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NEW APPLICATIONS ON FOURTH-ORDER DIFFERENTIAL SUBORDINATION FOR MEROMORPHIC UNIVALENT FUNCTIONS

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Abstract. In the present paper, we introduce new applications on fourth-order differential subordination associated with differential linear operator $I_{s,r,1}(n,\lambda)$ in the punctured open unit disk $\mathbb{U}^*$. Also, we obtain some new results.

1. Introduction, definitions, and preliminaries

Denote by $\mathbb{C}$ be a complex plane $\mathbb{H} = \mathbb{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers, let

$$\mathbb{H}[a,n] = \left\{f \in \mathbb{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... \quad \right\}, \text{ and } \mathbb{H}_1 = \mathbb{H}[1,1].$$

Let Σ denote the class of functions $f(z)$ of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and meromorphic univalent in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C},\, 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}.$$

Ali et al. [**2**] introduced and investigated the linear operator

$$J_1(n,\lambda) : \Sigma \longrightarrow \Sigma,$$

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that is obtained as follows:

$$J_1(n,\lambda)f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k,\quad (z\in\mathbb{U}^*,\ \lambda>1). \tag{1.2}$$

The general Hurwitz-Lerch Zeta function

$$\Phi(z,s,r)=\sum_{k=0}^{\infty}\frac{z^k}{(r+k)^s},\ r\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ s\in\mathbb{C}\ \text{when}\ 0<|z|<1.$$

A linear operator $I_{s,r,1}(n,\lambda):\Sigma\longrightarrow\Sigma$ (see [**9**]) is defined
(1.3)

$$I_{s,r,1}(n,\lambda)f(z)=\frac{\Phi(z,s,r)}{zr^{-s}}*J_1(n,\lambda)f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\left(\frac{r}{1+k+r}\right)^s\left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k.$$

It is easily verified from (1.3) that

$$z\left(I_{s,r,1}(n,\lambda)f(z)\right)'=(\lambda-1)I_{s,r,1}(n+1,\lambda)f(z)-\lambda I_{s,r,1}(n,\lambda)f(z). \tag{1.4}$$

$$I_{0,r,1}(n,\lambda)f(z)=J_1(n,\lambda)f(z)\ \text{and}\ I_{0,r,1}(0,\lambda)f(z)=f(z).$$

In 2011, Antonino and Miller [**3**] presented basic concepts and extended the theory of the second-order differential subordination in the open unit disk introduced by Miller and Mocanu [**13**] to the third-order case. Many scholar have discussed and dealt with second-order differential subordination and superordination theory in recent years, like [**1, 8, 9, 10, 11, 12, 14**]. There are many authors who discussed the theory of the third-order differential subordination for example [**4, 5, 15, 16, 17, 18, 19**], few authors introduced the theory of fourth-order differential subordination for example ([**6, 7**]). In this paper, using methods of fourth-order differential subordination, sufficient conditions obtained.

To prove our main results, we need the basic concepts in theory of the fourth-order.

Definition 1.1. ([**13**]) Let $f(z)$ and $F(z)$ be members of the analytic function class $\mathbb{H}$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is superordinate to $f(z)$, if there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=F(w(z))$ $(z\in\mathbb{U})$. In this case, we write $f\prec F$ or $f(z)\prec F(z)$. If the function $F(z)$ is univalent in $\mathbb{U}$, then

$$f(z)\prec F(z)\ (z\in\mathbb{U})\Longleftrightarrow f(0)=F(0),\ \text{and}\ f(\mathbb{U})\subset F(\mathbb{U}).$$

Definition 1.2. ([**3**])Let $\mathbb{Q}$ be the set of analytic and univalent functions q on the set $\overline{\mathbb{U}}\setminus E(q)$, where

$$E(q)=\left\{\zeta\in\partial\mathbb{U}:\ \lim_{z\longrightarrow\zeta}q(z)=\infty\right\},$$

and are such that $\min|q'(\zeta)|=\rho>0$ for $\zeta\in\partial\mathbb{U}\setminus E(q)$. Further, let the subclass of $\mathbb{Q}$ for which $q(0)=a$ be denoted by $\mathbb{Q}(a)$ with $\mathbb{Q}(0)=\mathbb{Q}_0$ and $\mathbb{Q}(1)=\mathbb{Q}_1$, $\mathbb{Q}_1=\{q\in\mathbb{Q}:q(0)=1\}$.

