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INVERT TRANSFORM AND RESTRICTED WORDS

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Abstract. We give combinatorial interpretations of several sequences defined recurrently in terms of restricted words over a finite alphabet. One of the main tools for such investigations is the notion of invert transform which allows us to enlarge the alphabet by one letter. The initial sequence f_0 is defined via a linear homogeneous recurrence of the second order. Then, we define, for each integer $n \geqslant 1$, the sequence f_n as the invert transform of f_{n-1}. For a number of such recurrences we find an explicit formula for its solutions as well as their interpretations in terms of restricted words. Explicit bijections between different sets of restricted words counted by the same Fibonacci number are constructed.

1. Introduction

Linear homogenous recurrences of the second order have been studied extensively and there is a vast amount of literature containing various formulas involving sequences defined recurrently (as an introduction to the topic, we recommend [**14**], [**12**], [**1**], [**13**], and [**6**]). Some well-known integer sequences are given by a linear homogeneous recurrence of the second order, for instance, Fibonacci numbers, Fibonacci polynomials, and Jacobsthal numbers.

In this paper, we continue our investigation of combinatorial interpretations of sequences defined recurrently in terms of restricted words ([**9**], [**10**], [**11**], [**4**]). The main tool in our investigation is that of invert transform which allows us to enlarge the alphabet by one letter.

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Let $f_0 = (f_0(1), f_0(2), \ldots)$ be an arithmetic function. The invert transform f_1 of f_0 is defined as follows

$$f_1(n) = \sum_{i=1}^{n} f_0(i) \cdot f_1(n-i), \quad n \geqslant 1, \tag{1.1}$$

where $f_1(0) = 1$.

Inductively, for $m \geqslant 1$, we define pairs (f_{m-1}, f_m) of arithmetic functions such that f_m is the invert transform of f_{m-1}.

For $m \geqslant 1$ and $0 \leqslant k \leqslant n$, we define a function $g_m(n,k)$ by the expansion

$$\left(\sum_{i=1}^{\infty} f_{m-1}(i) \cdot x^i\right)^k = \sum_{n=k}^{\infty} g_m(n,k) x^n. \tag{1.2}$$

It is easy to see that this formula is equivalent to the following

$$g_m(n,k) = \sum_{i_1+i_2+\cdots+i_k=n} f_{m-1}(i_1) \cdots f_{m-1}(i_k), \tag{1.3}$$

where the sum is over strictly positive $i_t, (t = 1, 2, \ldots, k)$. It is a well-known fact (see, for example, [**2**], [**5**], or Identity (1) and Identity (2) in [**10**]) that $g_m(n,k)$ is related to the Bell partial polynomials in the following way:

$$g_m(n,k) = \frac{k!}{n!} \cdot B_{n,k}(1! \cdot f_{m-1}(1), 2! \cdot f_{m-1}(2), \ldots).$$

It is obvious that the following recurrence holds

$$g_m(n,k) = \sum_{i=1}^{n-k+1} f_{m-1}(i) \cdot g_m(n-i, k-1), (1 \leqslant k \leqslant n), \tag{1.4}$$

with $g_m(0,0) = 1, g_m(n,0) = 0,\ n \neq 0$.

The next result from Proposition 6 in [**10**], associates g_m with g_{m-1} in the following way:

$$g_m(n,k) = \sum_{i=k}^{n} \binom{i-1}{k-1} \cdot g_{m-1}(n,i). \tag{1.5}$$

By applying the same formula to the right-hand side of this equation several times, we can express g_m as a function of g_1, i.e. we obtain

$$g_m(n,i_1) = \sum_{i_2=i_1}^{n} \cdots \sum_{i_{m-1}=i_{m-2}}^{n} \binom{i_2-i_1}{i_1-1} \cdots \binom{i_m-i_{m-1}}{i_{m-1}-i_{m-2}} \cdot g_1(n, i_{m-1}). \tag{1.6}$$

We investigate finite sequences over a finite alphabet $\alpha = \{0, 1, \ldots, a\}, (a \geqslant 1)$.

