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ON THE ENTIRE ZAGREB INDICES OF THE LINE GRAPH AND LINE CUT-VERTEX GRAPH OF THE SUBDIVISION GRAPH

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Abstract. Let $G = (V, E)$ be a graph. Then the first and second entire Zagreb indices of G are defined, respectively, as $M_1^\varepsilon(G) = \sum_{x \in V(G) \cup E(G)} (d_G(x))^2$ and $M_2^\varepsilon(G) = \sum_{\{x,y\} \in B(G)} d_G(x) d_G(y)$, where $B(G)$ denotes the set of all 2-element subsets $\{x, y\}$ such that $\{x, y\} \subseteq V(G) \cup E(G)$ and members of $\{x, y\}$ are adjacent or incident to each other. In this paper, we obtain the entire Zagreb indices of the line graph and line cut-vertex graph of the subdivision graph of the cycle-star graph.

1. Introduction

Throughout this paper, only the finite, undirected, and simple graphs are considered. Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. These two basic parameters n and m are called the *order* and *size* of G, respectively. The edge connecting the vertices u and v is denoted by uv. The *degree* of a vertex v, written $d_G(v)$, is the number of edges of G incident with v, each loop counting as two edges.

Among the oldest and most studied topological indices, there are two classical vertex-degree based topological indices - first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$. These two indices first appeared in [7], and were elaborated in [8]. The main properties of $M_1(G)$ and $M_2(G)$ were summarized in [10,16].

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The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a graph G are defined, respectively, as

$$M_1 = M_1(G) = \sum_{v \in V(G)} d_G(v)^2$$

$$M_2 = M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

During the past decades, numerous results concerning Zagreb indices have been put forward, see [2,3,4,9,11,12,16]; for historical details see [10].

Furtula et al. [5] introduced the forgotten index of G, written $F(G)$, as the sum of cubes of vertex degrees as follows:

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{e=uv \in E(G)} \left[d_G(u)^2 + d_G(v)^2 \right]$$

Milićević et al. [15] introduced the first and second reformulated Zagreb indices of a graph G as edge counterpart of the first and second Zagreb indices, respectively, as follows:

$$EM_1(G) = \sum_{e \sim f} [d_G(e) + d_G(f)] = \sum_{e \in E(G)} d_G(e)^2$$

$$EM_2(G) = \sum_{e \sim f} d_G(e) d_G(f),$$

where $d_G(e) = d_G(u) + d_G(v) - 2$ for the edge $e = uv$ and $e \sim f$ means that the edges e and f are incident.

Alwardi et al. [1] introduced the first and second entire Zagreb indices of a graph G as follows:

$$M_1^{\varepsilon}(G) = \sum_{x \in V(G) \cup E(G)} (d_G(x))^2$$

$$M_2^{\varepsilon}(G) = \sum_{\{x,y\} \in B(G)} d_G(x) d_G(y),$$

where $B(G)$ denotes the set of all 2-element subsets $\{x, y\}$ such that $\{x, y\} \subseteq V(G) \cup E(G)$ and members of $\{x, y\}$ are adjacent or incident to each other.

Jelena Sedlar [20] introduced the concept of *cycle-star graph* as follows.

The *cycle-star graph*, written $CS_{k,n-k}$, is a graph with n vertices consisting of the cycle-graph of length k and $n-k$ leafs appended to the same vertex of the cycle. The cycle-star graphs $CS_{3,4}$ and $CS_{4,3}$ are shown in Figure 1. The *subdivision graph* of a graph G, written $S(G)$, is the graph obtained from G by replacing each of its edges by a path of length 2, or equivalently by inserting an additional vertex into each edge of G.

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, line cut-vertex graphs; total graphs; middle graphs; and their generalizations.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.

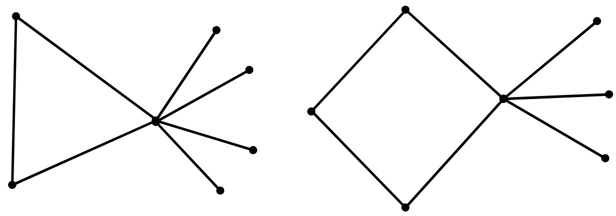

FIGURE 1

In [19], the Zagreb indices and coindices of the line graphs of the subdivision graphs were studied. The authors in [14] gave the following definition.

The *line cut-vertex graph* of G, written $L_c(G)$, is the graph whose vertices are the edges and cut-vertices of G, with two vertices of $L_c(G)$ adjacent whenever the corresponding edges of G have a vertex in common; or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G such that e_i is incident with c_j. Clearly, $L(G) \subseteq L_c(G)$, where $\subseteq$ is the subgraph notation.