DEFINITION 1.3. ([**6**]) Let $\varphi : \mathbb{C}^5 \times \mathbb{U} \longrightarrow \mathbb{C}$ and suppose that $h(z)$ be univalent function in $\mathbb{U}$. If $p(z)$ is analytic function in $\mathbb{U}$ and satisfies the following fourth-order differential subordination:

$$\varphi\left(p(z),\ zp'(z),\ z^2p''(z),\ z^3p^{(3)}(z),\ z^4p^{(4)}(z);\ z\right) \prec h(z), \tag{1.5}$$

then $p(z)$ is called a solution of the differential subordination (1.5). A univalent function $q(z)$ is called a dominant of the solution of (1.5), or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.5). A dominant $\tilde{q}(z)$ which satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.5) is said to be the best dominant.

LEMMA 1.1. *([**6**]) Let $z_0 \in \mathbb{U}$ with $r_0 = |z_0|$. For $n \geqslant 3$. Let*

$$f(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ...$$

be continuous on $\overline{\mathbb{U}_{r_0}}$ and analytic in $\mathbb{U}_{r_0} \cup \{z_0\}$, with $f(z) \neq 0$.If

$$|f(z_0)| = \max\left\{|f(z)| : z \in \overline{\mathbb{U}_{r_0}}\right\}, \tag{1.6}$$

then there exists $m \geqslant n$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = m, \tag{1.7}$$

$$Re\left\{\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right\} \geqslant m, \tag{1.8}$$

and

$$Re\left\{\frac{z_0 f'(z_0) + 3z_0^2 f''(z_0) + z_0^3 f^{(3)}(z_0)}{z_0 f'(z_0)}\right\} \geqslant m^2. \tag{1.9}$$

Then

$$Re\left\{\frac{z_0 f'(z_0) + 7z_0^2 f''(z_0) + 6z_0^3 f^{(3)}(z_0) + z_0^4 f^{(4)}(z_0)}{z_0 f'(z_0)}\right\} \geqslant m^3. \tag{1.10}$$

LEMMA 1.2. *([**6**]) Let $p \in \mathbb{H}[a, n]$ and $q \in \mathbb{Q}$ with $q(0) = a$ for $z \in \overline{\mathbb{U}_{r_0}}$. Let*

$$s = q^{-1}[p(z)] = f(z). \tag{1.11}$$

If there exists points $z_0 = r_0 e^{i\phi_0} \in \mathbb{U}$ and $s_0 \in \partial\mathbb{U} \setminus E(q)$ such that $p(z_0) = q(s_0)$ and $p(\overline{\mathbb{U}_{r_0}}) \subset q(\mathbb{U})$,

$$Re\left\{\frac{s_0 q''(s_0)}{q'(s_0)}\right\} \geqslant 0, \qquad \left|\frac{zp'(z)}{q'(s)}\right| \leqslant k, \tag{1.12}$$

and

$$Re\left\{\frac{s_0^2 q^{(3)}(s_0)}{q'(s_0)}\right\} \geqslant 0, \qquad \left|\frac{z^2 p''(z)}{q'(s)}\right| \leqslant k^2, \tag{1.13}$$

where $r_0 = |z_0|$. Then there exists $m \geqslant n \geqslant 3$ such that

$$z_0 p'(z_0) = m s_0 q'(s_0), \tag{1.14}$$

$$Re\left\{\frac{z_0 p''(z_0)}{p'(z_0)} + 1\right\} \geqslant m Re\left\{\frac{s_0 q''(s_0)}{q'(s_0)} + 1\right\}, \tag{1.15}$$

and

$$Re\left\{\frac{z_0p'(z_0)+3z_0^2p''(z_0)+z_0^3p^{(3)}(z_0)}{z_0p'(z_0)}\right\}\geqslant m^2Re\left\{\frac{s_0q'(s_0)+3s_0^2q''(s_0)+s_0^3q^{(3)}(s_0)}{s_0q'(s_0)}\right\}. \tag{1.16}$$

