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AN APPLICATION OF A MOMENT PROBLEM TO COMPLETELY MONOTONIC FUNCTIONS

Vladimir Jovanović [1] and Milanka Treml [1]

Abstract. We consider the following question: if a function of the form $\int_0^\infty \varphi(t)\, e^{-xt}dt$ is completely monotonic, is it then $\varphi \geqslant 0$? It turns out that the question is related to a moment problem. In the end we apply those results to answer some questions concerning complete monotonicity of certain functions raised in [**4**].

1. Introduction

At the beginning we review basic notions and facts related to *completely monotonic functions*. An infinitely differentiable function $f : (0, \infty) \to \mathbb{R}$ is called *completely monotonic*, if

$$(-1)^n f^{(n)} \geqslant 0, \quad n = 0, 1, 2, \ldots .$$

The crucial fact concerning this class of functions is *Bernstein theorem*: a function f is completely monotonic is and only if there exists a positive Borel measure μ on $[0, \infty)$, such that

$$f(x) = \int_{[0,\infty)} e^{-xt} d\mu(t), \tag{1.1}$$

for all $x > 0$. Furthermore, the measure μ is uniquely determined (see [**1**], p. 61). In many applications one comes up to the situation that a function of the form $\int_0^\infty \varphi(t)\, e^{-xt}dt$ is completely monotonic. Usually, in view of Bernstein theorem, it is tacitly assumed that the function φ is then necessarily non-negative. Our aim here is to clarify this question: we give a sufficient condition on φ which guarantees the claim and provide a complete proof. It turns out that our question has to

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do with uniqueness of measures in a moment problem which we consider in the next section. In the sequel we apply those results in order to answer the question (in a slightly more general form) raised in [**4**] on page 34 whether the functions $\psi'(x+1) - \sinh\frac{1}{x+1}$ and $\frac{1}{2}\sinh\frac{2}{x} - \psi'(x+1)$ are completely monotonic, where ψ is digamma function.

2. A moment problem

As we previously mentioned, our considerations are tightly related to the uniqueness question for measures in a moment problem, which we state in Theorem 2.1 (see below). It resembles the *Stieltjes moment problem*: if two nonnegative measures μ and ν with support on $[0,\infty)$ have the same moments, that is, if $\int_0^\infty t^n\,d\mu(t) = \int_0^\infty t^n\,d\nu(t)$, for all $n = 0, 1, \dots$, is it then $\mu = \nu$? In our case, we use a substitution and reduce it to the *Hausdorff moment problem*, where the support of measures is $[0,1]$. Now, we turn to our moment problem.

THEOREM 2.1. *Assume μ and ν are complex Borel measures on $[0,\infty)$ with the property*

$$\int_{[0,\infty)} e^{-nt}\,d\mu(t) = \int_{[0,\infty)} e^{-nt}\,d\mu(t), \quad n = 0, 1, 2 \dots.$$

Then, $\mu = \nu$.

We need the following change of variables formula.

PROPOSITION 2.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, $(Y, \mathcal{N})$ a measurable space and $F : X \to Y$ a measurable map. Then for every measurable function $f : Y \to \mathbb{C}$ and every $E \in \mathcal{N}$ we have*

$$\int_E f(y)\,dF_*\mu(y) = \int_{F^{-1}(E)} f(F(x))\,d\mu(x),$$

in the case either of two sides is defined. Here $F_*\mu = \mu \circ F^{-1}$ is a measure on $(Y, \mathcal{N})$, the so-called push-forward of μ.

For the proof, see [**3**, p. 30-31].

REMARK 2.1. We notice that the change of variable formula also holds for complex Borel measures.

Proof of Theorem 2.1.

Recall that the complex measures μ and ν are of bounded variation, $M_\mu := |\mu|([0,\infty)) < \infty$ and $M_\nu := |\nu|([0,\infty)) < \infty$ (see [**5**]). Let us define a homeomorphism $F : [0,\infty) \to (0,1]$, $F(t) = e^{-t}$. Applying Proposition 2.1 (more precisely Remark 2.1), we obtain

$$\int_{[0,\infty)} e^{-nt} d\mu(t) = \int_{(0,1]} s^n\,dF_*\mu(s), \qquad \int_{[0,\infty)} e^{-nt} d\nu(t) = \int_{(0,1]} s^n\,dF_*\nu(s).$$

