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REGULARITY OF ITERATED CROSSED PRODUCT OF MONOIDS

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Abstract. In this work, regularity of iterated crossed product of finite cyclic monoids from the point of Combinatorial Group Theory is studied. Here, it is determined necessary and sufficient conditions of the iterated crossed product $A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$ to be regular.

1. Introduction and Preliminaries

Recent developments in semigroup and group theory have raised the question of whether there exists a classification for some algebraic structures, such as regularity, strongly π-inverse, decision problems, orthodox. As an answer to the regularity of this algebraic structures, in [**15**], Skornjakov explained regularity of the wreath product of monoids. After that, in [**13**], it has been investigated regular property of semidirect products of monoids. After these works, Karpuz et al. [**12**] determined necessary and sufficient conditions for Schützenberger product of monoids and the new version of the Schützenberger product of monoids to be regular and strongly π-inverse. Also, in [**10**], the authors studied the regularity of a new monoid construction with combining crossed and Schützenberger product for any two monoids. In recent years, in [**5**], it has been investigated the regularity of crossed product. As a continuation of these studies, Çetinalp give the necessary and sufficient conditions for the regularity of n-generalized Schützenberger product of arbitrary monoids [**9**]. In this study, as the main result of this paper, we give necessary and sufficient conditions of the iterated crossed product

$$A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$$

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to be regular, where all A_i $(1 \leqslant i \leqslant n)$ are finite cyclic monoids.

DEFINITION 1.1. A monoid M is called *regular* if, for every $a \in M$, there exists $b \in M$ such that $aba = a$ and $bab = b$ (or, equivalently, for the set of inverses of a in M, that is, $a^{-1} = \{b \in B : aba = a \text{ and } bab = b\}$, M is regular if and only if, for all $a \in M$, the set a^{-1} is not equal to the emptyset).

Crossed product construction is an important structure from point of famous *extension problem*, which is one of the most interesting problem of algebra and first stated by O. L. Hölder in 1895 [**11**]. This problem consists of describing and classifying all groups C containing A as a normal subgroup such that $C/B \cong A$. The extension problem has been the starting point of new subjects in mathematics such as cohomology of groups, homological algebra, crossed products of groups acting on algebras, crossed products of Hopf algebras acting on algebras, crossed products for von Neumann algebras etc. In [**1, 2**] authors give some results on the crossed product about this extension problem. Also they say that the set of these $(C, .)$ group structures is a one to one correspondence with the set of all normalised crossed systems (A, B, φ, f).

Let A and B be two monoids. A crossed system of these monoids is a quadruple (A, B, φ, f), where $\varphi : B \to End(A)$ and $f : B \times B \to A$ are two maps such that the following compatibility conditions hold:

$$b_1 \triangleleft_\varphi (b_2 \triangleleft_\varphi a) = f(b_1, b_2)((b_1 b_2) \triangleleft_\varphi a) f(b_1, b_2)^{-1}, \tag{1.1}$$

$$f(b_1, b_2) f(b_1 b_2, b_3) = (b_1 \triangleleft_\varphi f(b_2, b_3)) f(b_1, b_2 b_3), \tag{1.2}$$

for all $b_1, b_2, b_3 \in B$ and $a \in A$. The crossed system (A, B, φ, f) is called normalized if $f(1, 1) = 1$. (A, B, φ, f) is normalized crossed system then $f(1, b) = f(b, 1) = 1$ and $1 \triangleleft_\varphi a = a$, for any $a \in A$ and $b \in B$. Here, the notation " $\triangleleft$ " is defined $g \triangleleft_\varphi h = \varphi_g(h)$ as semidirect product action.

Let $A \#_\varphi^f B := A \times B$ as a set with a binary operation defined by the formula:

$$(a_1, b_1)(a_2, b_2) = (a_1 (b_1 \triangleleft_\varphi a_2) f(b_1, b_2), b_1 b_2),$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $(A \#_\varphi^f B, \cdot)$ is a monoid with the identity $1_{A \#_\varphi^f B} = (1, 1)$ if and only if (A, B, φ, f) is a normalized crossed system [**1**]. The monoid $A \#_\varphi^f B$ is called *the crossed product of A and B associated to the crossed system* (A, B, φ, f). The reader is referred to [**7, 8, 10**] for recent studies on crossed product and its derivations.