Then

$$\begin{aligned}&Re\left\{\frac{z_0p'(z_0)+7z_0^2p''(z_0)+6z_0^3p^{(3)}(z_0)+z_0^4p^{(4)}(z_0)}{z_0p'(z_0)}\right\}\\ &\geqslant m^3Re\left\{\frac{s_0q'(s_0)+7s_0^2q''(s_0)+6s_0^3q^{(3)}(s_0)+s_0^4q^{(4)}(s_0)}{s_0q'(s_0)}\right\},\end{aligned} \tag{1.17}$$

or

$$Re\left\{\frac{z_0^3p^{(4)}(z_0)}{p'(z_0)}\right\}\geqslant k^3Re\left\{\frac{s_0^3q^{(4)}(s_0)}{q'(s_0)}\right\}. \tag{1.18}$$

Definition 1.4. ([**6**]) Let Ω be a set in $\mathbb{C}$, $q\in\mathbb{Q}$ and $n\in\mathbb{N}\backslash\{2\}$. The class $\Psi_n[\Omega,q]$ of admissible functions consists of those functions $\varphi:\mathbb{C}^5\times\mathbb{U}\longrightarrow\mathbb{C}$ that satisfy the following admissibility condition:

$$\varphi(r,s,t,u,v;z)\notin\Omega,$$

whenever

$$r=q(\zeta),\ s=k\zeta q'(\zeta),\ Re\left\{\frac{t}{s}+1\right\}\geqslant kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

and

$$Re\left\{\frac{u}{s}\right\}\geqslant k^2Re\left\{\frac{\zeta^2q^{(3)}(\zeta)}{q'(\zeta)}\right\},\quad Re\left\{\frac{v}{s}\right\}\geqslant k^3Re\left\{\frac{\zeta^3q^{(4)}(\zeta)}{q'(\zeta)}\right\},$$

where $z\in\mathbb{U}$, $\zeta\in\partial\mathbb{U}\setminus E(q)$ and $k\geqslant n$.

The next theorem is the foundation result in the theory of fourth-order differential subordinations.

Theorem 1.1. *(See [**6**]) Let $p\in\mathbb{H}[a,n]$ with $n\in\mathbb{N}\backslash\{2\}$. Also, let $q\in\mathbb{Q}(a)$ and satisfy the following admissibility conditions:*

$$Re\left\{\frac{\zeta^2q^{(3)}(\zeta)}{q'(\zeta)}\right\}\geqslant 0,\quad and\quad \left|\frac{z^2p''(z)}{q'(\zeta)}\right|\leqslant k^2, \tag{1.19}$$

where $z\in\mathbb{U}$, $\zeta\in\partial\mathbb{U}\setminus E(q)$ and $k\geqslant n$. If Ω is a set in $\mathbb{C}$, $\varphi\in\Psi_n[\Omega,q]$ and

$$\varphi\Big(p(z),\ zp'(z),\ z^2p''(z),\ z^3p^{(3)}(z),\ z^4p^{(4)}(z);\ z\Big)\in\Omega, \tag{1.20}$$

then

$$p(z)\prec q(z)\quad(z\in\mathbb{U}).$$

2. Fourth-order differential subordination with $I_{s,r,1}(n,\lambda)f(z)$

We first define the following class of admissible function, which are required in proving the differential subordination theorem involving the operator $I_{s,r,1}(n,\lambda)f(z)$ defined by (1.3).