The following theorem allows us to enlarge our alphabet by an additional letter (cf. Proposition 10 in [**10**]). As in Proposition 10 in [**10**], we assume that $g_m(n,k)$ counts the number of words satisfying some property that is preserved under the replacement of some of the instances of the letter a by a new symbol x.

THEOREM 1.1. *Assume that $g_{m-1}(n,k)$ equals the number of some words of length $n-1$ over a finite alphabet α that contain $k-1$ letters a. Then $g_m(n,k)$ equals the number of words of length $n-1$ over $\alpha \cup \{x\}$ containing $k-1$ letters a.*

PROOF. The number $g_{m-1}(n,i),(k \leqslant i \leqslant n)$, equals the number of words of length $n-1$ having $i-1$ letters equal to a. We replace $i-k$ of these a's by the letter x and obtain a word of length $n-1$ having $k-1$ letters equal to a. These $i-k$ letters may be chosen in $\binom{i-1}{k-1}$ different ways. Summing over i from k to n, we obtain the number of words of length $n-1$ over $\alpha \cup \{x\}$, having $k-1$ letters equal to a. By (1.5), this sum is equal to $g_m(n,k)$. $\square$

The following equation, given in Corollary 2 in [**10**], connects f_m to g_m:

$$f_m(n) = \sum_{k=1}^{n} g_m(n,k). \tag{1.7}$$

As a consequence, we obtain the following proposition.

PROPOSITION 1.1. *If $f_{m-1}(n)$ is the number of words of length $n-1$ over an alphabet α, and if $x \notin \alpha$, then $f_m(n)$ equals the number of words of length $n-1$ over the alphabet $\alpha \cup \{x\}$.*

Next, in Proposition 7 in [**10**] it was proved that (see also [**3**])

$$f_m(n) = \sum_{i=1}^{n} m^{i-1} \cdot g_1(n,i). \tag{1.8}$$

In the following statement, f_m is expressed in terms of f_0.

PROPOSITION 1.2 ([**8**], Corollary 9). *Let f_0 be defined as follows*

$$f_0(n+2) = x_0 \cdot f_0(n+1) + y_0 \cdot f_0(n),$$

where $x_0, y_0, f_0(1), f_0(2)$ are given numbers.

Then $f_m(1) = f_0(1), f_m(2) = m \cdot f_0(1)^2 + f_0(2)$, and

$$f_m(n+2) = x_m \cdot f_m(n+1) + y_m \cdot f_m(n),$$

where

$$x_m = x_0 + m \cdot f_0(1), y_m = y_0 - m \cdot x_0 \cdot f_0(1) + m \cdot f_0(2).$$

In the following sections, for the cases we consider, we proceed as follows. First, we use Proposition 1.2 to get a recursion for f_m, and by using this recursion we give a combinatorial interpretation of f_m in terms of restricted words. The next step is to use (1.2), (7.1), or (1.4) to compute $g_1(n,k)$ and to give a combinatorial description of $g_1(n,k)$. Once we know $g_1(n,k)$, we use Theorem 1.1 and a combinatorial description of $g_1(n,k)$ to give a combinatorial description of $g_m(n,k)$. Finally, we use (1.8) to compute f_m.

2. Case $f_0(1) = 1, f_0(2) = 1, x_0 = 0, y_0 = 0$

We start with the following proposition (cf. Example 16 in [**10**]).

Proposition 2.1. *The following is valid:*

(i) *The following recurrence holds*

$$f_m(1) = 1, f_m(2) = m + 1,$$
$$f_m(n + 2) = m \cdot f_m(n + 1) + m \cdot f_m(n), (n \geqslant 1).$$

(ii) *The number $f_m(n)$ equals the number of words of length $n - 1$ over the alphabet $\{0, 1, \ldots, m\}$ with no adjacent zeros.*

Proof. (i) This part follows from Proposition 1.2.

