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RIEČAN AND BOSBACH STATE OPERATORS ON SHEFFER STROKE MTL-ALGEBRAS

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Abstract. In this paper, we present Riečan and Bosbach states notions on Sheffer stroke MTL-algebras. We obtain some fundamental results on these operators. We put forward a connection between each other. Also, we give substantial relations among these state operators and very true operator. Besides, we acquire some constant conclusions by using these state operators and very true operator and handle some characteristic feature of these mentioned operators on Sheffer stroke MTL-algebras.

1. Introduction

The concept of monoidal t-norm-based logic (or shortly MTL) was firstly defined by Godo and Esteva [**14**]. Montogna and Jenei proved that MTL can be used for the logic of all left continuous t-norms and their residua [**18**]. In concordance with these works, MTL-algebras are identified as a counterpart of this logical system [**14**]. Recently, there are many important works, which have been published on the structure of MTL-algebras, such as [**20, 29**]. These studies take a constructional influence on its algebraic counterparts of monoidal t-norm-based logic. For example, Vetterlein indicated that MTL-algebras correspond to the positive cone of a partially ordered group [**29**]. Furthermore, he verified that this algebra is a bounded, integral, commutative and pre-linear residuated lattice [**29**]. Also, MTL-algebras are the basis residuated structures having all algebras induced by their residua and continuous t-norms. Therefore, MTL-algebras get an important location in different structures which are linked with fuzzy logic [**31**].

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Oner and Senturk described Sheffer stroke basic algebras at the first time in the literature [**21**]. Sheffer stroke basic algebras have an important role in many numbers of logics as many-valued Łukasiewicz logics, non-classical logics, fuzzy logics and etc. In parallel with this logical efforts, Senturk put forward a reduction for MTL-algebras via only Sheffer stroke operation which is called as Sheffer stroke MTL-algebras [**25**]. Senturk and Oner defined very true operator on Sheffer stroke MTL-algebras [**27**].

Munduci described the notion states on MV-algebras [**19**]. These states used to explain averaging processes for formulas in Łukasiewicz logics. They are not only a generalization of the usual probability measures on Boolean algebras, but also they are used for a semantical interpretation of the probability of fuzzy events. For a different perspective, Riečan put forward states on BL-algebras as functions described on these algebras with interval $[0, 1]$ [**23**]. Georgescu gave Bosbach and Riečan states consisting of a domain as a pseudo BL-algebra and a codomain as the real closed interval $[0, 1]$ [**15**]. Senturk examined state operators on sheffer stroke basic algebras [**26**]. As a consequences of the works, the concept of states is implemented to other logical algebraic structures such as equality algebras, pseudo equality algebras, psuedo-BCK algebras, BL-algebras, semi-divisible residuated lattice, residuated lattices, morphism algebras and etc. [**9, 10, 12, 11, 13, 24, 28, 4**].

The concept of "very true" was described by Hájek getting an answer to the demand "whether any natural axiomatization is possible and how far can even this sort of fuzzy logic be captured by standard methods of mathematical logic?" [**16**]. In other words, the concept very true operator is implemented to reduce the number of possible logical values in many-valued logic. In addition to these, this operator is not only effectively used in particular tasks in various fields of mathematics [**17, 7, 1, 32**] but also has been integreted to other logical algebras such as commutative basic algebras [**3**], effect algebras [**8**], Rℓ-monoids [**22**], equality algebras [**30**], etc.

In this work, we introduce Riečan and Bosbach states notions on Sheffer stroke MTL-algebras. We get some fundamental results on these operators. We put forward a connection between each other. The substantial contributions of this paper are to give important relations among these state operators and very true operator. Also, we support these relations with examples. In Section 2, we recall some fundamental concepts about Sheffer stroke MTL-algebras. In Section 3, we present the notions of Riečan and Bosbach state operators on Sheffer stroke MTL-algebras. We attain some fundamental conclusions about them and we explain that Riečan state is also a Bosbach state or vice versa. In Section 4, we obtain some constant conclusions by using these state operators and very true operator and handle some characteristic feature of the mentioned operators on Sheffer stroke MTL-algebras.

2. Preliminaries

In this section of the paper, we demonstrate fundamental concepts which are needed throughout the paper. They are taken from [**5**] and [**2**].

Definition 2.1. Let L be a non-empty set. The structure $\mathfrak{L} = (L; \wedge, \vee)$ is called a lattice if the binary operations $\vee$ and $\wedge$ satisfy the following statements for all $u_1, u_2, u_3 \in L$:
(L_1) $u_1 \wedge u_2 = u_2 \wedge u_1$ and $u_1 \vee u_2 = u_2 \vee u_1$,
(L_2) $u_1 \wedge (u_2 \wedge u_3) = (u_1 \wedge u_2) \wedge u_3$ and $u_1 \vee (u_2 \vee u_3) = (u_1 \vee u_2) \vee u_3$,
(L_3) $u_1 \wedge u_1 = u_1$ and $u_1 \vee u_1 = u_1$,
(L_4) $u_1 \wedge (u_1 \vee u_2) = u_1$ and $u_1 \vee (u_1 \wedge u_2) = u_1$.