From the assumptions of Theorem 2.1, we have

$$\int_{(0,1]} s^n\,dF_*\mu(s) = \int_{(0,1]} s^n\,dF_*\nu(s),$$

for all $n = 0, 1, 2 \dots$. Hence

$$\int_{(0,1]} P(s)\, dF_*\mu(s) = \int_{(0,1]} P(s)\, dF_*\nu(s), \tag{2.1}$$

for all polynomials P. Notice

$$|F_*\mu|((0,1]) = F_*|\mu|((0,1]) = |\mu|([0,\infty)) = M_\mu$$

and similarly $|F_*\nu|((0,1]) = M_\nu$. Therefore, each bounded and measurable (in Borel sense) function on $(0,1]$ is integrable with respect to the both measures $F_*\mu$ and $F_*\nu$. In view of

$$\left|\int_{(0,1]} g(s)\, dF_*\mu(s)\right| \leqslant M_\mu \,\|g\|_\infty, \quad \left|\int_{(0,1]} g(s)\, dF_*\nu(s)\right| \leqslant M_\nu \,\|g\|_\infty,$$

for all bounded measurable functions $g : (0,1] \to \mathbb{R}$, where $\|g\|_\infty = \sup\{|g(x)| : x \in (0,1]\}$, we conclude from Stone - Weierstrass theorem that

$$\int_{(0,1]} g(s)\, dF_*\mu(s) = \int_{(0,1]} g(s)\, dF_*\nu(s), \tag{2.2}$$

for all $g \in C[0,1]$. For small $\delta > 0$ introduce a continuous, piecewise linear function $I_\delta : (0,1] \to \mathbb{R}$,

$$I_\delta(t) = \begin{cases} 0, & t < a - \delta \\ \frac{t-(a-\delta)}{\delta}, & a - \delta \leqslant t \leqslant a \\ 1, & a \leqslant t \leqslant b \\ \frac{b+\delta-t}{\delta}, & b \leqslant t \leqslant b + \delta \\ 0, & b + \delta \leqslant t, \end{cases}$$

where $[a,b] \subset (0,1]$. From (2.2), we have

$$\int_{(0,1]} I_\delta(s)\, dF_*\mu(s) = \int_{(0,1]} I_\delta(s)\, dF_*\nu(s). \tag{2.3}$$

Taking into account that $I_\delta \to \chi_{[a,b]}$ pointwise as $\delta \to 0+$ (here χ denotes characteristic function) and $0 \leqslant I_\delta \leqslant 1$, one infers, applying Lebesgue dominant convergence theorem to integrals in (2.3), that $\int_{(0,1]} \chi_{[a,b]}(s)\, dF_*\mu(s) = \int_{(0,1]} \chi_{[a,b]}(s)\, dF_*\nu(s)$, or equivalently $F_*\mu([a,b]) = F_*\nu([a,b])$, for all $[a,b] \subset (0,1]$. Following a similar procedure one can also deduce $F_*\mu((0,b]) = F_*\nu((0,b])$, for all $(0,b] \subset (0,1]$. Therefore $F_*\mu(E) = F_*\nu(E)$ for all Borel sets $E \subset (0,1]$, which implies $F_*\mu = F_*\nu$. Finally, we obtain $\mu = \nu$, since F is a homeomorphism. $\Box$

Remark 2.2. Here we outline a more advanced proof of this theorem: if $\lambda = \mu - \nu$ and $F(z) = \int_0^\infty e^{-zt}\, d\lambda(t)$, then F is a bounded analytic function in the half-plane $\Re z > 0$ vanishing in $z = 0, 1, 2, \dots$ and by H^p - theory for $p = \infty$ follows $F = 0$, which easily implies $\lambda = 0$.

Proposition 2.2. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a continuous function with the property*

$$\int_0^\infty |\varphi(t)|\, dt < \infty. \tag{2.4}$$

If $f(x)=\int_0^\infty \varphi(t)\,e^{-xt}\,dt$ is completely monotonic, then $\varphi \geqslant 0$.