(ii) For $n = 1$, the statement holds because $f_m(1) = 1$ and the empty word has no adjacent zeros. Also, since $f_m(2) = m + 1$ and words of length 1 have no adjacent zeros, the statement holds for $n = 2$. If a word of length $n > 1$ begins with a non-zero letter, then we obviously have $m \cdot f_m(n - 1)$ such words. If a word begins by zero, then the next letter also must be different from zero. Hence, there are $mf_m(n - 2)$ such words and the recursion holds. □

We next derive an explicit formula for $g_1(n, k)$ and give a combinatorial interpretation (cf. Corollary 17 in [**10**]).

Proposition 2.2. *The following holds:*

(i) *The following formula holds*

$$g_1(n, k) = \binom{k}{n - k}.$$

(ii) *The number $g_1(n, k)$ equals the number of binary words of length $n - 1$ having $k - 1$ ones and no adjacent zeros.*

Proof. (i) Formula follows easily from (1.2) which, in this case, has the form

$$(x + x^2)^k = \sum_{n=k}^{\infty} g_1(n, k)x^n.$$

(ii) In this case, the recurrence (1.4) has the form

$$g_1(n, k) = g_1(n - 1, k - 1) + g_1(n - 2, k - 1), \ 1 \leqslant k \leqslant n.$$

If a word counted by $g_1(n, k)$ begins by 1, then the next letter can be arbitrary. So there are $g_1(n - 1, k - 1)$ such words. If a word begins by 0, then the next letter must be 1. Hence, there are $g_1(n - 2, k - 1)$ such words. □

As an immediate consequence of the previous proposition and Theorem 1.1, we obtain the following result.

Proposition 2.3. *The number $g_m(n, k)$ equals the number of words of length $n - 1$ over $\{0, 1, \ldots, m\}$ having $k - 1$ letters equal to m and no two adjacent zeros.*

Finally, from (1.8) we obtain the following explicit formula

$$f_m(n) = \sum_{i=1}^{n} m^{i-1} \cdot \binom{i}{n-i}.$$

Also, by using (1.6), we can obtain a formula for $g_m(n,k)$ as a convolution of binomial coefficients.

It is easy to see that $f_1(n) = F_{n+1}$, $(n = 1, 2, \ldots)$, where F_n is the nth Fibonacci number. We thus obtain a well-known formula

$$F_{n+1} = \sum_{k=1}^{n} \binom{k}{n-k}.$$

and a well-known property of Fibonacci numbers.

Corollary 2.1. *The number F_{n+1} equals the number of binary words of length $n-1$ with no two adjacent zeros.*

3. Case $f_0(1) = 1, f_0(2) = 2, x_0 = 0, y_0 = 0$

We start with a recursion for f_m and its combinatorial interpretation.

Proposition 3.1 (i). *The following recurrence holds*

$$f_m(1) = 1, f_m(2) = m + 2,$$
$$f_m(n+2) = m \cdot f_m(n+1) + 2 \cdot m \cdot f_m(n),\ n \geqslant 1.$$

(ii) *The number $f_m(n)$ equals the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, m+1\}$ avoiding subwords $00, 11, 01,$ and 10.*

Proof. (i) This part of the statement follows from Proposition 1.2.

(ii) Since $f_m(1) = 1$ and the empty word satisfies the condition, the statement is true for $n = 1$. Also, it is true for $n = 2$ as $f_m(2) = m + 2$ and each word of length 1 satisfies the condition.

Assume that $n > 2$. If a word of length $n-1$ begins with a letter from $\{2, 3, \ldots, m+1\}$, then the remaining part of that word can be an arbitrary word of length $n-2$, and the number of such words is $m \cdot f_m(n-1)$. If a word begins with either 0 or 1, then the next letter must be from $\{2, 3, \ldots, m+1\}$. Hence, there are $2 \cdot m \cdot f_m(n-2)$ such words. Hence, the recurrence from (i) holds. □

We next derive an explicit formula for $g_1(n,k)$ and give its combinatorial interpretation.