Figure 2 shows an example of a graph G and its line cut-vertex graph $L_c(G)$. In

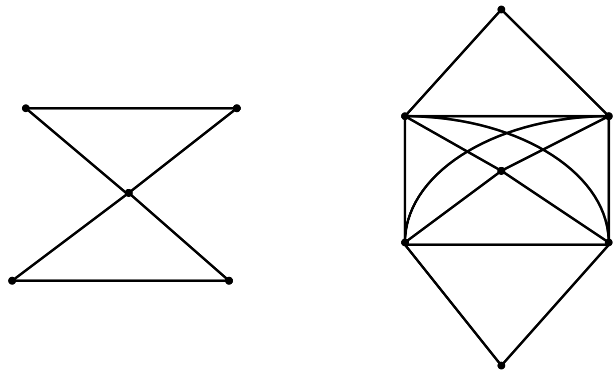

FIGURE 2

this paper we study the line graph and line cut-vertex graph of the subdivision graph of the cycle-star graph; and calculate the first and second entire Zagreb indices of the graphs $L(S(CS_{k,n-k}))$; and the first entire Zagreb index of $L_c(S(CS_{k,n-k}))$. Notations and definitions not introduced here can be found in [13].

2. Entire Zagreb indices of the line graph of the subdivision graph of the cycle-star $CS_{k,n-k}$

In this section we calculate the first and second entire Zagreb indices of the line graph of the subdivision graph of the cycle-star graph. The author in [17] established the following result.

THEOREM 2.1. *Let G be the line graph of the subdivision graph of the cycle-star graph. Then*
$M_1(G) = n^3 + (6-3k)n^2 + (3k^2 - 12k + 13)n - k^3 + 6k^2 - 5k$; *and*

$$M_2(G) = \frac{1}{2}\left(n^4 + (7-4k)n^3 + (6k^2 - 21k + 20)n^2 - (4k^3 - 21k^2 + 40k - 32)n\right) + \frac{1}{2}\left(k^4 - 7k^3 + 20k^2 - 16k\right).$$

We now find the forgotten index; and first and second reformulated Zagreb indices of the line graph of the subdivision graph of the cycle-star graph.

THEOREM 2.2. *Let G be the line graph of the subdivision graph of the cycle-star graph. Then*

$$F(G) = n^4 + k^4 - 4(n^3k + k^3n + 2k^3 - 2n^3) + 6(n^2k^2 + 4k^2 + 4n^2) - 12(2n^2k - 2nk^2 + 4nk) - 17k + 33n.$$

PROOF. The subdivision graph $S(CS_{k,n-k})$ contains $2n$ vertices and $2n$ edges, so that the line graph of $S(CS_{k,n-k})$ contains $2n$ vertices, out of which $2k-2$ vertices of are of degree 2; $n-k+2$ vertices are degree $n-k+2$; and the remaining $n-k$ vertices are of degree 1. Thus,

$$F(G) = 8(2k-2) + (n-k+2)(n-k+2)^3 + (n-k)$$

But,

$$(n-k+2)^4 = n^4 - 4n^3k + 8n^3 + 6n^2k^2 - 24n^2k + 24n^2 - 4nk^3 + 24nk^2 - 48nk + 32n + k^4 - 8k^3 + 24k^2 - 32k + 16.$$

Thus,

$$F(G) = n^4 + k^4 - 4(n^3k + k^3n + 2k^3 - 2n^3) + 6(n^2k^2 + 4k^2 + 4n^2) - 12(2n^2k - 2nk^2 + 4nk) - 17k + 33n.$$

□

Zhou et al. established in [21] the following relation.

THEOREM 2.3. *Let G be a graph with m edges. Then*

$$EM_1(G) = F(G) + 4m + 2M_2(G) - 4M_1(G).$$

We will use Theorem 2.3 to find the first reformulated Zagreb index of the line graph of the subdivision graph of the cycle-star graph.