Definition 2.2. The structure $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ is called bounded lattice if it verifies the following properties for each $u_1 \in L$:

(i) $u_1 \wedge 1 = u_1$ and $u_1 \vee 1 = 1$,
(ii) $u_1 \wedge 0 = 0$ and $u_1 \vee 0 = u_1$.

Also, the elements 0 and 1 are called the least element and the greatest element of the lattice, respectively.

Definition 2.3. Let the structure $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. A mapping $u_1 \mapsto u_1^{\perp}$ is said to be an *antitone involution* if it satisfies the following statements:

(i) $u_1^{\perp\perp} = u_1$ (involution),
(ii) $u_1 \leqslant u_2$ implies $u_2^{\perp} \leqslant u_1^{\perp}$ (antitone).

Definition 2.4. Let $\mathcal{L}$ be a bounded lattice with an antitone involution. If the following statements are verified, then $u_1^{\perp}$ is called the complement of u_1 and the lattice $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ is also called an ortholattice.

$$u_1 \vee u_1^{\perp} = 1 \quad \text{and} \quad u_1 \wedge u_1^{\perp} = 0,$$

Lemma 2.1. *Let $\mathcal{L} = (L; \vee, \wedge, ^{\perp})$ be a lattice with antitone involution. Then, the De Morgan laws are verified on this structure as follows:*

$$u_1^{\perp} \wedge u_2^{\perp} = (u_1 \vee u_2)^{\perp} \text{ and } u_1^{\perp} \vee u_2^{\perp} = (u_1 \wedge u_2)^{\perp}.$$

Definition 2.5. [**6**] Let $\mathcal{G} = (G, |)$ be a groupoid. If the following statements are verified, then the operation $| : G \times G \to G$ is called a *Sheffer stroke operation.*
$(S1)$ $\varrho_1|\varrho_2 = \varrho_2|\varrho_1$,
$(S2)$ $(\varrho_1|\varrho_1)|(\varrho_1|\varrho_2) = \varrho_1$,
$(S3)$ $\varrho_1|((\varrho_2|\varrho_3)|(\varrho_2|\varrho_3)) = ((\varrho_1|\varrho_2)|(\varrho_1|\varrho_2))|\varrho_3$,
$(S4)$ $(\varrho_1|((\varrho_1|\varrho_1)|(\varrho_1|\varrho_1)))|(\varrho_1|((\varrho_1|\varrho_1)|(\varrho_2|\varrho_2))) = \varrho_1$.
If also the following identity
$(S5)$ $\varrho_2|(\varrho_1|(\varrho_1|\varrho_1)) = \varrho_2|\varrho_2$,
is satisfied, then it is said to be an *ortho-Sheffer stroke operation.*

Lemma 2.2. [**6**] *Let $\mathcal{G} = (G, |)$ be a groupoid with Sheffer stroke operation. Then, the following statements are satisfied for each $\varrho_1, \varrho_2, \varrho_3 \in G$:*
(i) $(\varrho_1|\varrho_2)|(\varrho_1|(\varrho_2|\varrho_3)) = \varrho_1$,
(ii) $(\varrho_1|\varrho_1)|\varrho_2 = \varrho_2|(\varrho_1|\varrho_2)$,
(iii) $\varrho_1|((\varrho_2|\varrho_2)|\varrho_1) = \varrho_1|\varrho_2$.

Lemma 2.3. [**6**] *Let $\mathcal{G} = (G, |)$ be a groupoid. The binary relation $\leqslant$ is given on G as*

$$\varrho_1 \leqslant \varrho_2 \text{ if and only if } \varrho_1|(\varrho_2|\varrho_2) = 1.$$

Then, the relation $\leqslant$ is a partial order on G.

LEMMA 2.4. [**6**] *Let $|$ be a Sheffer stroke operation on G and $\leqslant$ order relation of $\mathcal{G}$. Then, the following statements are verified.*

(i) $\varrho_1 \leqslant \varrho_2$ *if and only if* $\varrho_2|\varrho_2 \leqslant \varrho_1|\varrho_1$,
(ii) $\varrho_1|(\varrho_2|(\varrho_1|\varrho_1)) = \varrho_1|\varrho_1$ *is the identity of* $\mathcal{G}$,
(iii) $\varrho_1 \leqslant \varrho_2$ *implies* $\varrho_2|\varrho_3 \leqslant \varrho_1|\varrho_3$, *for all* $\varrho_3 \in G$,
(iv) $\varrho_3 \leqslant \varrho_1$ *and* $\varrho_3 \leqslant \varrho_2$ *imply* $\varrho_1|\varrho_2 \leqslant \varrho_3|\varrho_3$.