Now, let us give the definition of iterated crossed product of monoids.

In [**8**], authors defined iterated crossed product which is more important than known group/monoid constructions since it possess direct, semidirect [**3, 6**], twisted [**14**], knit [**4**] and crossed products of algebraic structures. Also they obtained a presentation for iterated crossed product of cyclic groups and found a complete rewriting system and thus obtained normal form structure of elements of this new construction.

Definition 1.2. Let $A_1, A_2, \cdots, A_n$ be finite cyclic monoids of order $a_1, a_2, \cdots, a_n$, respectively. A crossed system of these monoids is a quadruple

$$(A_i, A_{i+1}\#^{f_{i+1}}_{\alpha_{i+1}} A_{i+2}\#^{f_{i+2}}_{\alpha_{i+2}} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n, \alpha_i, f_i) \quad (1 \leqslant i \leqslant n-1),$$

where $\alpha_i : A_{i+1}\#^{f_{i+1}}_{\alpha_{i+1}} A_{i+2}\#^{f_{i+2}}_{\alpha_{i+2}} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n \to End(A_i)$ and

$$f_i : (A_{i+1}\#^{f_{i+1}}_{\alpha_{i+1}} A_{i+2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n) \times (A_{i+1}\#^{f_{i+1}}_{\alpha_{i+1}} A_{i+2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n) \to A_i$$

are maps such that (1.1), (1.2) and the following compatability conditions hold:

$$\begin{aligned}
&a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} (a_{1,3}a_{2,3} \lhd_{\alpha_2} (\cdots \lhd_{\alpha_{n-2}} (a_{1,n}a_{2,n} \lhd_{\alpha_{n-1}} a_{3,1}) \cdots))\\
&f_1(a_{2,2}, a_{3,2})) = a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} (\cdots \lhd_{\alpha_{n-2}} (a_{1,n} \lhd_{\alpha_{n-1}}\\
&[a_{2,2} \lhd_{\alpha_1} (\cdots \lhd_{\alpha_{n-2}} (a_{2,n} \lhd_{\alpha_{n-1}} a_{3,1}) \cdots) f_1(a_{2,2}, a_{3,2})]) \cdots)).
\end{aligned}$$

For the other condition, we use the notation $f_i(a_{m,i+1}, a_{n,i+1})$ instead of $f_i((a_{m,i+1}, 1_{A_{i+2}}, \cdots, 1_{A_n}), (a_{n,i+1}, 1_{A_{i+2}}, \cdots, 1_{A_n}))$ to have more understandable expressions in multiplications. So,

$$f_i(a_{m,i+1}, a_{n,i+1}) = a_{m,i} \quad (2 \leqslant i \leqslant n-1),$$

where $a_{j,i}$ is the j th element of i th monoid.

The iterated crossed product of cyclic monoids $A_1, A_2, \cdots, A_n$ associated to the crossed system with respect to the actions given above is the set $A_1 \times A_2 \times \cdots \times A_n$ with the multiplication

$$\begin{aligned}
&(a_{1,1}, a_{1,2}, \cdots, a_{1,n})(a_{2,1}, a_{2,2}, \cdots, a_{2,n})\\
&= (a_{1,1}(a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} (\cdots \lhd_{\alpha_{n-2}} (a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)))f_1(f_2(\cdots\\
&(f_{n-1}(a_{1,n}, a_{2,n}), a_{2,n-1}), \cdots), a_{2,2}),\ a_{1,2}a_{2,2},\ a_{1,3}a_{2,3}, \cdots, a_{1,n}a_{2,n})
\end{aligned}$$

for all $a_{j,i} \in A_i$ $(1 \leqslant i \leqslant n)$.

This product is denoted by $A_1\#^{f_1}_{\alpha_1} A_2\#^{f_2}_{\alpha_2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n$.