THEOREM 2.4. *Let G be the line graph of the subdivision graph of the cycle-star graph. Then*

$$EM_1(G) = 2n^4 + 2k^4 - 8n^3k - 8k^3n - 11k^3 + 11n^3 + 12n^2k^2 + 22k^2 + 22n^2 - 33n^2k + 33nk^2 - 44nk - 15k + 23n.$$

PROOF. By definition, size of $L(S(CS_{k,n-k}))$ is

$$|E(L(S(CS_{k,n-k})))| = \frac{1}{2}[n^2 + k^2 + 5n - 2nk - k] \tag{2.1}$$

Theorems 2.1, 2.2, 2.3, and Expression 2.1, give us the result. □

THEOREM 2.5. *Let G be the line graph of the subdivision graph of the cycle-star graph. Then*

$$\begin{aligned} EM_2(G) = {} & 2k^4 - 8k^3n - 10k^3 + 12k^2n^2 + 30k^2n + 22k^2 - 8kn^3 - 30kn^2 - 44kn \\ & - 18k + \frac{(n-k)(n+2-k)(n+1-k)(2n+2-2k)^2}{2} + 2n^4 + 10n^3 \\ & + 22n^2 + 26n. \end{aligned}$$

PROOF. Let G be the line graph of the subdivision graph of the cycle-star graph. We consider the following five cases:
Case 1: There are $2(k-2)$ pairs of edges with degree 2. Then the second reformulated Zagreb index is $8(k-2)$.
Case 2: There are $\frac{(n-k)(n-k+1)(n-k+2)}{2}$ pairs of edges with degree $2n-2k+2$. Then the second reformulated Zagreb index is $(2n-2k+2)^2\left(\frac{(n-k)(n-k+1)(n-k+2)}{2}\right)$.
Case 3: There are $2(n-k+1)$ pairs of edges with degree $n-k+2$ and $2n-2k+2$. Then the second reformulated Zagreb index is $2(n-k+1)(n-k+2)(2n-2k+2)$.
Case 4: There are $(n-k)(n-k+1)$ pairs of edges with degree $n-k+1$ and $2n-2k+2$. Then the second reformulated Zagreb index is $(n-k)(n-k+1)^2(2n-2k+2)$.
Case 5: There are 2 pairs of edges with degree 2 and $n-k+2$. Then the second reformulated Zagreb index is $4(n-k+2)$.

By adding the expressions of all the cases mentioned above, we get

$$\begin{aligned} EM_2(G) = {} & 2k^4 - 8k^3n - 10k^3 + 12k^2n^2 + 30k^2n + 22k^2 - 8kn^3 - 30kn^2 - 44kn \\ & - 18k + \frac{(n-k)(n+2-k)(n+1-k)(2n+2-2k)^2}{2} + 2n^4 + 10n^3 \\ & + 22n^2 + 26n. \end{aligned}$$

□

Ali Ghalavand et al. in [6] established a complete set of relations between entire Zagreb indices with the Zagreb and reformulated Zagreb indices of graphs as follows:

THEOREM 2.6. *Let G be a graph with n vertices and m edges. Then*

$$M_1^{\varepsilon}(G) = M_1(G) + EM_1(G),$$
$$M_2^{\varepsilon}(G) = 3M_2(G) + EM_2(G) + F(G) - 2M_1(G).$$

We now give the expressions for the first and second entire Zagreb indices of the line graph of the subdivision graph of the cycle-star graph.

THEOREM 2.7. *Let G be the line graph of the subdivision graph of the cycle-star graph. Then*

$$M_1^{\varepsilon}(G) = 2n^4 - 8n^3k + 12n^3 + 12n^2k^2 - 36n^2k + 28n^2 - 8nk^3 + 36nk^2 - 56nk + 36n + 2k^4 - 12k^3 + 28k^2 - 20k$$

$$M_2^{\varepsilon}(G) = \frac{1}{2}(9n^4 - 36n^3k + 53n^3 + 54n^2k^2 - 159n^2k + 128n^2 - 36nk^3 + 159nk^2 - 256nk) + \frac{1}{2}(162n + 9k^4 - 53k^3 + 128k^2 - 98k + (n-k+1)^3(4n^2 - 8nk + 8n + 4k^2 - 8k)).$$

PROOF. Theorems 2.1, 2.2, 2.4, 2.5, and 2.6, give us the results. □

Figure 3 shows an example of the cycle-star graphs $CS_{3,1}$ and $S(CS_{3,1})$, and of the graph $L(S(CS_{3,1}))$. Here $n = 4$ and $k = 3$.

For $L(S(CS_{3,1}))$ of Figure 3, using Theorems 2.1, 2.2, 2.4, 2.5, and 2.7, we can find $M_1(G) = 44, M_2(G) = 54, F(G) = 114, EM_1(G) = 82$, $EM_2(G) = 132$, $M_1^{\varepsilon}(G) = 126$, and $M_2^{\varepsilon}(G) = 320$.