Proposition 3.2 (i). *The following formula holds*

$$g_1(n,k) = 2^{n-k} \cdot \binom{k}{n-k}.$$

(ii) *The number $g_1(n,k)$ equals the number of ternary words of length $n-1$ having $k-1$ letters equal to 2 and no subwords of the form $00, 11, 01, 10$.*

PROOF. (i) Formula follows easily from (1.2) which, in this case, has the form

$$(x+2x^2)^k = \sum_{n=k}^{\infty} g_1(n,k)x^n.$$

(ii) In this case, the recurrence (1.4) has the form

$$g_1(n,k) = g_1(n-1,k-1) + 2\cdot g_1(n-2,k-1),\ 1 \leqslant k \leqslant n.$$

If a word counted by $g_1(n,k)$ begins by the letter 2, then the next letter can be arbitrary. So, there are $g_1(n-1,k-1)$ such words. If a word begins by one of the letters 0 or 1, then the next letter must be 2. Hence, there are $g_1(n-2,k-1)$ such words. □

As an immediate consequence of the previous proposition and Theorem 1.1, we obtain the following proposition.

PROPOSITION 3.3. *The number $g_m(n,k)$ equals the number of words of length $n-1$ over $\{0,1,\ldots,m+1\}$ having $k-1$ letters equal to $m+1$ and no subwords* $00, 11, 01,$ *and* 10.

Finally, from (1.8) we obtain the following explicit formula

$$f_m(n) = \sum_{i=1}^{n} m^{i-1}\cdot 2^{n-i}\cdot \binom{i}{n-i}.$$

We finally note that, from (1.6), we can obtain a formula for $g_m(n,k)$ as a convolution of binomial coefficients.

In the case $m=1$, we obtain the well-known formula for Jacobsthal numbers J_n

$$J_{n+1} = \sum_{k=1}^{n} 2^{n-k}\cdot \binom{k}{n-k}.$$

Also, we have the following well-known result.

COROLLARY 3.1. *The number J_{n+1} equals the number of ternary words of length $n-1$ with no subwords* $00, 11, 10,$ *and* 01.

4. Case $f_0(n)=1$ if n is odd, and $f_0(n)=0$ if n is even

In this case, we have $f_0(1)=1, f_0(2)=0, x_0=0, y_0=1$. As in the previous cases, we start with the recursive formula for f_m and its combinatorial interpretation (cf., Corollary 28 in [**8**]).

PROPOSITION 4.1. (1) *For $m \geqslant 0$, the following recurrence holds*

$$f_m(1)=1, f_m(2)=m,$$
$$f_m(n+2) = m\cdot f_m(n+1) + f_m(n).$$

(2) *The number $f_m(n)$ equals the number of words of length $n-1$ over the alphabet $\{0,1,\ldots,m\}$ in which 0 avoids a run of odd length.*

PROOF. (1) The first part follows from Proposition 1.2.

(2) The statement holds for $n=1$ since the empty word satisfies the condition. It also holds for $n=2$ because $f_m(2)=m$ and words $1,2,\dots,m$ are of length 1, and obviously satisfy the condition. If a word of length $n>2$ begins with a non-zero letter, then the remaining part of the word can be an arbitrary word of length $n-1$. So, there are $mf_m(n-1)$ such words. If a word begins with 0, then the next letter is also equal to 0, and so, we have $f_m(n-2)$ such words. □

We next calculate $g_1(n,k)$. In this case, each term in the sum on the right-hand side of (1.4) is equal either to 1 or to 0, so that $g_1(n,k)$ equals the number of solutions of the Diophantine equation

$$2(j_1+j_2+\cdots+j_k)=n-k,\ j_t\geqslant 0,\ t=1,2,\dots,k.$$

It follows that $g_1(n,k)=0$ if $n-k$ is odd. If $n-k$ is even, then (see Proposition 24 in [**9**] and Example 18 in [**10**])

$$g_1(n,k)=\binom{\frac{n-k}{2}+k-1}{k-1}.$$

From this equation one easily deduces that the number $g_1(n,k)$ equals the number of binary words of length $n-1$ having $k-1$ letters equal to 1 and no runs of 0 of odd length.

By using induction on m and Theorem 1.1, we obtain the following proposition (see Proposition 19 and Corollary 21 in [**10**]).