LEMMA 2.5. [**21**] *Let $\mathcal{G} = (G; |)$ be a Sheffer stroke basic algebra with the constant element* 1*. Then, the following identities are verified.*

(i) $\varrho_1|(\varrho_1|\varrho_1) = 1$,
(ii) $\varrho_1|(1|1) = 1$,
(iii) $1|(\varrho_1|\varrho_1) = \varrho_1$,
(iv) $((\varrho_1|(\varrho_2|\varrho_2))|(\varrho_2|\varrho_2))|(\varrho_2|\varrho_2) = \varrho_1|(\varrho_2|\varrho_2)$,
(v) $(\varrho_2|(\varrho_1|(\varrho_2|\varrho_2)))|(\varrho_1|(\varrho_2|\varrho_2)) = 1$.

DEFINITION 2.6. [**31**] Let $M \neq \emptyset$. The operations $\vee$, $\wedge$, $\rightarrow$ and $\circledast$ be binary operations on M and the elements 0 and 1 be algebraic constant of M. If the following statements are verified for each $m_1, m_2, m_3 \in M$, then the structure $\mathcal{M} = (M; \vee, \wedge, \rightarrow, \circledast, 0, 1)$ is called an MTL-algebra.
(MTL_1) $(M; \wedge, \vee, 0, 1)$ is a bounded lattice,
(MTL_2) $(M; \circledast, 0, 1)$ is a commutative monoid,
(MTL_3) $m_1 \leqslant m_2 \rightarrow m_3$ if and only if $m_1 \circledast m_2 \leqslant m_3$,
(MTL_4) $(m_1 \rightarrow m_2) \vee (m_2 \rightarrow m_1) = 1$.

DEFINITION 2.7. [**31**] Let $\mathcal{M} = (M; \vee, \wedge, \rightarrow, \circledast, 0, 1)$ be an MTL-algebra. Then, the structure $\mathcal{M}$ is also
(i) a Gödel algebra if $m_1 \circledast m_1 = m_1$ for each $m_1 \in M$,
(ii) an MV-algebra if $(m_1 \rightarrow m_2) \rightarrow m_2 = (m_2 \rightarrow m_1) \rightarrow m_1$ for each $m_1, m_2 \in M$,
(i) a BL-algebra if $m_1 \wedge m_2 = m_1 \circledast (m_1 \rightarrow m_2)$ for each $m_1, m_2 \in M$.

THEOREM 2.1. [**25**] *Let $\mathcal{M} = (M; \vee, \wedge, \rightarrow, \circledast, 0, 1)$ be an MTL-algebra. If the operations are defined for each $m_1, m_2 \in M$ as follows:*
$m_1 \wedge m_2 := (((m_2|m_2)|m_1)|m_1)|(((m_2|m_2)|m_1)|m_1)$,
$m_1 \vee m_2 := (m_1|(m_2|m_2))|(m_2|m_2)$,
$m_1 \circledast m_2 := (m_1|m_2)|(m_1|m_2)$,
$m_1 \rightarrow m_2 := m_1|(m_2|m_2)$
then, the structure $\mathcal{M} = (M; |)$ is a Sheffer stroke reduction of MTL-algebra.

DEFINITION 2.8. [**27**] Let $\mathcal{M} = (M; |)$ be a Sheffer stroke MTL-algebra. If the following statements are satisfied $m_1, m_2 \in M$, then the mapping $\vartheta : M \rightarrow M$ is said to be a Sheffer stroke very true operator.
$(SV_{SM}1)$ $\vartheta(1) = 1$
$(SV_{SM}2)$ $\vartheta(m_1) \leqslant m_1$
$(SV_{SM}3)$ $\vartheta(m_1|(m_2|m_2)) \leqslant \vartheta(m_1)|(\vartheta(m_2)|\vartheta(m_2))$
$(SV_{SM}4)$ $\vartheta(m_1) \leqslant \vartheta(\vartheta(m_1))$

$(SV_{SM}5)$ $(\vartheta(m_1|(m_2|m_2))|(\vartheta(m_2|(m_1|m_1))|\vartheta(m_2|(m_1|m_1))))$
$|(\vartheta(m_2|(m_1|m_1))|\vartheta(m_2|(m_1|m_1))) = 1.$

PROPOSITION 2.1. [**27**] *Let $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then, the following statements are verified for each $m \in M$.*
(i) $\vartheta(0) = 0$,
(ii) $m = 1$ *if and only if* $\vartheta(m) = 1$,
(iii) ϑ *is increasing,*
(iv) $\vartheta(m|m) \leqslant \vartheta(m)|\vartheta(m)$.