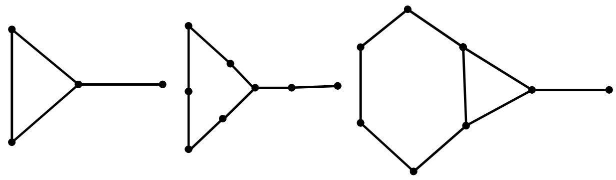

FIGURE 3

3. First entire Zagreb index of the line cut-vertex graph of the subdivision graph of the cycle-star graph $CS_{k,n-k}$

In this section we calculate the first entire Zagreb index of the line cut-vertex graph of the subdivision graph of the cycle-star graph. The author in [17] established the following result.

THEOREM 3.1. *Let G be the line cut-vertex graph of the subdivision graph of the cycle-star graph. Then*

$$M_1(G) = n^3 + (11 - 3k)n^2 + (3k^2 - 22k + 40)n - k^3 + 11k^2 - 32k + 14; \text{ and}$$

$$M_2(G) = \frac{1}{2}(n^4 + k^4 + (13 - 4k)n^3 + (6k^2 - 37k + 56)n^2 - (4k^3 - 35k^2 + 96k - 80)n) - \frac{11}{2}k^3 + 22k^2 - 34k + 9.$$

We now find the forgotten index and first reformulated Zagreb index of the line cut-vertex graph of the subdivision graph of the cycle-star graph.

THEOREM 3.2. *Let G be the line cut-vertex graph of the subdivision graph of the cycle-star graph. Then*

$$F(G) = k^4 - 4k^3n - 15k^3 + 6k^2n^2 + 45k^2n + 72k^2 - 4kn^3 - 45kn^2 - 144kn - 130k + n^4 + 15n^3 + 72n^2 + 146n + 46.$$

PROOF. The subdivision graph $S(CS_{k,n-k})$ contains $2n$ vertices and $2n$ edges, so that the line cut-vertex graph of $S(CS_{k,n-k})$ contains $2n+1+n-k = 3n-k+1$ vertices, out of which $2k-2$ vertices of are of degree 2; $2(n-k)$ vertices are degree 2; 2 vertices are of degree $n-k+3$; 1 vertex is of degree $n-k+2$; and the remaining $n-k$ vertices are of degree $n-k+4$. Thus,

$$F(G) = 8(2k-2) + 8(2n-2k) + 2(n-k+3)^3 + (n-k+2)^3 + (n-k)(n-k+4)^3.$$

But,

$$(n-k+2)^3 = n^3 - 3n^2k + 6n^2 + 3nk^2 - 12nk + 12n - k^3 + 6k^2 - 12k + 8,$$

$$(n-k+3)^3 = n^3 - 3n^2k + 9n^2 + 3nk^2 - 18nk + 27n - k^3 + 9k^2 - 27k + 27,$$

$$(n-k+4)^3 = n^3 - 3n^2k + 12n^2 + 3nk^2 - 24nk + 48n - k^3 + 12k^2 - 48k + 64,$$

$$(n-k)(n-k+4)^3 = n^4 - 4n^3k + 12n^3 + 6n^2k^2 - 36n^2k + 48n^2 - 4nk^3 + 36nk^2 - 96nk + 64n + k^4 - 12k^3 + 48k^2 - 64k.$$

Thus,

$$F(G) = k^4 - 4k^3n - 15k^3 + 6k^2n^2 + 45k^2n + 72k^2 - 4kn^3 - 45kn^2 - 144kn - 130k + n^4 + 15n^3 + 72n^2 + 146n + 46.$$

□

THEOREM 3.3. *Let G be the line cut-vertex graph of the subdivision graph of the cycle-star graph. Then*

$$EM_1(G) = 2k^4 - 8k^3n - 22k^3 + 12k^2n^2 + 68k^2n + 74k^2 - 8kn^3 - 70kn^2 - 156kn - 84k + 2n^4 + 24n^3 + 86n^2 + 88n + 16.$$

PROOF. By definition, the size of $L_c(S(CS_{k,n-k}))$ is

$$|E(L_c(S(CS_{k,n-k})))| = \frac{1}{2}(n^2 + k^2 + 11n - 7k - 2nk + 4) \quad (3.1)$$

Theorems 2.3, 3.1, 3.2, and Expression 3.1, give us the result. □

We now give the expression for the first entire Zagreb index of the line cut-vertex graph of the subdivision graph of the cycle-star graph.