Let us consider the actions given above. Then the iterated normalized crossed product $A_1\#^{f_1}_{\alpha_1} A_2\#^{f_2}_{\alpha_2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n$ defines a monoid with the identity $(1_{A_1}, 1_{A_2}, \cdots, 1_{A_n})$ if and only if $(A_i, A_{i+1}\#^{f_{i+1}}_{\alpha_{i+1}} A_{i+2}\#^{f_{i+2}}_{\alpha_{i+2}} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n, \alpha_i, f_i)$ $(1 \leqslant i \leqslant n-1)$ is a normalized crossed system. The reader is referred to [**8**] for more details.

2. Regularity for $A_1\#^{f_1}_{\alpha_1} A_2\#^{f_2}_{\alpha_2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n$

The target of this section is to present for the regularity of iterated crossed product of finite cyclic monoids $A_1, A_2, \cdots, A_{n-1}$ and A_n. In here, we use the notations e_i and c_i instead of $(a_{i,3}a_{k,3} \lhd_{\alpha_2} (a_{i,4}a_{k,4} \lhd_{\alpha_3} \cdots (a_{i,n}a_{k,n} \lhd_{\alpha_{n-1}} a_{i,1}) \cdots))$ and $(a_{i,3} \lhd_{\alpha_2} (a_{i,4} \lhd_{\alpha_3} \cdots (a_{i,n} \lhd_{\alpha_{n-1}} a_{k,1}) \cdots))$, respectively, to have more convenience expressions in multiplications, where $i = 1$, $k = 2$ or $i = 2$, $k = 1$.

Theorem 2.1. *Let $A_1, A_2, \cdots, A_{n-1}$ and A_n be finite cyclic monoids. Also, for $a_{1,i} \in a^{-1}_{2,i}$ $(2 \leqslant i \leqslant n)$, let us have $a_{i,1} \in A_1(a_{i,2} \lhd_{\alpha_1} (a_{k,2} \lhd_{\alpha_1} e_i))$ such that*

$$f_1(a_{i,2}, a_{k,2})(a_{i,2}a_{k,2} \lhd_{\alpha_1} e_i) f_1(a_{i,2}a_{k,2}, a_{i,2}) = a_{i,2} \lhd_{\alpha_1} (a_{k,2} \lhd_{\alpha_1} e_i),$$

where $i = 1,\ k = 2$ *or* $i = 2,\ k = 1$. *Then normalized iterated crossed product* $A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$ *is regular if and only if* $A_i\ (1 \leqslant i \leqslant n)$ *are regular.*

Proof. Let us suppose that $A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$ is regular. Thus for $(a_{1,1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_n}) \in A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$, there exists

$$(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \in A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$$

such that

$$\begin{aligned}
&(a_{1,1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_n})\\
&= (a_{1,1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})(a_{1,1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_n})\\
&= (a_{1,1}(1_{A_2} \triangleleft_{\alpha_1} (1_{A_3} \triangleleft_{\alpha_2} \cdots (1_{A_n} \triangleleft_{\alpha_{n-1}} a_{2,1}) \cdots))f_1(f_2(\cdots(f_{n-1}\\
&\quad (1_{A_n}, a_{2,n}), a_{2,n-1}), \cdots), a_{2,2}), a_{2,2}, a_{2,3}, \cdots, a_{2,n})(a_{1,1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_n})\\
&= (a_{1,1}a_{2,1},\ a_{2,2},\ a_{2,3}, \cdots, a_{2,n})(a_{1,1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_n})\\
&= (a_{1,1}a_{2,1}(a_{2,2} \triangleleft_{\alpha_1} (a_{2,3} \triangleleft_{\alpha_2} \cdots (a_{2,n} \triangleleft_{\alpha_{n-1}} a_{1,1}) \cdots))f_1(f_2(\cdots(f_{n-1}\\
&\quad (a_{2,n}, 1_{A_n}), 1_{A_{n-1}}), \cdots), 1_{A_2}), a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{1,1}a_{2,1}(a_{2,2} \triangleleft_{\alpha_1} (a_{2,3} \triangleleft_{\alpha_2} \cdots (a_{2,n} \triangleleft_{\alpha_{n-1}} a_{1,1}) \cdots)), a_{2,2}, a_{2,3}, \cdots, a_{2,n})
\end{aligned}$$