LEMMA 2.6. [**27**] *Let $\vartheta : M \to M$ be a Sheffer stroke very true operator. The following inequalities are verified for each $m_1, m_2 \in M$*

$$(\vartheta(m_1)|\vartheta(m_2))|(\vartheta(m_1)|\vartheta(m_2)) \leqslant (m_1|m_2)|(m_1|m_2) \leqslant \vartheta(m_1|m_2)|\vartheta(m_1|m_2).$$

3. Riečan and Bosbach state operators on Sheffer stroke MTL-algebras

In this part of the paper, we demonstrate Riečan state and Bosbach state operators on Sheffer stroke MTL-algebras (SMTL-algebras for short). We obtain some fundamental results of these operators. Moreover, we construct a bridge between Riečan state and Bosbach state on $\mathcal{M}$.

DEFINITION 3.1. The mapping $\tau^R_{SMTL} : M \to [0,1]$ is called Riečan state operator on SMTL-algebra if it satisfies the following statements:
$(\tau^R_{SMTL}1)$ $\tau^R_{SMTL}(1) = 1$,
$(\tau^R_{SMTL}2)$ $\tau^R_{SMTL}(m_1|m_2) = \tau^R_{SMTL}(m_1|m_1) + \tau^R_{SMTL}(m_2|m_2)$, where $(m_1|m_1)|(m_2|m_2) = 1$.

EXAMPLE 3.1. Let $M = \{0, m_1, m_2, m_3, m_4, m_5, m_6, 1\}$, where $0 < m_1 < m_5 < 1$, $0 < m_2 < m_6 < 1$ and $0 < m_3 < m_4 < 1$ but m_1, m_2, m_3 and m_4, m_5, m_6 are not comparable between each other, respectively. The partial order relation on M is described as Figure 1 and the operation $|$ on this structure is given as the Table 1.

The structure $\mathcal{M} = (M; |)$ corresponds to SMTL-algebra. The operation $\tau^R_{SMTL} : M \to [0,1]$ is given by

$$\tau^R_{SMTL}(m_i) := \begin{cases} 0, & m_i \in \{0, m_1, m_2\}, \\ 1, & m_i \in \{m_4, m_5, 1\}, \\ 2/5, & m_i = m_3, \\ 3/5, & m_i = m_6. \end{cases}$$

The statement $(\tau^R_{SMTL}1)$ is easily obtained. Satisfying the statement $(\tau^R_{SMTL}2)$, we need to handle all conditions which are listed as follows:
• For each $m_i \in M$, we have $(m_i|m_i)|(1|1) = (1|1)|(m_i|m_i) = 1$,

$$\begin{aligned}\tau^R_{SMTL}(1|m_i) = \tau^R_{SMTL}(m_i|m_i) &= \tau^R_{SMTL}(0) + \tau^R_{SMTL}(m_i|m_i) \\ &= \tau^R_{SMTL}(1|1) + \tau^R_{SMTL}(m_i|m_i).\end{aligned}$$

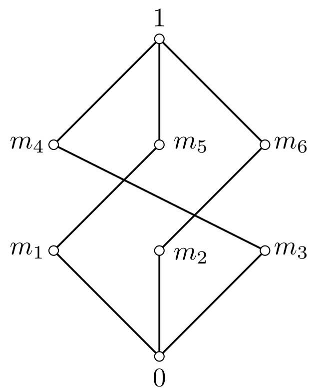

Figure 1. Diagram of M

$\vert$	0	m_1	m_2	m_3	m_4	m_5	m_6	1
0	1	1	1	1	1	1	1	1
m_1	1	m_4	1	1	1	m_4	m_4	m_4
m_2	1	1	m_5	1	m_1	1	m_5	m_5
m_3	1	1	1	m_6	m_6	m_2	1	m_6
m_4	1	1	m_1	m_6	m_1	m_2	m_3	m_1
m_5	1	m_4	1	m_2	m_2	m_2	m_3	m_2
m_6	1	m_4	m_5	1	m_3	m_3	m_3	m_3
1	1	m_4	m_5	m_6	m_1	m_2	m_3	0

Table 1. $\vert-$operation on M

- We have $(m_4|m_4)|(m_5|m_5) = (m_5|m_5)|(m_4|m_4) = 1$, then

$$\begin{aligned}\tau^R_{SMTL}(m_4|m_5) = \tau^R_{SMTL}(m_2) &= 0\\ &= \tau^R_{SMTL}(m_1) + \tau^R_{SMTL}(m_2)\\ &= \tau^R_{SMTL}(m_4|m_4) + \tau^R_{SMTL}(m_5|m_5).\end{aligned}$$