PROOF. Since f is completely monotonic, then according to Bernstein theorem, there exists a non-negative Borel measure μ on $[0,\infty)$ satisfying (1.1) for all $x>0$. Due to (2.4), we have

$$\mu([0,\infty)) = \int_{[0,\infty)} d\mu = f(0) = \int_0^\infty \varphi(t)\,dt < \infty,$$

and consequently, μ is a finite measure. Again, thanks to (2.4), we conclude that $\nu(E) = \int_E \varphi(t)\,dt$ is a Borel measure of bounded variation $|\nu|([0,\infty)) = \int_0^\infty |\varphi(t)|\,dt < \infty$. Taking into account that

$$\int_{[0,\infty)} e^{-xt}\,d\nu(t) = \int_0^\infty \varphi(t)\,e^{-xt}\,dt = f(x) = \int_{[0,\infty)} e^{-xt}\,d\nu(t),$$

for all $x \geqslant 0$, we see that the assumptions of Theorem 2.1 are fulfilled. Therefore, $\mu=\nu$. This implies

$$\int_a^b \varphi(t)\,dt = \nu([a,b]) = \mu([a,b]) \geqslant 0,$$

for all $[a,b] \subset [0,\infty)$. However, φ is continuous, whence $\varphi \geqslant 0$. □

3. Applications

We apply the results from the previous section with the aim to answer two questions stated in [**4**] on page 34, which concern complete monotonicity of functions $\psi'(x+1) - \sinh\frac{1}{x+1}$ and $\frac{1}{2}\sinh\frac{2}{x} - \psi'(x+1)$. We will actually prove slightly more general assertions. Here $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

PROPOSITION 3.1. *For all $m>0$ the function $f(x) = \psi'(x+1) - \frac{1}{m}\sinh\frac{m}{x+1}$ is not completely monotonic.*

PROOF. We employ the following representations

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\,e^{-xt}\,dt, \quad \psi'(x) = \int_0^\infty \frac{t}{1-e^{-t}}\,e^{-xt}\,dt, \tag{3.1}$$

for all $x>0$ and $n \in \mathbb{N}$. The latter one is due to S. Ramanujan (see [**2**, p. 374]). From

$$\frac{1}{m}\sinh\frac{m}{x+1} = \sum_{n=0}^\infty \frac{m^{2n}}{(2n+1)!}\,\frac{1}{(x+1)^{2n+1}},$$

we conclude that

$$\begin{aligned}\psi'(x+1) - \frac{1}{m}\sinh\frac{m}{x+1} &= \int_0^\infty \left(\frac{t}{1-e^{-t}} - \sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!}\right) e^{-(x+1)t}\,dt \\ &= \int_0^\infty \varphi(t)\,e^{-xt}\,dt,\end{aligned}$$

where $\varphi(t) = \left(\frac{t}{1-e^{-t}} - \sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!}\right)e^{-t}$. Owing to $\frac{t}{1-e^{-t}} \sim t$ as $t\to\infty$ and $\sum_{n=0}^\infty \frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} \geqslant \frac{m^2t^2}{4!\,5!}$, one obtains that φ is negative for large t. It is easy to see that $\int_0^\infty |\varphi(t)|\,dt < \infty$ and using Proposition 2.2, we infer that f is not completely monotonic. $\square$

Proposition 3.2. *For all $m > 0$ function the $f(x) = \frac{1}{m}\sinh\frac{m}{x} - \psi'(x+1)$ is completely monotonic.*

Proof. Using (3.1), we have

$$\frac{1}{m}\sinh\frac{m}{x} = \sum_{n=0}^{\infty}\frac{m^{2n}}{(2n+1)!\,x^{2n+1}} = \int_0^\infty \sum_{n=0}^{\infty}\frac{m^{2n}t^{2n}}{(2n)!\,(2n+1)!}\,e^{-tx}\,dt,$$

and

$$\frac{1}{m}\sinh\frac{m}{x} - \psi'(x+1) = \int_0^\infty\left(\sum_{n=0}^{\infty}\frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} - \frac{te^{-t}}{1-e^{-t}}\right)e^{-xt}\,dt,$$

for all $x > 0$. Hence

$$f(x) = \int_0^\infty \varphi(t)\,e^{-xt}\,dt, \tag{3.2}$$

where $\varphi(t) = \sum_{n=0}^{\infty}\frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} - \frac{te^{-t}}{1-e^{-t}}$. Further, it is

$$\varphi(t)(e^t-1) = (e^t-1)\sum_{n=0}^{\infty}\frac{m^{2n}t^{2n}}{(2n)!(2n+1)!} - t \geqslant t\cdot 1 - t = 0,$$

for all $t \geqslant 0$. Consequently, $\varphi \geqslant 0$ on $[0,\infty)$ and by (3.2) one concludes the proof. $\square$

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[1] Faculty of Sciences and Mathematics, University of Banja Luka, Banja Luka, Republic of Srpska, Bosnia and Herzegovina

Email address: vladimir.jovanovic@pmf.unibl.org

Email address: milanka.treml@pmf.unibl.org