and

$$\begin{aligned}
&(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})(a_{1,1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{2,1}(a_{2,2} \triangleleft_{\alpha_1} (a_{2,3} \triangleleft_{\alpha_2} \cdots (a_{2,n} \triangleleft_{\alpha_{n-1}} a_{1,1}) \cdots))f_1(f_2(\cdots(f_{n-1}(a_{2,n}, 1_{A_n}),\\
&\quad 1_{A_{n-1}}), \cdots), 1_{A_2}), a_{2,2}, a_{2,3}, \cdots, a_{2,n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{2,1}(a_{2,2} \triangleleft_{\alpha_1} (a_{2,3} \triangleleft_{\alpha_2} \cdots (a_{2,n} \triangleleft_{\alpha_{n-1}} a_{1,1}) \cdots)), a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&\quad (a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{2,1}(a_{2,2} \triangleleft_{\alpha_1} (a_{2,3} \triangleleft_{\alpha_2} \cdots (a_{2,n} \triangleleft_{\alpha_{n-1}} a_{1,1}) \cdots))\\
&\quad (a_{2,2} \triangleleft_{\alpha_1} (a_{2,3} \triangleleft_{\alpha_2} \cdots (a_{2,n} \triangleleft_{\alpha_{n-1}} a_{2,1}) \cdots))\\
&\quad f_1(f_2(\cdots(f_{n-1}(a_{2,n}, a_{2,n}),\quad a_{2,n-1}), \cdots), a_{2,2}), a_{2,2}^2,\ \ a_{2,3}^2, \cdots, a_{2,n}^2).
\end{aligned}$$

Therefore, we obtain that $\forall a_{2,i} = 1_{A_i}(2 \leqslant i \leqslant n)$. This gives that $a_{1,1}a_{2,1}a_{1,1} = a_{1,1}$ and $a_{2,1}a_{1,1}a_{2,1} = a_{2,1}$. Hence A_1 is regular. By using the similar argument, for $(1_{A_1}, a_{1,2}, 1_{A_3}, \cdots, 1_{A_n}) \in A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$, there exists

$$(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \in A_1\#_{\alpha_1}^{f_1}A_2\#_{\alpha_2}^{f_2}\cdots\#_{\alpha_{n-1}}^{f_{n-1}}A_n$$

such that

$$\begin{aligned}
&(1_{A_1}, a_{1,2}, 1_{A_3}, \cdots, 1_{A_n}) \\
&= (1_{A_1}, a_{1,2}, 1_{A_3}, \cdots, 1_{A_n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \\
&\qquad (1_{A_1}, a_{1,2}, 1_{A_3}, \cdots, 1_{A_n}) \\
&= (1_{A_1}(a_{1,2} \lhd_{\alpha_1} (1_{A_3} \lhd_{\alpha_2} \cdots (1_{A_n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)) f_1(f_2(\cdots(f_{n-1} \\
&\quad (1_{A_n}, a_{2,n}), a_{2,n-1}), \cdots), a_{2,2}),\ a_{1,2}a_{2,2}, \cdots, a_{2,n})(1_{A_1}, a_{1,2}, 1_{A_3}, \cdots, 1_{A_n}) \\
&= (a_{1,2} \lhd_{\alpha_1} a_{2,1}), a_{1,2}a_{2,2}, \cdots, a_{2,n})(1_{A_1}, a_{1,2}, 1_{A_3}, \cdots, 1_{A_n}) \\
&= ((a_{1,2} \lhd_{\alpha_1} a_{2,1})(a_{1,2}a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} 1_{A_1}) \cdots)) \\
&\quad f_1(f_2(\cdots(f_{n-1}(a_{2,n}, 1_{A_n}), 1_{A_{n-1}}), \cdots) a_{1,2}), a_{1,2}a_{2,2}a_{1,2},\ a_{2,3}, \cdots, a_{2,n}) \\
&= ((a_{1,2} \lhd_{\alpha_1} a_{2,1}),\ a_{1,2}a_{2,2}a_{1,2},\ a_{2,3}, \cdots, a_{2,n}).
\end{aligned}$$