- We have $(m_4|m_4)|(m_6|m_6) = (m_6|m_6)|(m_4|m_4) = 1$, then

$$\begin{aligned}\tau^R_{SMTL}(m_4|m_6) = \tau^R_{SMTL}(m_3) &= 2/5\\ &= \tau^R_{SMTL}(m_1) + \tau^R_{SMTL}(m_3)\\ &= \tau^R_{SMTL}(m_4|m_4) + \tau^R_{SMTL}(m_6|m_6).\end{aligned}$$

- We have $(m_4|m_4)|(m_1|m_1) = (m_1|m_1)|(m_4|m_4) = 1$, then

$$\begin{aligned}&= \tau^R_{SMTL}(m_1) + \tau^R_{SMTL}(m_4)\\ &= \tau^R_{SMTL}(m_4|m_4) + \tau^R_{SMTL}(m_1|m_1).\end{aligned}$$

• We have $(m_5|m_5)|(m_6|m_6) = (m_6|m_6)|(m_5|m_5) = 1$, then

$$\begin{aligned}\tau^R_{SMTL}(m_5|m_6) = \tau^R_{SMTL}(m_3) &= 2/5\\ &= \tau^R_{SMTL}(m_2) + \tau^R_{SMTL}(m_3)\\ &= \tau^R_{SMTL}(m_5|m_5) + \tau^R_{SMTL}(m_6|m_6).\end{aligned}$$

• We have $(m_5|m_5)|(m_2|m_2) = (m_2|m_2)|(m_5|m_5) = 1$, then

$$\begin{aligned}\tau^R_{SMTL}(m_5|m_2) = \tau^R_{SMTL}(1) &= 1\\ &= \tau^R_{SMTL}(m_2) + \tau^R_{SMTL}(m_5)\\ &= \tau^R_{SMTL}(m_5|m_5) + \tau^R_{SMTL}(m_2|m_2).\end{aligned}$$

• We have $(m_3|m_3)|(m_6|m_6) = (m_6|m_6)|(m_3|m_3) = 1$, then

$$\begin{aligned}\tau^R_{SMTL}(m_3|m_6) = \tau^R_{SMTL}(1) &= 1\\ &= \tau^R_{SMTL}(m_6) + \tau^R_{SMTL}(m_3)\\ &= \tau^R_{SMTL}(m_3|m_3) + \tau^R_{SMTL}(m_6|m_6).\end{aligned}$$

By using commutativity of the $|$ and $+$ operators, we handle one sided of the above conditions. This operation verifies the statement $(\tau^R_{SMTL}2)$. As a result, it is a Riečan state operator on $\mathcal{M}$.

PROPOSITION 3.1. *Let* $\tau^R_{SMTL} : M \to [0,1]$ *is called Riečan state operator on SMTL-algebra. Then, the following conclusions are obtained:*
(i) $\tau^R_{SMTL}(0) = 0$,
(ii) $1 = \tau^R_{SMTL}(m) + \tau^R_{SMTL}(m|m)$, *for each* $m \in M$.

PROOF. (i) Since $0|1 = 1$, we get $\tau^R_{SMTL}(0|1) = 1$ from $(\tau^R_{SMTL}1)$. Also, the equality $(0|0)|(1|1) = 1$ is verified in SMTL-algebra. Therefore, we obtain by the help of $(\tau^R_{SMTL}2)$:

$$1 = \tau^R_{SMTL}(0|1) = \tau^R_{SMTL}(0|0) + \tau^R_{SMTL}(1|1) = \tau^R_{SMTL}(1) + \tau^R_{SMTL}(0).$$

So, we conclude that $\tau^R_{SMTL}(0) = 0$.
(ii) The equalities $((m|m)|(m|m))|(m|m) = m$ and $(m|m)|m = 1$ is verified for each $m \in M$. Then, we get the following conclusion

$$\begin{aligned}1 = \tau^R_{SMTL}(1) = \tau^R_{SMTL}((m|m)|m) &= \tau^R_{SMTL}((m|m)|(m|m)) + \tau^R_{SMTL}(m|m)\\ &= \tau^R_{SMTL}(m) + \tau^R_{SMTL}(m|m).\end{aligned}$$

□

LEMMA 3.1. *Let* $\tau^R_{SMTL} : M \to [0,1]$ *is called Riečan state operator on SMTL-algebra. If* $m_1|m_1 \leqslant m_2$, *then* $\tau^R_{SMTL}(m_1) + \tau^R_{SMTL}(m_2) = 2 - \tau^R_{SMTL}(m_1|m_2)$.

