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COMPLETE MONOTONICITY OF SOME EXPONENTIAL AND TRIGAMMA RELATED FUNCTIONS

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Abstract. Induced by the inequalities in [**4**], we show that the functions $\theta(x,m)-\psi'(x+1)$ and $\psi'(x+1)-\theta(x,m)$ are not completely monotonic for any $m>0$, where $\theta(x,m)=\frac{1}{2m}\left(e^{\frac{m}{x+1}}-e^{\frac{-m}{x}}\right)$ and ψ is digamma function.

1. Introduction

Recall that completely monotonic function (shortly CM) is an infinitely differentiable function $f:(0,\infty)\to\mathbb{R}$ with the property

$$(-1)^n f^{(n)}\geqslant 0,\quad n=0,1,2,\ldots.$$

Fundamental for CM functions is *Bernstein theorem*: a function f is completely monotonic if and only if

$$f(x)=\int_{[0,\infty)} e^{-xt}d\mu(t),\;\; x>0 \tag{1.1}$$

for a non-negative Borel measure μ on $[0,\infty)$. The measure μ is unique (see [**2**], p. 61). Principle motivation for our paper are the inequalities

$$\theta(x,p)<\psi'(x+1)<\theta(x,q),\;\; x>0, \tag{1.2}$$

where $\theta(x,m)=\frac{1}{2m}\left(e^{\frac{m}{x+1}}-e^{-\frac{m}{x}}\right)$ and ψ is digamma function. It is proved in [**4**] that $p=1$ and $q=2$ are the best possible constants in (1.2) among all $p,q\in(0,\infty)$. Now, one is tempted to ask whether $\theta(x,m)-\psi'(x+1)$ and $\psi'(x+1)-\theta(x,m)$ are CM functions for some $m>0$. We show that, however, it is never the case. In order

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to accomplish the proof, we represent both functions in the form $\int_0^\infty \varphi(t)\, e^{-xt}\, dt$ and show that φ may be negative in both cases. Our considerations rely on the results in [**3**].

2. Formulations and proofs

The main ingredient in the proofs of our assertions is the following Proposition.

PROPOSITION 2.1. *Let* $\varphi : [0,\infty) \to \mathbb{R}$ *be a continuous function such that* $\int_0^\infty |\varphi(t)|\, e^{-\varepsilon t}\, dt < \infty$ *for some* $\varepsilon > 0$. *If* $f(x) = \int_0^\infty \varphi(t)\, e^{-xt}\, dt$ *is CM, then* $\varphi \geqslant 0$.

PROOF. Then $f(x+\varepsilon) = \int_0^\infty \varphi(t) e^{-\varepsilon t}\, e^{-xt}\, dt$ is also CM and according to Proposition 2.2 in [**3**], we have $\varphi(t)\, e^{-\varepsilon t} \geqslant 0$. Hence, $\varphi \geqslant 0$. □

Next, we need

LEMMA 2.1. *If* $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!}$, *then for all* $x > 0$

$$g(x) = \frac{2}{\pi\sqrt{x}} \int_0^1 \frac{t \sinh(2t\sqrt{x})}{\sqrt{1-t^2}}\, dt, \tag{2.1}$$

$$g(-x) = \frac{2}{\pi\sqrt{x}} \int_0^1 \frac{t \sin(2t\sqrt{x})}{\sqrt{1-t^2}}\, dt. \tag{2.2}$$

PROOF. Thanks to Cauchy integral formula for derivatives of exp, one obtains $\frac{1}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{e^z}{z^{n+1}}\, dz$, where Γ is the positively oriented circle $|z| = R$. Consequently, for all $x \neq 0$

$$\begin{aligned} g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!} &= \frac{1}{2\pi i x} \int_\Gamma e^z \left(\sum_{n=0}^{\infty} \frac{(x/z)^{n+1}}{(n+1)!} \right) dz \\ &= \frac{1}{2\pi i x} \int_\Gamma e^z \left(e^{\frac{x}{z}} - 1 \right) dz \\ &= \frac{1}{2\pi i x} \int_\Gamma \exp\left(z + \frac{x}{z} \right) dz. \end{aligned}$$

