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AMICABLE SETS OF GENERALIZED ALMOST DISTRIBUTIVE FUZZY LATTICES

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ABSTRACT. In this paper we introduce the concept of a compatible set, maximal set and amicable set of a Generalized Almost Distributive Fuzzy Lattices (GADFLs) and we study their properties in a GADFL. We also show any two amicable sets are isomorphic in an associative GADFL.

1. Introduction

The concept of Generalized almost distributive lattices (GADLs) was introduced by Rao, Bandaru and Rafi [**6**] as a generalization of Almost distributive lattices (ADLs) [**8**] which was a common abstraction of almost all the existing ring theoretic generalization of a Boolean algebra on one hand and Distributive lattice on the other. The concept of amicable set was first introduced by Subrahmanyam [**7**] in the class of triple systems. Later studied by Swamy and Rao [**8**] in the class of ADLs. Bandaru [**2**] also studied about a compatible set, maximal set and amicable set and their properties in a GADL. On the other hand, L. A. Zadeh [**9**] in 1965 introduced the notion of fuzzy set. Again in 1971, Zadeh [**10**] defined a fuzzy ordering as a generalization of the concept of ordering, that is, a fuzzy ordering is a fuzzy relation that is transitive. In particular, a fuzzy partial ordering is a fuzzy ordering that is reflexive and antisymmetric. In 1994, Ajmal and Thomas [**1**] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. In 2009, Chon [**5**], considering the notion of fuzzy order of Zadeh [**10**], introduced a

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new notion of fuzzy lattice and studied the level sets of fuzzy lattices. He also introduced the notions of distributive and modular fuzzy lattices and considered some basic properties of fuzzy lattices. In 2017, Berhanu *et al.* [**3**] introduce the concept of Almost distributive fuzzy lattices (ADFLs) as a generalization of Distributive fuzzy lattices and characterized some properties of an ADL using the fuzzy partial order relations and fuzzy lattices defined by I. Chon. Later on Berhanu and Yohannes [**4**] introduce the concept of Generalized almost distributive fuzzy lattices (GADFLs) as a generalization of Almost distributive fuzzy lattices (ADFLs). As a continuation in this paper we introduce the concept of compatible set, maximal set and amicable set of a Generalized almost distributive fuzzy lattices.

2. Preliminaries

First we recall certain definitions and properties of a Generalized almost distributive lattices.

Definition 2.1. ([**2**]) An algebra $(L, \vee, \wedge)$ of type (2, 2) is called a Generalized almost distributive lattice if it satisfies the following axioms:

(As $\wedge$) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
(LD $\wedge$) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(LD $\vee$) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
(A_1) $x \wedge (x \vee y) = x$,
(A_2) $(x \vee y) \wedge x = x$,
(A_3) $(x \wedge y) \vee y = y$

for all $x, y, z \in L$.

Lemma 2.1 ([**2**]). *For any $a \in L$,*

(1) $a \vee a = a$
(2) $a \wedge a = a$.

In addition to the 3 absorption laws A_1, A_2, A_3 given in Definition 2.1, we also get the following:

Lemma 2.2 ([**2**]). *For any $a, b \in L$,*

(A_4) $a \vee (a \wedge b) = a$,
(A_5) $a \vee (b \wedge a) = a$.

Definition 2.2 ([**2**]). For any $a, b \in L$ we say that a is less than or equal to b and write $a \leqslant b$, if $a \wedge b = a$ or equivalently, $a \vee b = b$.

Lemma 2.3 ([**2**]). *For any $a, b, c \in R$, $a \wedge b \wedge c = b \wedge a \wedge c$*

Definition 2.3. ([**5**]) Let X be a set. A function $A : X \times X \longrightarrow [0,\ 1]$ is called a fuzzy relation in X. The fuzzy relation A in X is reflexive iff $A(x, x) = 1$ for all $x \in X$, A is transitive iff $A(x, z) \geqslant \sup_{y \in X} \min(A(x, y), A(y, z))$, and A is antisymmetric iff $A(x, y) > 0$ and $A(y, x) > 0$ imply $x = y$. A fuzzy relation A is a fuzzy partial order relation if A is reflexive, antisymmetric and transitive. A fuzzy partial order relation A is a fuzzy total order relation iff $A(x, y) > 0$ or $A(y, x) > 0$ for all $x, y \in X$. If A is a fuzzy partial order relation in a set X, then (X, A)

is called a fuzzy partially ordered set or a fuzzy poset. If B is a fuzzy total order relation in a set X, then (X, B) is called a fuzzy totally ordered set or a fuzzy chain.