PROOF. Assume that $m_1|m_1 \leqslant m_2$. By the help of Lemma 2.3, we have $(m_1|m_1)|(m_2|m_2) = 1$. Using the statement $(\tau^R_{SMTL}2)$, we get

$$\tau^R_{SMTL}(m_1|m_2) = \tau^R_{SMTL}(m_1|m_1) + \tau^R_{SMTL}(m_2|m_2). \tag{3.1}$$

Moreover, we obtain the following equalities via Proposition 3.1 (ii):

$$\tau^R_{SMTL}(m_1) + \tau^R_{SMTL}(m_1|m_1) = 1, \tag{3.2}$$

and

$$\tau^R_{SMTL}(m_2) + \tau^R_{SMTL}(m_2|m_2) = 1. \tag{3.3}$$

By combining the Equalities (3.1), (3.2) and (3.3), we attain

$$\tau^R_{SMTL}(m_1) + \tau^R_{SMTL}(m_2)2 - \tau^R_{SMTL}(m_1|m_2).$$

□

In the following part of this chapter, we define Bosbach state operator on SMTL-algebras. We put forward some fundamental conclusions. Besides, we prove that a Bosbach state is a Riečan state or vice versa on $\mathcal{M}$.

Definition 3.2. The mapping $\tau^B_{SMTL} : M \to [0,1]$ is called Bosbach state operator on a SMTL-algebra if it satisfies the following statements for all $m_1, m_2 \in M$:
($\tau^B_{SMTL}1$) $\tau^B_{SMTL}(1) = 1$,
($\tau^B_{SMTL}2$) $\tau^B_{SMTL}(m_1|m_1) + \tau^B_{SMTL}((m_1|m_1)|m_2) = \tau^B_{SMTL}(m_2|m_2) + \tau^B_{SMTL}((m_2|m_2)|m_1)$,
($\tau^B_{SMTL}3$) $\tau^B_{SMTL}(m_3) = 0$ such that there exists any element $m_3 \in M$.

Example 3.2. Let $M = \{0, m_1, m_2, 1\}$, where $0 < m_1 < 1$ and $0 < m_2 < 1$ but m_1 but m_2 are not comparable with each other. The operation $|$ on this structure is given as the Table 2.

$\|$	0	m_1	m_2	1
0	1	1	1	1
m_1	1	m_2	1	m_2
m_2	1	1	m_1	1
1	1	m_2	m_1	0

Table 2. $|-$operation on M

The structure $\mathcal{M} = (M; |)$ corresponds to SMTL-algebra. The operation $\tau^B_{SMTL} : M \to [0,1]$ is given by

$$\tau^B_{SMTL}(m_i) := \begin{cases} 0, & m_i = 0, \\ 1, & m_i = 1, \\ 2/5, & m_i = m_1, \\ 3/5, & m_i = m_2. \end{cases}$$

The statement ($\tau^B_{SMTL}1$) and ($\tau^B_{SMTL}3$) are easily obtained. Satisfying the statement ($\tau^B_{SMTL}2$), we need to handle all conditions which are listed as follows:
• For each $m_i, m_j \in M$, the following condition is verified when $m_i = m_j$ because of the symmetry:

$$\tau^B_{SMTL}(m_i|m_i) + \tau^B_{SMTL}((m_i|m_i)|m_j) = \tau^B_{SMTL}(m_j|m_j) + \tau^B_{SMTL}((m_j|m_j)|m_i).$$

- Assume that $m_i = 0$ and $m_j = m_1$. Then we obtain

$$\begin{aligned}\tau^B_{SMTL}(0|0) + \tau^B_{SMTL}((0|0)|m_1) &= \tau^B_{SMTL}(1) + \tau^B_{SMTL}(m_2)\\ &= \tau^B_{SMTL}(m_1|m_1) + \tau^B_{SMTL}((m_1|m_1)|0).\end{aligned}$$

- Assume that $m_i = 0$ and $m_j = m_2$. Then we obtain

$$\begin{aligned}\tau^B_{SMTL}(0|0) + \tau^B_{SMTL}((0|0)|m_2) &= \tau^B_{SMTL}(1) + \tau^B_{SMTL}(m_1)\\ &= \tau^B_{SMTL}(m_2|m_2) + \tau^B_{SMTL}((m_2|m_2)|0).\end{aligned}$$

- Assume that $m_i = 1$ and $m_j = m_1$. Then we obtain

$$\begin{aligned}\tau^B_{SMTL}(1|1) + \tau^B_{SMTL}((1|1)|m_1) &= 1\\ &= \tau^B_{SMTL}(m_2) + \tau^B_{SMTL}(m_1)\\ &= \tau^B_{SMTL}(m_1|m_1) + \tau^B_{SMTL}((m_1|m_1)|1).\end{aligned}$$

- Assume that $m_i = 1$ and $m_j = m_2$. Then we obtain

$$\begin{aligned}\tau^B_{SMTL}(1|1) + \tau^B_{SMTL}((1|1)|m_2) &= 1\\ &= \tau^B_{SMTL}(m_1) + \tau^B_{SMTL}(m_2)\\ &= \tau^B_{SMTL}(m_2|m_2) + \tau^B_{SMTL}((m_2|m_2)|1).\end{aligned}$$

- Assume that $m_i = m_1$ and $m_j = m_2$. Then we obtain

$$\begin{aligned}\tau^B_{SMTL}(m_1|m_1) + \tau^B_{SMTL}((m_1|m_1)|m_2) &= \tau^B_{SMTL}(m_2) + \tau^B_{SMTL}(m_1)\\ &= \tau^B_{SMTL}(m_2|m_2) + \tau^B_{SMTL}((m_2|m_2)|m_1).\end{aligned}$$

By the using commutativity of the $|$ and $+$ operators, we handle one sided of the above conditions. This operation verifies the statement ($\tau^B_{SMTL}2$). As a result, it is a Bosbach state operator on $\mathcal{M}$.