Let $x > 0$ and Γ_x be the positively oriented circle $|z| = \sqrt{x}$. Note that for all $z \in \Gamma_x$ the numbers z and $\frac{x}{z}$ are complex-conjugate and therefore $z + \frac{x}{z} = 2\Re(z)$. If we use the parametrization $\gamma(t) = \sqrt{x}(t \pm i\sqrt{1-t^2})$, $t \in [-1,1]$ of Γ_x, we have

$$\begin{aligned} \int_{\Gamma_x} \exp\left(z + \frac{x}{z} \right) dz &= \sqrt{x} \int_{-1}^{1} e^{2t\sqrt{x}} \left(1 + i \frac{t}{\sqrt{1-t^2}} \right) dt \\ &\quad - \sqrt{x} \int_{-1}^{1} e^{2t\sqrt{x}} \left(1 - i \frac{t}{\sqrt{1-t^2}} \right) dt \\ &= 2i\sqrt{x} \int_{-1}^{1} e^{2t\sqrt{x}} \frac{t}{\sqrt{1-t^2}}\, dt = 4i\sqrt{x} \int_0^1 \frac{t \sinh(2t\sqrt{x})}{\sqrt{1-t^2}}\, dt, \end{aligned}$$

whence (2.1) for $x > 0$. Since g is an entire function, then owing to the uniqueness theorem

$$g(z) = \frac{2}{\pi\sqrt{z}} \int_0^1 \frac{t \sinh(2t\sqrt{z})}{\sqrt{1-t^2}}\, dt \tag{2.3}$$

for all $z \in \mathbb{C}$. Note that the right-hand side in (2.3) is even in $\sqrt{z}$, so it is independent on the branch we use for $\sqrt{z}$. Now, from $\sqrt{-x} = i\sqrt{x}$ for $x > 0$ and $\sinh(iz) = i \sin z$, one obtains (2.2). $\square$

Our first assertion concerning complete monotonicity is

Proposition 2.2. *The function $f(x) = \theta(x, m) - \psi'(x + 1)$ is not completely monotonic for any $m > 0$, where*

$$\theta(x, m) = \frac{1}{2m} \left(e^{\frac{m}{x+1}} - e^{-\frac{m}{x}}\right)$$

and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is digamma function.

Proof. From $\frac{1}{x^{n+1}} = \frac{1}{n!} \int_0^\infty t^n\, e^{-xt}\, dt$ for all $x > 0$ and $n = 0, 1, 2, \dots,$ we have

$$\begin{aligned}\theta(x, m) &= \frac{1}{2m} \left(e^{\frac{m}{x+1}} - e^{-\frac{m}{x}}\right)\\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{m^n}{(n+1)!} \left(\frac{1}{(x+1)^{n+1}} + \frac{(-1)^n}{x^{n+1}}\right)\\ &= \frac{1}{2} \int_0^\infty \sum_{n=0}^{\infty} \frac{(mt)^n}{n!(n+1)!} \left(e^{-t} + (-1)^n\right) e^{-xt}\, dt,\end{aligned}$$

so we obtain

$$\theta(x, m) = \int_0^\infty \varphi_m(t)\, e^{-xt}\, dt, \quad x > 0, \tag{2.4}$$

where $\varphi_m(t) = \frac{g(mt)e^{-t} + g(-mt)}{2}$. We will use Proposition 2.1 to show that $\theta(x, m)$ is not CM for any $m > 0$. In order to achieve that task, we prove that for any $m > 0$ there exists $t > 0$ such that $\varphi_m(t) < 0$. By using (2.1), (2.2) and introducing the substitution $s = \sqrt{mt}$, we get

$$\begin{aligned}2\varphi_m(t) = g(s^2)e^{-s^2/m} + g(-s^2) &= \frac{2}{\pi s} \left(\int_0^1 \frac{t \sinh(2ts)}{\sqrt{1-t^2}} e^{-\frac{s^2}{m}} + \int_0^1 \frac{t \sin(2ts)}{\sqrt{1-t^2}}\right) dt\\ &= \frac{2}{\pi s} I_1(s) + \frac{2}{\pi s} I_2(s).\end{aligned}$$