Definition 2.4. ([**5**]) Let (X, A) be a fuzzy poset and let $B \subseteq X$. An element $u \in X$ is said to be an upper bound for a subset B iff $A(b, u) > 0$ for all $b \in B$. An upper bound u_0 for B is the least upper bound of B iff $A(u_0, u) > 0$ for every upper bound u for B. An element $v \in X$ is said to be a lower bound for a subset B iff $A(v, b) > 0$ for all $b \in B$. A lower bound v_0 for B is the greatest lower bound of B iff $A(v, v_0) > 0$ for every lower bound v for B.

We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$.

Definition 2.5. ([**5**]) Let (X, A) be a fuzzy poset. (X, A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.

Definition 2.6. ([**5**]) Let (X, A) be a fuzzy lattice. (X, A) is distributive if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.

We define a Generalized Almost Distributive Fuzzy Lattice (GADFL) as follows:

Definition 2.7. ([**4**]) Let $(R, \vee, \wedge)$ be an algebra of type (2, 2) and (R, A) be a fuzzy poset. Then we call (R, A) is a Generalized almost distributive fuzzy lattice if it satisfies the following axioms:

(1) $A((a \wedge b) \wedge c, a \wedge (b \wedge c)) = A(a \wedge (b \wedge c), (a \wedge b) \wedge c) = 1$;
(2) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$;
(3) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$;
(4) $A(a \wedge (a \vee b), a) = A(a, a \wedge (a \vee b)) = 1$;
(5) $A((a \vee b) \wedge a, a) = A(a, (a \vee b) \wedge a) = 1$;
(6) $A((a \wedge b) \vee b, b) = A(b, (a \wedge b) \vee b) = 1$

for all $a, b, c \in R$.

Now, we give some elementary properties of a GADFL.

Theorem 2.1 ([**4**]). *Let (R, A) be a fuzzy poset . Then R is a GADL iff (R, A) is a GADFL.*

Theorem 2.2 ([**4**]). *Let (R, A) be a GADFL . Then $a = b \Leftrightarrow A(a, b) = A(b, a) = 1$.*

Definition 2.8. ([**4**]) Let (R, A) be a GADFL . Then for any $a, b \in R$, $a \leqslant b$ if and only if $A(a, b) > 0$.

In view of the above definition, we have the following theorem.

Theorem 2.3 ([**4**]). *If (R, A) is a GADFL then $a \wedge b = a$ if and only if $A(a, b) > 0$.*

Lemma 2.4 ([**4**]). *Let (R, A) be a GADFL and $a, b \in R$such that $a \neq b$. If $A(a, b) > 0$ then $A(b, a) = 0$.*

LEMMA 2.5 ([**4**]). *Let (R, A) be a GADFL. Then for each a and b in R*

(i) $A(a \wedge b, b) > 0$ *and* $A(b \wedge a, a) > 0$.
(ii) $A(a, a \vee b) > 0$ *and* $A(b, b \vee a) > 0$.

DEFINITION 2.9. ([**4**]) The fuzzy poset (R, A) is directed above if and only if the poset $(R, \leqslant)$ is directed above.

Now, we give the following conditions for a GADFL to become a distributive fuzzy lattice.

THEOREM 2.4 ([**4**]). *Let (R, A) be a GADFL. Then the following are equivalent.*

(1) (R, A) *is distributive fuzzy lattice;*
(2) *The fuzzy poset* (R, A) *is directed above;*
(3) $A(a \wedge (b \vee a), a) > 0$ *and* $A(a, a \wedge (b \vee a)) > 0$;
(4) $A(a \vee b, b \vee a) > 0$ *and* $A(b \vee a, a \vee b) > 0$;
(5) $A(a \wedge b, b \wedge a) > 0$ *and* $A(b \wedge a, a \wedge b) > 0$;
(6) *The relation* $\theta = \{(a, b) \in R \times R \mid A(b, a \wedge b) > 0\}$ *is antisymmetric.*

3. Amicable Sets of a GADFL

In this section we define compatible set, maximal set and amicable set in a Generalized Almost Distributive Fuzzy Lattices and we study their properties.

DEFINITION 3.1. Let (R, A) be a GADFL. For any $a, b \in R$, we say that a is compatible with b (written $a \sim_A b$) if $A(a \wedge b\ ,\ b \wedge a) > 0$ and $A(b \wedge a,\ a \wedge b) > 0$ or equivalently $A(a \vee b,\ b \vee a) > 0$ and $A(b \vee a,\ a \vee b) > 0$. A subset S_A of R is said to be compatible if $a \sim_A b$ for all $a, b \in S_A$. By a maximal set, we mean a maximal compatible set where maximal in the usual sense.