THEOREM 3.1. *The Riečan state operator* τ^R_{SMTL} *corresponds to the Bosbach state operator* τ^B_{SMTL} *in SMTL-algebras, or vice versa.*

PROOF. It can be proved by using similar technique in [**26**]. □

Since τ^R_{SMTL} and τ^B_{SMTL} correspond to each other, we use τ_{SMTL} in the rest of the paper for these two state operators.

4. Relations between very true operator and state operators

In this part of this paper, we give substantial relations among these state operators and very true operator. Besides, we acquire some constant conclusions by using these state operator and very true operator, and also we handle some characteristic feature of these states and this very true operator on SMTL-algebras.

LEMMA 4.1. *Let* ϑ *be a Sheffer stroke very true operator on SMTL-algebras. Then, the following identity is verified for each SMTL-algebra*

$$\tau_{SMTL}(\vartheta(m)|\vartheta(m|m)) = 1.$$

Proof. By Proposition 2.1 (iv), we have $\vartheta(m|m) \leqslant \vartheta(m)|\vartheta(m)$. Using Lemma 2.3 and Definition 2.5 $(S2)$, we get $\vartheta(m)|\vartheta(m|m) = 1$. As a result, we attain $\tau_{SMTL}(\vartheta(m)|\vartheta(m|m)) = \tau_{SMTL}(1) = 1$. □

Lemma 4.2. *The following identity is verified for each $m \in M$*

$$\tau_{SMTL}((\vartheta(m)|\vartheta(m))|m) = \tau_{SMTL}(\vartheta(m)) + \tau_{SMTL}(m|m).$$

Proof. From Definition 2.8 $(SV_{SM}2)$, we have $\vartheta(m) \leqslant m$ for each $m \in M$. We achieve

$$((\vartheta(m)|\vartheta(m))|(\vartheta(m)|\vartheta(m)))|(m|m) = 1$$

via Lemma 2.3 and Definition 2.5 $(S2)$. By Definition 3.1 $(\tau^R_{SMTL}2)$, we achieve for each $m \in M$

$$\tau_{SMTL}((\vartheta(m)|\vartheta(m))|m) = \tau_{SMTL}(\vartheta(m)) + \tau_{SMTL}(m|m).$$

□

Corollary 4.1. *Let $m_1 \leqslant m_2$. Then the identity is verified*

$$\tau_{SMTL}((\vartheta(m_1)|\vartheta(m_1))|\vartheta(m_2)) = \tau_{SMTL}(\vartheta(m_1)) + \tau_{SMTL}(\vartheta(m_2)|\vartheta(m_2)).$$

Proof. It is clearly obtained by using Lemma 4.2 and the Proposition 2.1 (iii). □

Proposition 4.1. *The following statements are verified for each $m \in M$*
(i) $\vartheta(\tau_{SMTL}(m)) \leqslant \tau_{SMTL}(m)$,
(ii) $\vartheta(\tau_{SMTL}(m)|\tau_{SMTL}(m)) \leqslant \vartheta(\tau_{SMTL}(m))|\vartheta(\tau_{SMTL}(m))$,
(iii) $\vartheta(\tau_{SMTL}(m)) = \vartheta^2(\tau_{SMTL}(m))$.

Proposition 4.2. *The following identities are satisfied for each $m \in M$*
(i) $\tau_{SMTL}((m|m)|\vartheta(m)) = 1$,
(ii) $\tau_{SMTL}(m|\vartheta(m|m)) = 1$.

Proof. It is straightforward from Lemma 2.6. □

Lemma 4.3. *The following identities are satisfied for each $m_1, m_2 \in M$*
(i) $\tau_{SMTL}(\vartheta(m_1|m_1)|(\vartheta(m_1|m_2)|\vartheta(m_1|m_2))) = 1$,
(ii) $\tau_{SMTL}((\vartheta(m_1|m_2)|\vartheta(m_1|m_2))|(\vartheta(m_1)|\vartheta(m_1))) = 1$.