DEFINITION 2.1. Let Ω be a set in $\mathbb{C}$, and $q \in \mathbb{Q}_1$. The class $\mathbb{A}_l[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^5 \times \mathbb{U} \longrightarrow \mathbb{C}$ which satisfy the following admissibility condition:

$$\phi(a, b, c, d, e; z) \notin \Omega,$$

whenever

$$a = q(\zeta),\ b = \frac{k\zeta q'(\zeta) + \lambda q(\zeta)}{\lambda - 1},$$

$$Re\left\{\frac{(\lambda-1)(\lambda-2)c-\lambda(\lambda-1)a}{(\lambda-1)b-\lambda a} - (2\lambda - 1)\right\} \geqslant kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$$Re\left\{\frac{(\lambda-1)(\lambda-2)(\lambda-3)d-3\lambda(\lambda-1)(\lambda-2)c+2\lambda(\lambda^2-1)a}{(\lambda-1)b-\lambda a} + 3\lambda(\lambda+1)\right\} \geqslant k^2 Re\left\{\frac{\zeta^2 q^{(3)}(\zeta)}{q'(\zeta)}\right\},$$

and

$$Re\left\{\frac{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)e-4\lambda(\lambda-1)(\lambda-2)(\lambda-3)d+6\lambda(\lambda^2-1)(\lambda-2)c-3\lambda(\lambda^3+2\lambda^2-\lambda+2)a}{(\lambda-1)b-\lambda a}\right.$$
$$\left. -4\lambda(\lambda^2+9\lambda+2)\right\} \geqslant k^3 Re\left\{\frac{\zeta^3 q^{(4)}(\zeta)}{q'(\zeta)}\right\},$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geqslant 3$.

THEOREM 2.1. *Let $\phi \in \mathbb{A}_l[\Omega, q]$. If the function $f \in \Sigma$ and $q \in \mathbb{Q}_1$ satisfy the following conditions:*

$$Re\left\{\frac{\zeta^2 q^{(3)}(\zeta)}{q'(\zeta)}\right\} \geqslant 0,\ \left|\frac{I_{s,r,1}(n+2,\lambda)f(z)}{q'(\zeta)}\right| \leqslant k^2, \tag{2.1}$$

and

$$\begin{aligned}\{\phi\,(I_{s,r,1}(n,\lambda)f(z),\ I_{s,r,1}(n+1,\lambda)f(z),\ I_{s,r,1}(n+2,\lambda)f(z),\\ I_{s,r,1}(n+3,\lambda)f(z), I_{s,r,1}(n+4,\lambda)f(z); z) : z \in \mathbb{U}\} \subset \Omega,\end{aligned} \tag{2.2}$$

then

$$I_{s,r,1}(n,\lambda)f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

PROOF. Define the analytic function $F(z)$ in $\mathbb{U}$ by

$$F(z) = I_{s,r,1}(n,\lambda)f(z). \tag{2.3}$$

By differentiating (1.3) with respect to z with using (2.3), we deduce that

$$I_{s,r,1}(n+1,\lambda)f(z) = \frac{zF'(z) + \lambda F(z)}{\lambda - 1}. \tag{2.4}$$

By a similar argument, we get

$$I_{s,r,1}(n+2,\lambda)f(z) = \frac{z^2F''(z) + 2\lambda zF'(z) + \lambda(\lambda-1)F(z)}{(\lambda-1)(\lambda-2)}, \tag{2.5}$$

$$I_{s,r,1}(n+3,\lambda)f(z) = \frac{z^3F^{(3)}(z)+3\lambda z^2F''(z)+3\lambda(\lambda-1)zF'(z)+\lambda(\lambda^2-3\lambda+2)F(z)}{(\lambda-1)(\lambda-2)(\lambda-3)}, \tag{2.6}$$

and

$$\begin{aligned}I_{s,r,1}(n+4,\lambda)f(z) = &\frac{z^4F^{(4)}(z)+4\lambda z^3F^{(3)}(z)+6\lambda(\lambda-1)z^2F''(z)}{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}\\ &+\frac{4\lambda(\lambda^2-3\lambda+2)zF'(z)+\lambda(\lambda^3-6\lambda^2+11\lambda-6)F(z)}{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}.\end{aligned} \tag{2.7}$$

