then $\{s, 1\}$ is not a SBE-filter of S from Lemma 4.1, and so, $\{s, 1\}$ is not a SBE-subalgebra of S. Since $s \circ (1 \circ 1) = 1 \in \{s, 1\}$ and $1 \circ (s \circ s) = s \in \{s, 1\}$ from Lemma 2.1 (i)-(ii), $\{s, 1\}$ is a SBE-subalgebra of S, which is a contradiction. Thus, every element of S is a tile of S, i.e., S contains only tiles. $\square$

LEMMA 4.3. *Let s_1 and s_2 be any elements of a SBE-algebra S such that $s_1 \neq 1 \neq s_2$. If s_1 and s_2 are tiles of S, then $s_2 \circ (s_1 \circ s_1) = s_2$ and $s_1 \circ (s_2 \circ s_2) = s_2$.*

PROOF. Let s_1 and s_2 be tiles of S such that $s_1 \neq 1 \neq s_2$. By Lemma 4.1, $\{s_1, 1\}$ and $\{s_2, 1\}$ are SBE-filters of S. Since $s_1 \preceq s_2 \circ (s_1 \circ s_1)$ and $s_2 \preceq s_1 \circ (s_2 \circ s_2)$ from Lemma 2.2 (2), it follows from Lemma 2.3 (ii) that $s_2 \circ (s_1 \circ s_1) \in \{s_1, 1\}$ and $s_1 \circ (s_2 \circ s_2) \in \{s_2, 1\}$. If $s_2 \circ (s_1 \circ s_1) = 1$ or $s_1 \circ (s_2 \circ s_2) = 1$, then $s_1 = s_2$ or $s_1 = 1$ or $s_2 = 1$, which is a contradiction. Thus, $s_2 \circ (s_1 \circ s_1) = s_1$ and $s_1 \circ (s_2 \circ s_2) = s_2$. □

However, the inverse of Lemma 4.3 does not usually hold.

EXAMPLE 4.2. Consider the SBE-algebra S in Example 4.1. Then $u \circ (v \circ v) = v$ and $v \circ (u \circ u) = u$ but the elements u and v of S are not tiles of S since $R(S) = \{1\}$.

LEMMA 4.4. *$R(S)$ is a SBE-subalgebra of a SBE-algebra S.*

PROOF. Let $x, y \in R(S)$. If $x = y$, then $x \circ (x \circ x) = 1 \in R(S)$ from (SBE-1) and Remark 4.1. If $x \neq y$, then $x \circ (y \circ y) = y$ and $y \circ (x \circ x) = x$ from Lemma 4.3, and so, $x \circ (y \circ y) \in R(S)$ and $y \circ (x \circ x) \in R(S)$. Thus, $R(S)$ is a SBE-subalgebra of S. □

$R(S)$ is not a SBE-filter of a SBE-algebra S in general.

EXAMPLE 4.3. Consider the SBE-algebra $\langle S; \circ, 1\rangle$ where $S = \{0, u, v, 1\}$ and Sheffer stroke $\circ$ with Cayley table as below [**4**]:

TABLE 3. Table of the Sheffer stroke $\circ$ on S

$\circ$	0	u	v	1
0	1	1	1	1
u	1	v	1	v
v	1	1	u	u
1	1	v	u	0

Then $R(S) = \{u, v, 1\}$ is a SBE-subalgebra of S but it is not a SBE-filter of S since $0 \notin R(S)$ when $u \in R(S)$ and $u \circ (0 \circ 0) = v \in R(S)$.

DEFINITION 4.2. Let S be a SBE-algebra. Then a subset $B(s) = \{x \in S : x \circ (s \circ s) = 1\}$ of S is called a branch of S, where s is a tile of S. The element s is called ultimate for $B(s)$. If there exists $y \neq x$ such that $B(x) \subset B(y)$, then $B(x)$ is called improper. If there does not exist $y \in S$ such that $y \neq x$ and $x \circ (y \circ y) = 1$, then $B(x)$ is called proper. The set of all ultimate elements of proper branches of S is denoted by $I(S)$. Obviously, $I(S) \subseteq R(S)$.

EXAMPLE 4.4. Consider the SBE-algebra S in Example 4.3. Then $R(S) = \{u, v, 1\}$, $B(u) = \{0, u\}$, $B(v) = \{0, v\}$, $B(1) = S$ and $I(S) = \{1\}$. Thus, $B(1)$ is a proper branch of S but $B(u)$ and $B(v)$ are improper branches of S. $I(S)$ is a SBE-filter (and a SBE-subalgebra) of S, and $R(S)$ is a SBE-subalgebra of S but it is not a SBE-filter of S.

LEMMA 4.5. *Let S be a SBE-algebra. Then $B(1) = S$ and it is a proper branch of S.*

PROOF. Let S be a SBE-algebra. Since we have from Lemma 2.4 (i) that $x \circ (1 \circ 1) = 1$, for all $x \in S$, it follows that $B(1) = S$ and it is a proper branch of S. □

THEOREM 4.1. *Let S be a SBE-algebra. Then $1 \circ 1 \in B(s)$, for all ultimate elements $s \in S$.*

PROOF. Let S be a SBE-algebra. Since $(1 \circ 1) \circ (s \circ s) = (s \circ s) \circ (1 \circ 1) = 1$ from (S1) and Lemma 2.1 (i), it follows that $1 \circ 1 \in B(s)$, for all ultimate elements $s \in S$. □