Applying $\sinh x \leqslant e^x$ for $x \geqslant 0$ and $\displaystyle\int_0^1 \frac{t}{\sqrt{1-t^2}}\, dt = 1$, we have

$$I_1(s) = \int_0^1 \frac{t \sinh(2ts)}{\sqrt{1-t^2}} e^{-\frac{s^2}{m}}\, dt \leqslant e^{2s - \frac{s^2}{m}}.$$

Let $s = n\pi$ for $n \in \mathbb{N}$. Then, it is

$$I_2(s) = \int_0^1 \frac{t\sin(2tn\pi)}{\sqrt{1-t^2}}\,dt = \sum_{k=0}^{n-1}\left(\int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}} \frac{t\sin(2tn\pi)}{\sqrt{1-t^2}}\,dt + \int_{\frac{2k+1}{2n}}^{\frac{2k+2}{2n}} \frac{t\sin(2tn\pi)}{\sqrt{1-t^2}}\,dt\right).$$

If we apply substitution $t = y + \frac{1}{2n}$ in the second integral above, then, thanks to $\sin(2tn\pi) = \sin(2(y+\frac{1}{2n})n\pi) = -\sin(2yn\pi)$, one obtains

$$I_2(s) = \sum_{k=0}^{n-1}\int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}}\left(\frac{t}{\sqrt{1-t^2}} - \frac{t+\frac{1}{2n}}{\sqrt{1-\left(t+\frac{1}{2n}\right)^2}}\right)\sin(2tn\pi)\,dt.$$

For $t \in (\frac{2k}{2n}, \frac{2k+1}{2n})$ it is $2tn\pi \in (2k\pi, (2k+1)\pi)$ and therefore $\sin(2tn\pi) \geqslant 0$. On the other hand, if we apply Lagrange theorem, it follows that for certain $\eta \in (t, t+\frac{1}{2n})$ it is

$$\frac{t}{\sqrt{1-t^2}} - \frac{t+\frac{1}{2n}}{\sqrt{1-\left(t+\frac{1}{2n}\right)^2}} = -\frac{1}{2n}\frac{1}{\sqrt{(1-\eta^2)^3}} \leqslant -\frac{1}{2n}.$$

Consequently,

$$I_2(s) \leqslant \sum_{k=0}^{n-1}\frac{-1}{2n}\int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}}\sin(2nt\pi)\,dt = \sum_{k=0}^{n-1}\frac{-1}{2n^2\pi} = -\frac{1}{2n\pi} = -\frac{1}{2s}$$

and for $s = n\pi$, where $n \in \mathbb{N}$, we finally deduce

$$2\varphi_m(t) = \frac{2}{\pi s}\left(I_1(s) + I_2(s)\right) \leqslant \frac{2}{\pi s}\left(e^{2s-s^2/m} - \frac{1}{2s}\right) < 0,$$

for n large enough. It is certainly $\int_0^\infty |\varphi_m(t)|\,e^{-\varepsilon t}\,dt < \infty$ for all $\varepsilon > 0$. Therefore, from Proposition 2.1 and (2.4) we conclude that $\theta(x,m)$ is not CM. On the other hand, we know that $\psi'(x+1)$ is completely monotonic. This follows from the representation $\psi'(x) = \int_0^\infty \frac{t}{1-e^{-t}}\,e^{-xt}\,dt$, which is due to S. Ramanujan (see [**1**, p. 260]). If $f(x) = \theta(x,m) - \psi'(x+1)$ were CM, then $\theta(x,m) = f(x) + \psi'(x+1)$ would also be CM (as a sum of two CM functions), which is not. $\square$