Define the transformation from $\mathbb{C}^5$ to $\mathbb{C}$ by

$$a(r,s,t,u,v)=r,\ b(r,s,t,u,v)=\frac{s+\lambda r}{\lambda-1},\ c(r,s,t,u,v)=\frac{t+2\lambda s+\lambda(\lambda-1)r}{(\lambda-1)(\lambda-2)},$$

$$d(r,s,t,u,v)=\frac{u+3\lambda t+3\lambda(\lambda-1)s+\lambda(\lambda^2-3\lambda+2)r}{(\lambda-1)(\lambda-2)(\lambda-3)},$$

and

$$e(r,s,t,u,v)=\frac{v+4\lambda u+6\lambda(\lambda-1)t+4\lambda(\lambda^2-3\lambda+2)s+\lambda(\lambda^3-6\lambda^2+11\lambda-6)r}{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}.$$

Let

$$\begin{aligned}&\psi(r,s,t,u,v;z)=\phi(a,b,c,d,e;z)\\ &=\phi\Big(r,\tfrac{s+\lambda r}{\lambda-1},\tfrac{t+2\lambda s+\lambda(\lambda-1)r}{(\lambda-1)(\lambda-2)},\tfrac{u+3\lambda t+3\lambda(\lambda-1)s+\lambda(\lambda^2-3\lambda+2)r}{(\lambda-1)(\lambda-2)(\lambda-3)},\\ &\tfrac{v+4\lambda u+6\lambda(\lambda-1)t+4\lambda(\lambda^2-3\lambda+2)s+\lambda(\lambda^3-6\lambda^2+11\lambda-6)r}{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)};z\Big).\end{aligned}\tag{2.8}$$

The proof will make use of Lemma 1.1. Using (2.3) to (2.7) and from (2.8), we have

$$\begin{aligned}&\psi\left(F(z),\ zF'(z),\ z^2F''(z),\ z^3F^{(3)}(z),\ z^4F^{(4)}(z);\ z\right)\\ &=\phi\left(I_{s,r,1}(n,\lambda)f(z),\ I_{s,r,1}(n+1,\lambda)f(z),\ I_{s,r,1}(n+2,\lambda)f(z),\right.\\ &\left.I_{s,r,1}(n+3,\lambda)f(z),I_{s,r,1}(n+4,\lambda)f(z);z\right).\end{aligned}\tag{2.9}$$

Hence (2.2) becomes

$$\psi\left(p(z),\ zp'(z),\ z^2p''(z),\ z^3p^{(3)}(z),\ z^4p^{(4)}(z);\ z\right)\in\Omega.$$

Note that

$$\frac{t}{s}+1=\frac{(\lambda-1)(\lambda-2)c-\lambda(\lambda-1)a}{(\lambda-1)b-\lambda a}-(2\lambda-1),$$

$$\frac{u}{s}=\frac{(\lambda-1)(\lambda-2)(\lambda-3)d-3\lambda(\lambda-1)(\lambda-2)c+2\lambda(\lambda^2-1)a}{(\lambda-1)b-\lambda a}+3\lambda(\lambda+1),$$

and

$$\begin{aligned}\frac{v}{s}&=\frac{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)e-4\lambda(\lambda-1)(\lambda-2)(\lambda-3)d+6\lambda(\lambda^2-1)(\lambda-2)c-3\lambda(\lambda^3+2\lambda^2-\lambda+2)a}{(\lambda-1)b-\lambda a}\\ &\quad-4\lambda(\lambda^2+9\lambda+2).\end{aligned}$$

Thus, the admissibility condition for $\phi\in\mathbb{A}_l[\Omega,q]$ in Definition 2.1 is equivalent to the admissibility condition for $\varphi\in\Psi_3[\Omega,q]$ as given in Definition 1.4 with $n=3$. Therefore, by using (2.1) and Lemma 1.1, we have

$$F(z)=I_{s,r,1}(n,\lambda)f(z)\prec q(z).$$

This completes the proof of Theorem 2.1. □

Our next corollary is an extension of Theorem 2.1 to the case when the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 2.1. *Let* $\Omega \subset \mathbb{C}$, *and let the function* q *be univalent in* $\mathbb{U}$ *with* $q(0) = 1$. *Let* $\phi \in \mathbb{A}_l[\Omega, q_\rho]$ *for some* $\rho \in (0,1)$, *where* $q_\rho(z) = q(\rho z)$. *If the function* $f \in \Sigma$ *and* q_ρ *satisfy the following conditions:*