and

$$\begin{aligned}
&(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \\
&= (a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})(1_{A_1}, a_{1,2}, 1_{A_3}, \cdots, 1_{A_n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \\
&= (a_{2,1}(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} 1_{A_1}) \cdots)) f_1(f_2(\cdots(f_{n-1}(a_{2,n}, 1_{A_n}), \\
&\quad 1_{A_{n-1}}), \cdots), a_{1,2}), a_{2,2}a_{1,2},\ a_{2,3}, \cdots, a_{2,n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \\
&= (a_{2,1},\ a_{2,2}a_{1,2},\ a_{2,3}, \cdots, a_{2,n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \\
&= (a_{2,1}(a_{2,2}a_{1,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)) f_1(f_2(\cdots(f_{n-1} \\
&\quad (a_{2,n}, a_{2,n}), a_{2,n-1}), \cdots), a_{2,2}), a_{2,2}a_{1,2}a_{2,2},\ a_{2,3}, \cdots, a_{2,n}).
\end{aligned}$$

Here, we obtain that $\forall a_{2,i} = 1_{A_i}$ $(1 \leqslant i \leqslant n,\ i \neq 2)$. This gives that $a_{1,2}a_{2,2}a_{1,2} = a_{1,2}$ and $a_{2,2}a_{1,2}a_{2,2} = a_{2,2}$. Hence A_2 is regular.

Now, we apply the operations given above for the elements

$$\left.\begin{array}{c}
(1_{A_1}, 1_{A_2}, a_{1,3}, \cdots, 1_{A_{n-1}}, 1_{A_n}), \\
\cdots \\
(1_{A_1}, 1_{A_2}, 1_{A_3}, \cdots, a_{1,n-1}, 1_{A_n}), \\
(1_{A_1}, 1_{A_2}, 1_{A_3}, \cdots, 1_{A_{n-1}}, a_{1,n})
\end{array}\right\} \in A_1 \#_{\alpha_1}^{f_1} A_2 \#_{\alpha_2}^{f_2} \cdots \#_{\alpha_{n-1}}^{f_{n-1}} A_n.$$

For the element $(1_{A_1}, 1_{A_2}, \cdots, a_{1,k}, \cdots, 1_{A_{n-1}}, 1_{A_n}) \in A_1 \#_{\alpha_1}^{f_1} A_2 \#_{\alpha_2}^{f_2} \cdots \#_{\alpha_{n-1}}^{f_{n-1}} A_n$, we obtain

$$\forall a_{2,i} = 1_{A_i}\ (1 \leqslant i \leqslant n,\ i \neq k).$$

After all above processes , we see that $A_i (3 \leqslant i \leqslant n)$ are regular.

Conversely, suppose that $A_1,\ A_2, \cdots, A_{n-1}$ and A_n are regular. Then by assumption we have for $a_{i,1} \in A_1(a_{i,2} \lhd_{\alpha_1} (a_{k,2} \lhd_{\alpha_1} e_i))$ such that

$$f_1(a_{i,2}, a_{k,2})(a_{i,2}a_{k,2} \lhd_{\alpha_1} e_i) f_1(a_{i,2}a_{k,2}, a_{i,2}) = a_{i,2} \lhd_{\alpha_1} (a_{k,2} \lhd_{\alpha_1} e_i),$$

where $i = 1,\ k = 2$ or $i = 2,\ k = 1$. Then, for the case $i = 1$ and $k = 2$, there are some $u_1 \in A_1,\ a_{1,1} = u_1(a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} e_1))$ and also for