Proof. (i) Let m_1 and m_2 be an elements of M such that $m_1 \leqslant 1$ and $m_2 \leqslant 1$. By the help of Lemma 2.4, Lemma 2.5 and Proposition 2.1, we get that $\vartheta(m_1|m_1) \leqslant \vartheta(m_1|m_2)$. So, we attain that $\vartheta(m_1|m_1)|(\vartheta(m_1|m_2)|\vartheta(m_1|m_2)) = 1$. As a result, we conclude that $\tau_{SMTL}(\vartheta(m_1|m_1)|(\vartheta(m_1|m_2)|\vartheta(m_1|m_2))) = 1$.
(ii) Let m_1 and m_2 be an elements of M such that $m_1 \leqslant 1$ and $m_2 \leqslant 1$. By the help of Lemma 2.4 and Lemma 2.5, we get that $m_1|m_1 \leqslant m_1|m_2$. By means of Lemma 4.2 and Definition 2.5, we obtain $(m_1|m_2)|(m_1|m_2) \leqslant m_1$. Since the very true operator is increasing , we conclude that $\vartheta((m_1|m_2)|(m_1|m_2)) \leqslant \vartheta(m_1)$, i.e., $\vartheta((m_1|m_2)|(m_1|m_2))|(\vartheta(m_1)|\vartheta(m_1)) = 1$. Therefore, we achieve that

$$\tau_{SMTL}((\vartheta(m_1|m_2)|\vartheta(m_1|m_2))|(\vartheta(m_1)|\vartheta(m_1))) = 1.$$

□

Theorem 4.1. *Let ϑ be a Sheffer stroke very true operator and let τ_{SMTL} be an any state operator. Let* inf *and* sup *be the greatest lower bound and least upper bound functions, respectively. Then the following statements are verified for each $m_1, m_2 \in M$*

$$\sup\{\vartheta(\tau_{SMTL}(m_1)), \vartheta(\tau_{SMTL}(m_2))\} = \vartheta(\sup\{\tau_{SMTL}(m_1), \tau_{SMTL}(m_2)\})$$

and

$$\inf\{\vartheta(\tau_{SMTL}(m_1)), \vartheta(\tau_{SMTL}(m_2))\} = \vartheta(\inf\{\tau_{SMTL}(m_1), \tau_{SMTL}(m_2)\}).$$

Proof. It is clearly obtained by using similar technique in [**27**]. □

Corollary 4.2. *Let ϑ be a Sheffer stroke very true operator and let τ_{SMTL} be an any state operator. Let* inf *and* sup *be the greatest lower bound and least upper bound functions, respectively. Then the following statements are verified for each $m_1, m_2 \in M$*

$$\sup\{\vartheta(\tau_{SMTL}(m_1)), \vartheta(\tau_{SMTL}(m_2))\} = \vartheta(\sup\{\vartheta(\tau_{SMTL}(m_1)), \vartheta(\tau_{SMTL}(m_2))\})$$

and

$$\inf\{\vartheta(\tau_{SMTL}(m_1), \vartheta(\tau_{SMTL}(m_2))\} = \vartheta(\inf\{\vartheta(\tau_{SMTL}(m_1)), \vartheta(\tau_{SMTL}(m_2))\}).$$

Example 4.1. Let τ_{SMTL} be a state operator on M and let ϑ be very true operator on $\tau_{SMTL}(M)$. The state operator is defined as in Example 3.2 and the very true operator is defined as $\vartheta(m_j) = m_j$ for each $m_j \in \tau_{SMTL}(M)$. Then, we have

$$\begin{aligned}
\sup\{\vartheta(\tau_{SMTL}(m_i)) \ : \ m_i \in M\} &= sup\{\vartheta(\tau_{SMTL}(0)), \vartheta(\tau_{SMTL}(1)), \\
&\quad \vartheta(\tau_{SMTL}(m_1)), \vartheta(\tau_{SMTL}(m_2))\} \\
&= sup\{\vartheta(0), \vartheta(1), \vartheta(2/5), \vartheta(3/5)\} \\
&= sup\{0, 1, 2/5, 3/5\} \\
&= 1 \\
&= \tau_{SMTL}(1) \\
&= \vartheta(\sup\{\tau_{SMTL}(m_i) \ : \ m_i \in M\}).
\end{aligned}$$

By using similar technique, we obtain that

$$\inf\{\vartheta(\tau_{SMTL}(m_i)) \ : \ m_i \in M\} = 0 = \vartheta(\inf\{\tau_{SMTL}(m_i) \ : \ m_i \in M\}).$$

5. Conclusion

In this work, we presented Riečan and Bosbach states notions on SMTL-algebras. We obtained some fundamental results on these operators. We attained a connection between each other. Besides, we gave substantial relations among these state operators and very true operator. We achieved some constant conclusions by using these state operator and very true operator and handle some characteristic feature of these mentioned operators on SMTL-algebras. After this paper, we will examine these relations among other algebraic structures. Due to these reasons, we will integrate this work for future papers.

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