(2.10)
$$Re\left\{\frac{\zeta^2 q_\rho^{(3)}(\zeta)}{q'_\rho(\zeta)}\right\} \geqslant 0,\ \left|\frac{I_{s,r,1}(n+2,\lambda)f(z)}{q'_\rho(\zeta)}\right| \leqslant k^2,\ (z \in \mathbb{U}, k \geqslant 3, \zeta \in \partial\mathbb{U} \setminus E(q_\rho)),$$

and

$$\phi\left(I_{s,r,1}(n,\lambda)f(z),\ I_{s,r,1}(n+1,\lambda)f(z),\ I_{s,r,1}(n+2,\lambda)f(z),\right.$$
$$\left.I_{s,r,1}(n+3,\lambda)f(z), I_{s,r,1}(n+4,\lambda)f(z); z\right) \in \Omega,$$

then

$$I_{s,r,1}(n+2,\lambda)f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Proof. By using Theorem 1.1, we get

$$I_{s,r,1}(n,\lambda)f(z) \prec q_\rho(z) \quad (z \in \mathbb{U}).$$

Then, we obtain the result from

$$q_\rho(z) \prec q(z) \quad (z \in \mathbb{U}).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\mathbb{A}_l[h(\mathbb{U}), q]$ is written as $\mathbb{A}_l[h, q]$. The following theorem is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let* $\phi \in \mathbb{A}_l[h, q]$. *If the function* $f \in \Sigma$ *and* $q \in \mathbb{Q}_0$ *satisfy the condition (2.1), and*

$$\phi\left(I_{s,r,1}(n,\lambda)f(z),\ I_{s,r,1}(n+1,\lambda)f(z),\ I_{s,r,1}(n+2,\lambda)f(z),\right.$$
$$\left.I_{s,r,1}(n+3,\lambda)f(z), I_{s,r,1}(n+4,\lambda)f(z); z\right) \prec h(z), \tag{2.11}$$

then $I_{s,r,1}(n,\lambda)f(z) \prec q(z) \quad (z \in \mathbb{U})$.

The next result is an immediate consequence of Corollary 2.1.

Corollary 2.2. *Let* $\Omega \subset \mathbb{C}$, *and* q *be univalent function in* $\mathbb{U}$ *with* $q(0) = 1$. *Let* $\phi \in \mathbb{A}_l[h, q_\rho]$ *for some* $\rho \in (0,1)$, *where* $q_\rho(z) = q(\rho z)$. *If the function* $f \in \Sigma$ *and* q_ρ *satisfy conditions (2.10), and (2.11), then*

$$I_{s,r,1}(n,\lambda)f(z) \prec q_\rho(z) \quad (z \in \mathbb{U}).$$

The following theorem yield the best dominant of the differential subordination (2.11).

Theorem 2.3. *Let* h *be univalent function in* $\mathbb{U}$. *Also, let* $\phi : \mathbb{C}^5 \times \mathbb{U} \longrightarrow \mathbb{C}$ *and suppose that the differential equation:*

(2.12)
$$\varphi\left(q(z), \frac{zq'(z)+\lambda q(z)}{\lambda-1}, \frac{z^2q''(z)+2\lambda zq'(z)+\lambda(\lambda-1)q(z)}{(\lambda-1)(\lambda-2)},\right.$$
$$\frac{z^3q^{(3)}(z)+3\lambda z^2q''(z)+3\lambda(\lambda-1)zq'(z)+\lambda(\lambda^2-3\lambda+2)q(z)}{(\lambda-1)(\lambda-2)(\lambda-3)},$$
$$\left.\frac{z^4q^{(4)}(z)+4\lambda z^3q^{(3)}(z)+6\lambda(\lambda-1)z^2q''(z)+4\lambda(\lambda^2-3\lambda+2)zq'(z)+\lambda(\lambda^3-6\lambda^2+11\lambda-6)q(z)}{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}; z\right) = h(z)$$

has a solution $q(z)$ *with* $q(0) = 1$*, which satisfies the condition (2.1). If the function* $f \in \Sigma$ *that satisfies condition (2.11), and if the function*