Our last assertion is

PROPOSITION 2.3. *The function $f(x) = \psi'(x+1) - \theta(x,m)$ is not CM for any $m > 0$, where the functions θ and ψ are as in the previous proposition.*

PROOF. Following procedure from the proof of the preceding proposition, we see that in this case $f(x) = \int_0^\infty \varphi_m(t)\,e^{-xt}\,dt$, where

$$\varphi_m(t) = \frac{te^{-t}}{1-e^{-t}} - \frac{g(mt)e^{-t} + g(-mt)}{2}$$

and φ_m obeys the condition $\int_0^\infty |\varphi_m(t)|\,e^{-\varepsilon t}\,dt < \infty$ for all $\varepsilon > 0$. Again, we prove that for any $m > 0$ it is $\varphi_m(t) < 0$ for certain $t > 0$. Since $g(mt) \geqslant 0$, it follows $2\varphi_m(t) \leqslant \frac{2t}{e^t-1} - g(-mt)$ and the substitution $s = \sqrt{mt}$ together with (2.2) yields

$$(2.5)\quad 2\varphi_m(t) \leqslant h(s) = \frac{2s^2}{m(e^{\frac{s^2}{m}}-1)} - g(-s^2) = \frac{2s^2}{m(e^{\frac{s^2}{m}}-1)} - \frac{2}{\pi s}\int_0^1 \frac{t\sin(2ts)}{\sqrt{1-t^2}}\,dt.$$

Now, if we set $s = n\pi + \frac{\pi}{2}$ for $n \in \mathbb{N}$ and carry out similar steps as in the proof of the previous proposition, we obtain

$$\begin{aligned}
\int_0^1 \frac{t\sin(2ts)}{\sqrt{1-t^2}}\,dt &= \int_0^1 \frac{t\sin((2n+1)\pi t)}{\sqrt{1-t^2}}\,dt \\
&= \int_0^{\frac{1}{2n+1}} \frac{t\sin((2n+1)\pi t)}{\sqrt{1-t^2}}\,dt + \sum_{k=1}^{n}\int_{\frac{2k-1}{2n+1}}^{\frac{2k}{2n+1}} \frac{t\sin((2n+1)\pi t)}{\sqrt{1-t^2}}\,dt \\
&+ \sum_{k=1}^{n}\int_{\frac{2k}{2n+1}}^{\frac{2k+1}{2n+1}} \frac{t\sin((2n+1)\pi t)}{\sqrt{1-t^2}}\,dt \\
&\geqslant \sum_{k=1}^{n}\int_{\frac{2k-1}{2n+1}}^{\frac{2k}{2n+1}} \frac{t\sin((2n+1)\pi t)}{\sqrt{1-t^2}}\,dt + \int_{\frac{2k}{2n+1}}^{\frac{2k+1}{2n+1}} \frac{t\sin((2n+1)\pi t)}{\sqrt{1-t^2}}\,dt \\
&= \sum_{k=1}^{n}\int_{\frac{2k}{2n+1}}^{\frac{2k+1}{2n+1}} \left(\frac{t}{\sqrt{1-t^2}} - \frac{t-\frac{1}{2n+1}}{\sqrt{1-(t-\frac{1}{2n+1})^2}}\right)\sin((2n+1)\pi t)\,dt \\
&\geqslant \frac{1}{2n+1}\sum_{k=1}^{n}\int_{\frac{2k}{2n+1}}^{\frac{2k+1}{2n+1}} \sin((2n+1)\pi t)\,dt \\
&= \frac{2n}{(2n+1)^2\pi} = \frac{2s-\pi}{4s^2}.
\end{aligned}$$

Therefore, if $s = n\pi + \frac{\pi}{2}$ for $n \in \mathbb{N}$, then owing to (2.5), we have

$$2\varphi_m(t) \leqslant h(s) \leqslant \frac{2s^2}{m(e^{\frac{s^2}{m}}-1)} - \frac{2s-\pi}{2s^3\pi} < 0,$$

for large n, and again, Proposition 2.1 concludes the proof. □

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