$c_1 = (a_{1,3} \lhd_{\alpha_2} \cdots (a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)$, there exist $c_1 = a_{2,2} \lhd_{\alpha_1} e_1$, so we get

$$\begin{aligned}
& (a_{1,1}(a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} \cdots (a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots))f_1(a_{1,2}, a_{2,2})(a_{1,2}a_{2,2} \lhd_{\alpha_1} \\
& \quad (a_{1,3}a_{2,3} \lhd_{\alpha_2} \cdots (a_{1,n}a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots))f_1(a_{1,2}a_{2,2}, a_{1,2})), \\
= \; & u_1(a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} e_1))(a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} \cdots (a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)) \\
& \quad \underline{f_1(a_{1,2}, a_{2,2})(a_{1,2}a_{2,2} \lhd_{\alpha_1} (a_{1,3}a_{2,3} \lhd_{\alpha_2} \cdots (a_{1,n}a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots))} \\
& \quad f_1(a_{1,2}a_{2,2}, a_{1,2})), \\
= \; & u_1(a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} e_1))(a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} \cdots (a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)) \\
& \quad (a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} (a_{1,3}a_{2,3} \lhd_{\alpha_2} \cdots (a_{1,n}a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots))f_1(a_{1,2}, a_{2,2}) \\
& \quad f_1(a_{1,2}a_{2,2}, a_{1,2}) \\
= \; & u_1(a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} e_1))(a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} \cdots (a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)) \\
& \quad (a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} e_1))f_1(a_{1,2}, a_{2,2})f_1(a_{1,2}a_{2,2}, a_{1,2}) \\
= \; & u_1(a_{1,2} \lhd_{\alpha_1} ((a_{2,2} \lhd_{\alpha_1} e_1)(a_{1,3} \lhd_{\alpha_2} \cdots (a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots)(a_{2,2} \lhd_{\alpha_1} e_1))) \\
& \quad f_1(a_{1,2}, a_{2,2})f_1(a_{1,2}a_{2,2}, a_{1,2}) \\
= \; & u_1(a_{1,2} \lhd_{\alpha_1} ((a_{2,2} \lhd_{\alpha_1} e_1)c_1(a_{2,2} \lhd_{\alpha_1} e_1)))f_1(a_{1,2}, a_{2,2})f_1(a_{1,2}a_{2,2}, a_{1,2}) \\
= \; & u_1\underline{(a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} e_1))f_1(a_{1,2}, a_{2,2})}f_1(a_{1,2}a_{2,2}, a_{1,2}) \\
= \; & u_1f_1(a_{2,2})(a_{1,2}a_{2,2} \lhd_{\alpha_1} e_1)f_1(a_{1,2}a_{2,2}, a_{1,2}) \\
= \; & u_1(a_{1,2} \lhd_{\alpha_1} (a_{2,2} \lhd_{\alpha_1} e_1)) = a_{1,1}.
\end{aligned}$$

Also, while $i = 2$ and $k = 1$, there are some $u_2 \in A_2$, $a_{2,1} = u_2(a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} e_2))$ and also for $c_2 = (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots)$, there exist $c_2 = a_{1,2} \lhd_{\alpha_1} e_2$, so we get