THEOREM 3.4. *Let G be the line cut-vertex graph of the subdivision graph of the cycle-star graph. Then*

$$M_1^\varepsilon(G) = 2n^4 + 25n^3 - 8n^3k + 12n^2k^2 + 97n^2 - 73n^2k + 71nk^2 + 128n - 8nk^3 - 178nk + 2k^4 + 85k^2 + 30 - 23k^3 - 116k$$

PROOF. Theorems 2.6, 3.1, 3.3, give us the result. □

Figure 4 shows an example of the cycle-star graphs $CS_{3,1}$ and $S(CS_{3,1})$, and of the graph $L_c(S(CS_{3,1}))$. Here $n = 4$ and $k = 3$.

For $L_c(S(CS_{3,1}))$ of Figure 4, using Theorems 3.1, 3.2, 3.3, and 3.4, we can find $M_1(G) = 90$, $M_2(G) = 147$, $F(G) = 328$, $EM_1(G) = 318$, and $M_1^\varepsilon(G) = 408$.

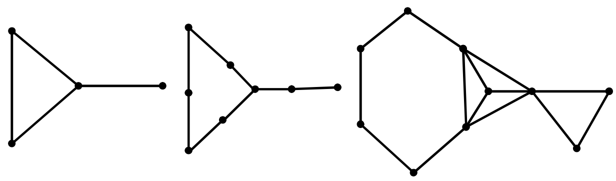

FIGURE 4

4. Conclusion

In this paper we have investigated the first and second entire Zagreb indices of the line graph of the subdivision graph of the cycle-star graph; and the first entire Zagreb index of the line cut-vertex graph of the subdivision graph of the cycle-star graph. However, to determine the entire Zagreb indices of some other graph operators [18] still remain open and challenging problem for researchers.

References

[1] A. Alwardi, A. Alqesmah, R. Rangarajan, I. N. Cangul, Entire Zagreb indices of graphs, *Discr. Math. Alg. Appl.* **10** (2018) # 1850037 (16 pages).

[2] T. Došlić, Vertex-weighted Weiner polynomials for composite graphs, *Ars Math. Contemp.* **1** (2008) 66-80.

[3] K. C. Das, I. Gutman, Some properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem.* **52** (2004) 103-112.

[4] B. Furtula, I. Gutman, M. Dehmer, On structure-sensitivity of degree-based topological indices, *Appl. Math. Comput.* **219** (2013) 8973-8978.

[5] B. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.* **53** (2015) 1184-1190.

[6] A. Ghalavand, A. R. Ashrafi, Bounds on the entire Zagreb indices of graphs, *MATCH Commun. Math. Comput. Chem.* **81** (2019) 371-381.

[7] I. Gutman, N. Trinajstić N, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem Phys Lett.* **17** (1972) 535-538.

[8] I. Gutman, B. Ruščić, N. Trinajstić, C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J Chem Phys.* **62** (1975) 3399-3405.

[9] I. Gutman, Degree-based topological indices, *Croat. Chem, Acta.* **86** (2013) 351-361.

[10] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 83-92.

[11] I. Gutman, J.Tošvić, Testing the quality of molecular structure descriptors. Vertex-degree based topological indices, *J. Serb. Chem. Soc.* **78** (2013) 805-810.
[12] I. Gutman, On the origin of two degree-based topological indices, *Bull. Acad. Serbe Sci. Arts (Cl. Sci. Math. Natur.).* **146** (2014) 39-52.
[13] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass (1969).
[14] V. R. Kulli, M. H. Muddebihal, On lict and litact graph of a graph, *Proceeding of the Indian National Science Academy.* **41**(1975) 275-280.
[15] A. Milićević, S. Nikolić, N. Trinajstić, On reformulated Zagreb indices, *Mol. Divers.* **8** (2004) 393-399.
[16] S. Nikolić, G. Kovačević, A. Milićević, N. Trinajstić, The Zagreb indices 30 years after, *Croat Chem Acta.* **76** (2003) 113-124.
[17] H. M. Nagesh, On the Zagreb indices of the line graph and line cut-vertex graph of the subdivision graph, *submitted.*
[18] E. Prisner, *Graph dynamics*, Chapman & Hall/CRC, **338** (1995).
[19] P. S. Ranjini, V. Lokesha, I. N. Cangul, On the Zabreb indices of the line graphs of the subdivision graphs, *Appl. Math. Comput.* **210** (2011) 699-702.
[20] J. Sedlar, Extremal unicyclic graphs with respect to additively weighted Harary index, *Miskolic mathematical Notes.* **16(2)** (2013) 1-16.
[21] B. Zhou, N. Trinajstić, Some properties of the reformulated Zagreb indices, *J.Math.Chem.* **48** (2010) 714-719.

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