PROPOSITION 4.2. *The number $g_m(n,k)$ equals the number of words of length $n-1$ over $\{0,1,\dots,m\}$ having $k-1$ letters equal to m and no runs of 0 of odd length.*

By using (1.5), we again get an expression for $g_m(n,k)$ as a convolution of the binomial coefficients.

For $m=1$, the recurrence from Proposition 4.1 becomes the recurrence for Fibonacci numbers, that is $f_1(n)=F_n$.

COROLLARY 4.1. *The Fibonacci number F_n equals the number of binary words of length $n-1$ in which 0 avoids a run of odd length.*

5. Case $f_0(n)=n$

This is the case when $f_0(1)=1, f_0(2)=2, x_0=2, y_0=-1$. As before, we first give a recursive formula for $f_m(n)$ and its combinatorial interpretation in terms of restricted words (see Corollary 37 in [**8**]).

PROPOSITION 5.1. (1) *The following recurrence holds*

$$f_m(1)=1, f_m(2)=m+2,$$
$$f_m(n+2)=(m+2)\cdot f_m(n+1)-f_m(n).$$

(2) *The number $f_m(n)$ is the number of 01-avoiding words of length $n-1$ over $\{0,1,\dots,m+1\}$.*

Proof. (1) The first assertion follows from Proposition 1.2.

(2) Since the empty word satisfies the condition, we conclude that the statement holds for $n = 1$. Also, it holds for $n = 2$, because it is clear that $f_m(2) = m+1$ and each word of length 1 satisfies the condition. Assume the assertion holds for words of length $n-1$. By placing an arbitrary letter in front of such a word, we obtain $(m+2)\cdot f_m(n+1)$ words of length n. Among them are all desired words of length n. To count all such words, we must subtract the number of words beginning by 01. The number of such words is $f_m(n)$. □

In the next proposition, we prove that $g_1(n, k)$ has the desired combinatorial interpretation (see Corollary 24 in [**11**]).

Proposition 5.2. (1) *The number $g_1(n, k)$ equals the number of ternary words of length $n-1$ having $k-1$ letters equal to 2 and avoiding the subword 01.*

(2) *The following formula holds*

$$g_1(n, k) = \binom{n+k-1}{2k-1}.$$

Proof. (1) The recurrence for $g_1(n, k)$ is given by (1.4):

$$g_1(n, k) = \sum_{i=1}^{n-k+1} i \cdot g_1(n-i, k-1).$$

We count the number of words of length $n-1$ having $k-1$ letters equal to 2 according to the first appearance, from left to right, of the letter 2. If 2 is the first letter, then we have $g_1(n-1, k-1)$ such words. This is the first term on the right side of the equation (1.4). If the first 2 appears at the jth place, then this word starts by the string of length $j-1$ consisting of zeros and ones. They are of the form

$$(1, \dots, 1), (1, \dots, 1, 0), (1, \dots, 1, 0, 0), \dots, (0, 0, \dots, 0). \tag{5.1}$$

So we have j such words, which gives the term $j \cdot g_{m-1}(n-j, k-1)$ in the formula (1.4). Summing over all j, we obtain the desired result.

(2) It is not too difficult to prove the formula directly by induction on n. See also [**7**] (Case (iii), page 123). □

By applying Theorem 1.1 and induction, we obtain the following result (see Corollary 23 in [**11**]).

Proposition 5.3. *The number $g_m(n, k)$ equals the number of words of length $n-1$ over $\{0, 1, \dots, m+1\}$ having $k-1$ letters equal to 2 and avoiding the subword 01.*

From Proposition 1.2, we obtain

$$f_1(1) = 1, f_1(2) = 3,$$
$$f_1(n+2) = 3f_1(n+1) - f_1(n),\ n \geqslant 1.$$

Clearly, this is a recurrence for the bisection of the Fibonacci numbers. Namely, it is easy to see that the following equation holds

$$f_1(n) = F_{2n},\ n \geqslant 1.$$

The following result is a combinatorial description of the bisection of Fibonacci numbers.

COROLLARY 5.1. *The number F_{2n} equals the number of ternary words of length $n-1$ avoiding* 01.