$$\begin{aligned}
& (a_{2,1}(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots))f_1(a_{2,2}, a_{1,2}) \\
& (a_{2,2}a_{1,2} \lhd_{\alpha_1} (a_{2,3}a_{1,3} \lhd_{\alpha_2} \cdots (a_{2,n}a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots))f_1(a_{2,2}a_{1,2}, a_{2,2}) \\
= \; & u_2(a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} e_2))(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots)) \\
& \underline{f_1(a_{2,2}, a_{1,2})(a_{2,2}a_{1,2} \lhd_{\alpha_1} (a_{2,3}a_{1,3} \lhd_{\alpha_2} \cdots (a_{2,n}a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots))} \\
& f_1(a_{2,2}a_{1,2}, a_{2,2}) \\
= \; & u_2(a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} e_2))(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots)) \\
& (a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} (a_{2,3}a_{1,3} \lhd_{\alpha_2} \cdots (a_{2,n}a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1}) \cdots))) \\
& f_1(a_{2,2}, a_{1,2})f_1(a_{2,2}a_{1,2}, a_{2,2}) \\
= \; & u_2(a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} e_2))(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots (a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1}) \cdots)) \\
& (a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} e_2)f_1(a_{2,2}, a_{1,2})f_1(a_{2,2}a_{1,2}, a_{2,2}) \\
= \; & u_2(a_{2,2} \lhd_{\alpha_1} ((a_{1,2} \lhd_{\alpha_1} e_2)c_2(a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} e_2))f_1(a_{2,2}, a_{1,2}) \\
& f_1(a_{2,2}a_{1,2}, a_{2,2}) \\
= \; & u_2\underline{(a_{2,2} \lhd_{\alpha_1} ((a_{1,2} \lhd_{\alpha_1} e_2)f_1(a_{2,2}, a_{1,2})}f_1(a_{2,2}a_{1,2}, a_{2,2}) \\
= \; & u_2f_1(a_{2,2}, a_{1,2})(a_{2,2}a_{1,2} \lhd_{\alpha_1} e_2)f_1(a_{2,2}a_{1,2}, a_{2,2}) \\
= \; & u_2a_{2,2} \lhd_{\alpha_1} (a_{1,2} \lhd_{\alpha_1} e_2)) = a_{2,1}.
\end{aligned}$$

Consequently, as seen above, for every
$(a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n}) \in A_1\#^{f_1}_{\alpha_1} A_2\#^{f_2}_{\alpha_2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n$, there exists
$(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}) \in A_1\#^{f_1}_{\alpha_1} A_2\#^{f_2}_{\alpha_2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n$ such that

$$\begin{aligned}
&(a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})(a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n})\\
&= (a_{1,1}(a_{1,2} \lhd_{\alpha_1} (\cdots(a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1})\cdots))f_1(f_2(\cdots(f_{n-1}(a_{1,n}, a_{2,n}),\\
&\quad a_{2,n-1}),\cdots), a_{2,2}), a_{1,2}a_{2,2}, a_{1,3}a_{2,3}, \cdots, a_{1,n}a_{2,n})\\
&(a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n})\\
&= (a_{1,1}(a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} \cdots(a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1})\cdots))f_1(a_{1,2}, a_{2,2}),\\
&\quad a_{1,2}a_{2,2},\ a_{1,3}a_{2,3}, \cdots, a_{1,n}a_{2,n})(a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n})\\
&= (a_{1,1}(a_{1,2} \lhd_{\alpha_1} (a_{1,3} \lhd_{\alpha_2} \cdots(a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1})\cdots))f_1(a_{1,2}, a_{2,2})(a_{1,2}a_{2,2} \lhd_{\alpha_1}\\
&\quad (a_{1,3}a_{2,3} \lhd_{\alpha_2} \cdots(a_{1,n}a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1})\cdots))f_1(a_{1,2}a_{2,2}, a_{1,2}),\\
&\quad a_{1,2}a_{2,2}a_{1,2},\ a_{1,3}a_{2,3}a_{1,3}, \cdots, a_{1,n}a_{2,n}a_{1,n})\\
&= (a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n})
\end{aligned}$$

and

$$\begin{aligned}
&(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})(a_{1,1}, a_{1,2}, a_{1,3}, \cdots, a_{1,n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{2,1}(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots(a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1})\cdots))f_1(f_2(\cdots(f_{n-1}(a_{2,n}, a_{1,n}),\\
&\quad a_{1,n-1}),\cdots), a_{1,2}), a_{2,2}a_{1,2}, \cdots, a_{2,n}a_{1,n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{2,1}(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots(a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1})\cdots))f_1(a_{2,2}, a_{1,2}),\\
&\quad a_{2,2}a_{1,2}, a_{2,3}a_{1,3}, \cdots, a_{2,n}a_{1,n})(a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n})\\
&= (a_{2,1}(a_{2,2} \lhd_{\alpha_1} (a_{2,3} \lhd_{\alpha_2} \cdots(a_{2,n} \lhd_{\alpha_{n-1}} a_{1,1})\cdots))f_1(a_{2,2}, a_{1,2})\\
&\quad (a_{2,2}a_{1,2} \lhd_{\alpha_1} (a_{2,3}a_{1,3} \lhd_{\alpha_2} \cdots(a_{2,n}a_{1,n} \lhd_{\alpha_{n-1}} a_{2,1})\cdots))f_1(a_{2,2}a_{1,2}, a_{2,2}),\\
&\quad a_{2,2}a_{1,2}a_{2,2},\ \ a_{2,3}a_{1,3}a_{2,3}, \cdots, a_{2,n}a_{1,n}a_{2,n})\\
&= (a_{2,1}, a_{2,2}, a_{2,3}, \cdots, a_{2,n}).
\end{aligned}$$