COROLLARY 4.1. *Let S be a SBE-algebra. Then $S = \bigcup B(s)$, for all ultimate elements $s \in S$.*

LEMMA 4.6. *Let S be a SBE-algebra. Then $1 \in I(S)$.*

PROOF. Let S be a SBE-algebra. Since $B(1) = S$ is a proper branch of S from Lemma 4.5 and $1 \in R(S)$ is ultimate element of S from Remark 4.1, it is obtained that $1 \in I(S)$. □

DEFINITION 4.3. Let S be a SBE-algebra. Then

•it is called a branchwise commutative if $(x \circ (y \circ y)) \circ (y \circ y) = (y \circ (x \circ x)) \circ (x \circ x)$, and

•it is called a branchwise self-distributive if $x \circ ((y \circ (z \circ z)) \circ (y \circ (z \circ z))) = (x \circ (y \circ y)) \circ ((x \circ (z \circ z)) \circ (x \circ (z \circ z)))$,

for all x, y and z in the same branch.

EXAMPLE 4.5. In Example 4.4, the branches $B(u) = \{0, u\}$, $B(v) = \{0, v\}$ and $B(1) = S$ are branchwise commutative and branchwise self-distributive. However, $B(1) = S$ is not a branchwise commutative and a branchwise self-distributive for the SBE-algebra S in Example 4.1 since $(u \circ (w \circ w)) \circ (w \circ w) = w \neq u = (w \circ (u \circ u)) | (u|u)$ and $t \circ ((u \circ (w \circ w)) \circ (u \circ (w \circ w))) = 1 \neq w = (t \circ (u \circ u)) \circ ((t \circ (w \circ w)) \circ (t \circ (w \circ w)))$.

DEFINITION 4.4. A nonempty subset T of a SBE-algebra S is called a chain if $x \circ (y \circ y) = 1$ or $y \circ (x \circ x) = 1$, for $x, y \in T$. A chain initiated by s is denoted by $C(s)$, i.e., $s \circ (x \circ x) = 1$, for all $x \in C(s)$.

EXAMPLE 4.6. Consider the SBE-algebra S in Example 4.3. Then $C_1(0) = \{0, u, 1\}$, $C_2(0) = \{0, v, 1\}$, $C(u) = \{u, 1\}$, $C(v) = \{v, 1\}$ and $C(1) = \{1\}$. Also, $S = C_1(0) \cup C_2(0)$.

LEMMA 4.7. *Let S be a SBE-algebra. Then $C(s)$ is a SBE-filter of S, for tiles $s \in S$.*

PROOF. Let s be a tile of S. Since it is known from Lemma 2.1 (i) that $x \circ (1 \circ 1) = 1$, for $x \in C(s)$, we have that $1 \in C(s)$. Assume that $x, x \circ (y \circ y) \in C(s)$. Since $C(s)$ is initiated by s, it is obtained that $s \circ (a \circ a) = 1$, for all $a \in C(s)$. Thus, $s \circ (x \circ x) = 1$ and $s \circ ((x \circ (y \circ y)) \circ (x \circ (y \circ y))) = 1$, and so, $x = s$ or $x = 1$, and $x \circ (y \circ y) = s$ or $x \circ (y \circ y) = 1$. If $x = s$, then $s \circ (y \circ y) = 1$. Hence, $y = s$ or $y = 1$, i.e., $y \in C(s)$. If $x = 1$, then $1 \circ (y \circ y) = s$ or $1 \circ (y \circ y) = 1$, and so, $y = s$ or $y = 1$ from Lemma 2.1 (ii). Thereby, $y \in C(s)$. Therefore, $C(s)$ is a SBE-filter of S. □

5. Conclusion

In this study, an obstinate SBE-filter, a tile, a branch and a chain of a SBE-algebra are introduced and some properties are investigated. The statements equivalent to the definition of an obstinate SBE-filter of a SBE-algebra are presented. It is illustrated that every obstinate SBE-filter of a SBE-algebra is its SBE-filter but the inverse is not true in general. We define a tile of a SBE-algebra and it is shown that $1 \neq s$ is a tile of a SBE-algebra if and only if a subset $\{s, 1\}$ of the algebraic structure is its SBE-filter. Indeed, it is proved that A SBE-algebra contains only tiles if and only if every SBE-subalgebra of this algebraic structure is the SBE-filter. Infact, we demonstrate that $s_2 \circ (s_1 \circ s_1) = s_2$ and $s_1 \circ (s_2 \circ s_2) = s_2$ if s_1 and s_2 are different tiles of a SBE-algebra, but the inverse does not mostly hold. Also, it is stated that the set $R(S)$ of all tiles of a SBE-algebra is its SBE-subalgebra. We describe a branch of a SBE-algebra by means of a tile of this algebraic structure and this tile is called ultimate for the branch. We show that the branch $B(1)$ of a SBE-algebra equals to the algebraic structure and is a proper branch of the algebraic structure. Besides, it is indicated that every branch of a SBE-algebra has an element $1 \circ 1$ of the algebra and the algebra equals to an union of its branches, for all ultimate elements. By giving definitions of improper and proper branches of a SBE-algebra, the set $I(S)$ of all ultimate elements of proper branches of a SBE-algebra have the element 1 of the algebra. A chain of a SBE-algebra is introduced and it is propounded that a chain initiated by a tile of a SBE-algebra is its SBE-filter. Finally, we determine a branchwise commutative and a branchwise self-distributive branch of a SBE-algebra.

In the future works, we want to study on different SBE-filters and algebraic neighborhoods of SBE-algebras.

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