6. Case $f_0(1) = 0, f_0(n) = 1,\ n > 0$

In the following proposition, we give a recurrence for $f_m(n)$ and its combinatorial interpretation (see Example 26 in [**10**] and Corollary 24 in [**8**]).

PROPOSITION 6.1. (1) *The following recurrence holds*

$$f_m(1) = 0, f_m(2) = 1,$$
$$f_m(n+2) = f_m(n+1) + m \cdot f_m(n).$$

(2) *For $n > 1$, the number $f_m(n+2)$ equals the number of words of length $n-1$ over $\{0, 1, \ldots, m\}$ in which no two consecutive letters are nonzero.*
In particular case $m = 1$, we have

$$f_1(n) = F_{n-1},\ n \geqslant 1.$$

PROOF. The first assertion follows from Proposition 1.2.

For the second statement, note that $f_m(3) = f_m(2) = 1$. Next, $f_m(4) = 1 + m$ is the number of required words of length 1. If $n > 2$, then there are $f_m(n+1)$ words of length $n-1$ beginning by 0. If a word of length $n-1$ begins by a non-zero letter, then the next letter must be 0, so that we have $m \cdot f_n(n)$ such letters.

Finally, for $m = 1$, we obviously obtain the recurrence for Fibonacci numbers. □

From Proposition 13 in [**9**] it follows that

$$g_1(n, k) = \binom{n-k-1}{k-1}.$$

Note that $g_1(n, k) = 0$ for $n < 2k$, and subsequently that $g_m(n, k) = 0$ when $n < 2k$.

COROLLARY 6.1 ([**8**], Corollary 28). *The number $g_m(n+3, k)$ equals the number of words of length n having $k-1$ ones and no adjacent nonzero letters.*

From this we get a combinatorial interpretation for $g_1(n, k)$.

COROLLARY 6.2. *The number $g_1(n+3, k)$ equals the number of binary words of length n having $k-1$ ones which are all isolated.*

7. Case $f_0(1) = 1, f_0(2) = r, r \geqslant 1,\ x_0 = y_0 = 0$

This case generalizes the first two cases where we derived formulas for Fibonacci and Jacobsthal numbers. In order to emphasize the importance of the two special cases, we postponed this general case until now.

PROPOSITION 7.1. (1) *The following recurrence holds*

$$f_m(1) = 1, f_m(2) = m + r,$$
$$f_m(n + 2) = m \cdot f_m(n + 1) + r \cdot m \cdot f_m(n).$$

(2) *The number $f_m(n)$ equals the number of words of length $n - 1$ over the alphabet $\{0, 1, \ldots, m + r - 1\}$ such that each letter from $\{0, 1, \ldots, r - 1\}$ must be followed by one of the remaining m letters.*

PROOF. (1) This part easily follows from Proposition 1.2.

(2) Since $f_m(1) = 1$ and only the empty word has length 0, the claim holds for $n = 1$. Also, since $f_m(2) = m + r$ and our alphabet has $m + r$ letters, it holds for $n = 2$. Assume that $n > 2$. If a word of length $n - 1$ begins by a letter from $\{r, r + 1, \ldots, m + r - 1\}$, then the next letter may be arbitrary, so that we have $m \cdot f_m(n + 1)$ such words. If a word begins by a letter from $\{0, 1, \ldots, r - 1\}$, then the next letter must be one of the remaining m letters. So, we have $r \cdot m \cdot f_m(n)$ such letters. □

We next derive an explicit formula for $g_1(n, k)$ and give its combinatorial interpretation.