$$\phi\left(I_{s,r,1}(n,\lambda)f(z),\ I_{s,r,1}(n+1,\lambda)f(z),\ I_{s,r,1}(n+2,\lambda)f(z),\right.$$
$$\left. I_{s,r,1}(n+3,\lambda)f(z), I_{s,r,1}(n+4,\lambda)f(z); z\right)$$

is analytic in $\mathbb{U}$, *then*

$$I_{s,r,1}(n,\lambda)f(z) \prec q(z),$$

and $q(z)$ *is the best dominant.*

Proof. From Theorem 1.1, we see that $q(z)$ is a dominant of (2.11). Since $q(z)$ satisfies (2.12), it has also a solution of (2.11) and therefore q will be dominated by all dominants. Hence $q(z)$ is the best dominant. $\square$

In view of Definition 2.1 and in a special case when $q(z) = Mz, (M > 0)$, the class of admissible functions $\mathbb{A}_l[\Omega, q]$, denoted by $\mathbb{A}_l[\Omega, M]$, is described as follows.

Definition 2.2. Let Ω be a set in $\mathbb{C}$, and $M > 0$. The class $\mathbb{A}_l[\Omega, M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^5 \times \mathbb{U} \longrightarrow \mathbb{C}$ such that
(2.13)

$$\phi\Big(Me^{i\theta}, \tfrac{k+\lambda}{\lambda-1}Me^{i\theta}, \tfrac{L+[2\lambda k+\lambda(\lambda-1)]Me^{i\theta}}{(\lambda-1)(\lambda-2)}, \tfrac{N+3\lambda L+[3\lambda(\lambda-1)k+\lambda(\lambda^2-3\lambda+2)]Me^{i\theta}}{(\lambda-1)(\lambda-2)(\lambda-3)},$$
$$\tfrac{A+4\lambda N+6\lambda(\lambda-1)L+[4\lambda(\lambda^2-3\lambda+2)k+\lambda(\lambda^3-6\lambda^2+11\lambda-6)]Me^{i\theta}}{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}; z\Big) \notin \Omega,$$

whenever $z \in \mathbb{U}$, $Re(Le^{-i\theta}) \geqslant (k-1)kM$, $Re(Ne^{-i\theta}) \geqslant 0$ and $Re(Ae^{-i\theta}) \geqslant 0\, (\theta \in \mathbb{R};\ k \geqslant 3)$.

Corollary 2.3. *Let* $\phi \in \mathbb{A}_l[\Omega, M]$*. If the function* $f \in \Sigma$ *that satisfies*

$$|I_{s,r,1}(n+2,\lambda)| \leqslant k^2 M \quad (z \in \mathbb{U},\ k \geqslant 3,\ M > 0),$$

and

$$\phi\left(I_{s,r,1}(n,\lambda)f(z),\ I_{s,r,1}(n+1,\lambda)f(z),\ I_{s,r,1}(n+2,\lambda)f(z),\right.$$
$$\left. I_{s,r,1}(n+3,\lambda)f(z), I_{s,r,1}(n+4,\lambda)f(z); z\right) \in \Omega,$$

then

$$|I_{s,r,1}(n,\lambda)| < M.$$

In the special case, when $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\mathbb{A}_l[\Omega, M]$ is simply denoted by $\mathbb{A}_l[M]$. Corollary 2.3 can now be written in the following form:

Corollary 2.4. *Let* $\phi \in \mathbb{A}_l[\Omega, M]$*. If the function* $f \in \Sigma$ *that satisfies*

$$|I_{s,r,1}(n+2,\lambda)f(z)| \leqslant k^2 M \quad (z \in \mathbb{U},\ k \geqslant 3,\ M > 0),$$

and

$$\left|\phi\left(I_{s,r,1}(n,\lambda)f(z),\ I_{s,r,1}(n+1,\lambda)f(z),\ I_{s,r,1}(n+2,\lambda)f(z),\right.\right.$$
$$\left.\left. I_{s,r,1}(n+3,\lambda)f(z), I_{s,r,1}(n+4,\lambda)f(z); z\right)\right| < M,$$

then

$$|I_{s,r,1}(n,\lambda)f(z)| < M.$$

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