Hence the result. □

Now, we give the following results according to the cases of maps $\alpha_i (1 \leqslant i \leqslant n)$ and f_i $(1 \leqslant i \leqslant n)$. It is known that if we take f_i $(1 \leqslant i \leqslant n)$ trivial maps, the iterated crossed product $A_1\#^{f_1}_{\alpha_1} A_2\#^{f_2}_{\alpha_2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n$ becomes iterated semidirect product $A_1 \rtimes_{\alpha_1} A_2 \rtimes_{\alpha_2} \cdots \rtimes_{\alpha_{n-1}} A_n$ [**6**]. Also, if we take α_i $(1 \leqslant i \leqslant n)$ and f_i $(1 \leqslant i \leqslant n)$ trivial actions, then iterated crossed product $A_1\#^{f_1}_{\alpha_1} A_2\#^{f_2}_{\alpha_2} \cdots \#^{f_{n-1}}_{\alpha_{n-1}} A_n$ becomes n-direct product. This product is denoted by $A_1 \times A_2 \times \cdots \times A_n$.

So we can give the other results in below.

Corollary 2.1. *Let $A_1, A_2, \cdots, A_{n-1}$ and A_n be finite cyclic monoids. For $a_{1,i} \in a^{-1}_{2,i}$ $(2 \leqslant i \leqslant n)$, let us have*

$$a_{i,1} \in A_1(a_{i,2} \lhd_{\alpha_1} (a_{k,2} \lhd_{\alpha_1} e_i)),$$

where $i = 1$, $k = 2$ or $i = 2$, $k = 1$. Then iterated semidirect product $A_1 \rtimes_{\alpha_1} A_2 \rtimes_{\alpha_2} \cdots \rtimes_{\alpha_{n-1}} A_n$ is regular if and only if each A_i $(1 \leqslant i \leqslant n)$ regular.

COROLLARY 2.2. *Let $A_1, A_2, \cdots, A_{n-1}$ and A_n be cyclic monoids. Then n-direct product of finite cyclic monoids $A_1 \times A_2 \times \cdots \times A_n$ is regular if and only if each A_i $(1 \leqslant i \leqslant n)$ regular.*

We also note that iterated crossed product $A_1 \#_{\alpha_1}^{f_1} A_2 \#_{\alpha_2}^{f_2} \cdots \#_{\alpha_{n-1}}^{f_{n-1}} A_n$ includes crossed product $A \#_{\alpha}^{f} B$, semidirect product $A \rtimes_\alpha B$ and twisted product $A \times^f B$. So, we can give the following result.

COROLLARY 2.3. *Let us consider the conditions obtained for the regularity of iterated crossed product $A_1 \#_{\alpha_1}^{f_1} A_2 \#_{\alpha_2}^{f_2} \cdots \#_{\alpha_{n-1}}^{f_{n-1}} A_n$ given in Theorem 2.1. Assume that finite cyclic monoids $A_3, A_4, \cdots, A_n$ are trivial monoids, then the conditions obtained for the regularity of $A_1 \#_{\alpha_1}^{f_1} A_2 \#_{\alpha_2}^{f_2} \cdots \#_{\alpha_{n-1}}^{f_{n-1}} A_n$ can be reduced to the conditions obtained for $A \#_{\alpha}^{f} B$, $A \rtimes_\alpha B$ and $A \times^f B$ in* [**5**].

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