PROPOSITION 7.2 (i). *If $m = 1$, then*

$$g_1(n, k) = r^{n-k}\binom{k}{n-k}.$$

(ii) *The number $g_1(n, k)$ equals the number of words over $\{0, 1, \ldots, r\}$ of length $n - 1$ having $k - 1$ letters equal to r and no subwords of the form ij where $i, j \in \{0, 1, \ldots, r - 1\}$.*

PROOF. (i) From (7.1) follows that

$$g_1(n, k) = \sum_{i_1+i_2+\cdots+i_k=n} f_0(i_1)\cdots f_0(i_k), \tag{7.1}$$

where the sum is over positive i_j. Since $f_0(n) = 0$ for $n > 2$, it follows that we need to find solutions of $i_1 + i_2 + \cdots + i_k = n$, where each i_j is either 1 or 2. Let j be the number of 2's and $k - j$ the number of 1's appearing in this equation. Since $2j + k - j = n$, it follows that $j = n - k$. We can choose $n - k$ 2's in $\binom{k}{n-k}$ ways. It is obvious that if $n > 2k$, then this equation has no solutions.

(ii) In this case, the recurrence (1.4) has the form

$$g_1(n, k) = g_1(n - 1, k - 1) + r \cdot g_1(n - 2, k - 1),\ 1 \leqslant k \leqslant n.$$

If a word counted by $g_1(n, k)$ begins by the letter r, then the next letter can be arbitrary. So, there are $g_1(n - 1, k - 1)$ such words. If a word begins by one of the letters $0, 1, \ldots, r-1$, then the next letter must be r. Hence, there are $r \cdot g_1(n-2, k-1)$ such words. $\square$

As an immediate consequence of the previous proposition and Theorem 1.1, we obtain the following proposition.

PROPOSITION 7.3. *The number $g_m(n, k)$ equals the number of words of length $n-1$ over $\{0, 1, \ldots, r+m-1\}$ having $k-1$ letters equal to $r+m-1$ and no subwords of the form ij where $i, j \in \{0, 1, \ldots, r-1\}$.*

From (1.8) we obtain the following explicit formula

$$f_m(n) = \sum_{i=1}^{n} m^{i-1} \cdot r^{n-i} \cdot \binom{i}{n-i}.$$

We finally note that, from (1.6), we can obtain a formula for $g_m(n, k)$ as a convolution of binomial coefficients.

8. Bijections between sets of restricted words

In this section we gather the results about sets of restricted words that are counted by the Fibonacci numbers F_n and we construct explicit bijections between them. Before proceeding, we give another set of restricted words counted by the Fibonacci numbers.

PROPOSITION 8.1. *The number $F_n, (n \geqslant 0)$, equals the number of binary words of length $n+1$ beginning by 0 in which each 0 is followed by 1.*

PROOF. Let f_n count the number of such words of length $n+1$. For $n=0$, we have $F_0 = 0$, and since there are no words of length 1 beginning by 0 in which each 0 is followed by 1, the statement holds for $n=0$. It also holds for $n=1$ because it is clear that 01 is the only binary word satisfying given conditions, thus $f_1 = 1 = F_1$.

If $n \geqslant 2$, then, the last two letters of given words of length $n+1$ are either 01 or 11. By omitting 01 at the end of the words ending by 01, we obtain f_{n-2} words of length $n-1$, and by omitting 1 at the end of the words ending by 11, we obtain f_{n-1} words of length n. Hence, $f_n = f_{n-1} + f_{n-2} = F_n$. $\square$

By comparing the results related to Fibonacci numbers, we obtain the following proposition.

PROPOSITION 8.2. *Each of the following sets has F_n, where $n \geqslant 2$, elements:*

1. *The set A_{n-2} of binary words of length $n-2$ with no two adjacent zeros.*
2. *The set B_{n-1} of binary words of length $n-1$ in which 0 avoids a run of odd length.*
3. *The set C_{n+1} of binary words of length $n+1$ beginning by 0 in which each 0 is followed by 1.*

Furthermore, for even indices, F_{2n} is the number of elements in the set D_{n-1} of ternary words of length $n-1$ avoiding 01.

In the following proposition we construct explicit bijections between the sets A_{n-2}, B_{n-1}, C_{n+1}, and $D_{\frac{n}{2}-1}$ (when n is even) of restricted words from the previous proposition.

Proposition 8.3. (1) *A bijection between the sets B_{n-1} and A_{n-2} is given as follows: for a given word of length $n-1$ replace the leftmost occurrence of* 00 *by* 0*, and other occurrences of* 00 *by* 10 *to obtain a word of length $n-2$ with no two adjacent zeros. If there are no zeros, remove one instance of* 1*.*

(2) *A bijection between the sets C_{n+1} and A_{n-2} is given as follows: for a given word of length $n+1$ remove* 01 *from the beginning and* 1 *from the end of the word to obtain a word of length $n-2$ with no two adjacent zeros.*

(3) *A bijection between the sets D_{n-1} and A_{2n-2} is given as follows: for a given ternary word, each* 0 *is replaced by* 10*, each* 1 *is replaced by* 01*, and each* 2 *is replaced by* 11*.*

Proof. (1) By construction, the resulting word is of length $n-2$ and it has no two adjacent zeros. It is obvious that the given map is injective.

(2) Note that given words of length $n+1$ have to end by 1. Since the subwords of such words do not have adjacent zeros, the given map is a bijection.

(3) Since given ternary words do not have subwords of the form 01, there are no consecutive zeros in the words that are in the image of the given map. Again, by construction, the given map is injective. □

Example 8.1. The Fibonacci number F_6 counts the number of binary words of length 4 with no two adjacent zeros, the number of binary words of length 5 in which 0 avoids a run of odd length, the number of binary words of length 7 beginning by 0 in which each 0 is followed by 1, and the number of ternary words of length 2 avoiding subwords 01. The bijections from the previous proposition are given as follows:

$$\begin{array}{ccccc}
11111 & & 1111 & & 0111111 \\
00111 & & 0111 & & 0101111 \\
10011 & & 1011 & & 0110111 \\
11001 & \to & 1101 & \leftarrow & 0111011 \\
11100 & & 1110 & & 0111101 \\
10000 & & 1010 & & 0110101 \\
00001 & & 0101 & & 0101011 \\
00100 & & 0110 & & 0101101
\end{array}, \qquad
\begin{array}{ccc}
22 & & 1111 \\
12 & & 0111 \\
02 & & 1011 \\
21 & \to & 1101 \\
20 & & 1110 \\
00 & & 1010 \\
11 & & 0101 \\
10 & & 0110
\end{array}.$$

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References

[1] A.T. Benjamin and J.J. Quinn. *Proofs That Really Count.The Art of Combinatorial Proof*, Washington, DC: The Mathematical Association of America, 2003.

[2] D. Birmajer, J. B. Gil and M. D. Weiner. Linear recurrence sequences and their convolutions via Bell polynomials, *J. Integer Sequences* **18**(2015), Article 15.1.2

[3] D. Birmajer, J. B. Gill, and M. D. Weiner. On the enumeration of restricted words over a finite alphabet, *J. Integer Sequences*, **19**(2016), Article 16.1.3

[4] D. Bogdanić and M. Janjić. Arithmetics of some sequences via 2-determinants, *Bull. Int. Math. Virtual Inst.*, **11**(3) (2021), 517–526.

[5] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, 1974.

[6] R.A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, Singapore: World Scientific, 1997.

[7] V. E. Hoggatt and D. A. Lind. Fibonacci and binomial properties of weighted compositions, *J. Combin. Theory* **4** (1968), 121–124.

[8] M. Janjić. On linear recurrence arising from compositions of positive integers. *J. Integer Sequences*, **18** (2015), Article 15.4.7

[9] M. Janjić. Binomial coefficients and enumeration of restricted words. *J. Integer Sequences*, **19** (2016), Article 16.7.3

[10] M. Janjić. Some formulas for number of restricted words. *J. Integer Sequences*, **20** (2017), Article 17.6.5

[11] M. Janjić. Words and linear Recurrences. *J. Integer Sequences*, **21** (2018), Article 18.1.4

[12] R. C. Johnson. Fibonacci numbers and matrices. *manuscript available at* http://www.dur.ac.uk/bob.johnson/fibonacci/, 2008.

[13] T. Koshy. *Fibonacci and Lucas Numbers With Applications*, Wiley-Interscience, 2001.

[14] S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section. Theory and Applications*, Chichester: Ellis Horwood Ltd., Halsted Press, New York, 1989.

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