DEFINITION 3.2. Let M_A be a maximal set in a GADFL (R, A). Then an element $x \in R$ is said to be M_A- amicable if there exists $d \in M_A$ such that

$$A(x, d \wedge x) > 0.$$

Now we give the definition of an amicable set in a GADFL

DEFINITION 3.3. A maximal set M_A is said to be an amicable set in a GADFL (R, A) if every element of R is M_A- amicable.

EXAMPLE 3.1. Let $R = \{a, b, c\}$ and $M_A = \{a, b\}$. Define two binary operations $\vee$ and $\wedge$ on R as follows:

$\vee$	a	b	c
a	a	a	a
b	a	b	b
c	c	c	c

and

$\wedge$	a	b	c
a	a	b	c
b	b	b	c
c	b	b	c

Define a fuzzy relation $A : R \times R \longrightarrow [0, 1]$ as follows: $A(a, a) = A(b, b) = A(c, c) = 1$, $A(b, a) = A(b, c) = A(c, a) = A(c, b) = 0$, $A(a, b) = 0.2$ and $A(a, c) = 0.4$. Clearly, (R, A) is a GADFL.

Now, consider $M_A \subseteq R$. For $a, b \in M_A$, $A(a \wedge b,\, b \wedge a) > 0$ and $A(b \wedge a,\, a \wedge b) > 0$. Hence M_A is a compatible set and it is a maximal compatible set. Now, let $x \in R$, then there exists d in M_A such that $A(x, d \wedge x) > 0$. Hence every element of R is M_A- amicable. Therefore M_A is amicable set in a GADFL (R, A).

LEMMA 3.1. *Let (R, A) be a GADFL. For any $a, b \in R$,*

$$a \wedge b \sim_A a \;\Rightarrow\; A(a \wedge b, b \wedge a) = 1.$$

PROOF. Suppose (R, A) is a GADFL and $a, b \in R$. Assume $a \wedge b \sim_A a$. Then $A(a \wedge b, a) > 0$ and $A(a, a \wedge b) > 0$ by definition 3.1, hence by antisymmetry property of A, $a \wedge b = a$. Now,

$$A(a \wedge b,\, b \wedge a) = A(a \wedge b,\, b \wedge a \wedge b) = A(a \wedge b,\, a \wedge b \wedge b) = A(a \wedge b,\, a \wedge b) = 1.$$

Therefore $A(a \wedge b,\, b \wedge a) = 1$. □

Now we prove the following

LEMMA 3.2. *Let (R, A) be a GADFL and $a, b, c \in R$. Then*

$$a \sim_A b \;\Rightarrow\; A((a \vee c) \wedge b, (a \wedge b) \vee (c \wedge b)) = 1.$$

PROOF. Let (R, A) be a GADFL and $a, b, c \in R$. Let $a \sim_A b$. Then $A(a \wedge b,\, b \wedge a) > 0$ and $A(b \wedge a,\, a \wedge b) > 0$ then by antisymmetry propperty of A, $a \wedge b = b \wedge a$. Now,

$$\begin{aligned}
&\mathrm{A}((a \vee c) \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A((a \vee c) \wedge b \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A(b \wedge (a \vee c) \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A(\{(b \wedge a) \vee (b \wedge c)\} \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A(\{(a \wedge b) \vee (b \wedge c)\} \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A(\{[(a \wedge b) \vee b] \wedge [(a \wedge b) \vee c]\} \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A(b \wedge [(a \wedge b) \vee c] \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A([(a \wedge b) \vee c] \wedge b \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A(\{(a \wedge b) \vee c)\} \wedge b,\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A(\{(a \wedge b) \vee c\} \wedge \{(a \wedge b) \vee b\},\, (a \wedge b) \vee (c \wedge b))\\
&\quad= A((a \wedge b) \vee (c \wedge b),\, (a \wedge b) \vee (c \wedge b))\\
&\quad= 1.
\end{aligned}$$

Therefore $A((a \vee c) \wedge b,\, (a \wedge b) \vee (c \wedge b)) = 1$. □

LEMMA 3.3. *Let (R, A) be a GADFL. For any $a, b \in R$, $a \sim_A b$ if and only if $A(\, a \vee x,\, b \vee x) > 0$ and $A(b \vee x,\, a \vee x) > 0$ for some x in R.*

PROOF. Suppose (R, A) is a GADFL and $a, b \in R$,

($\Rightarrow$) Suppose $a \sim_A b$. Then $A(a \vee b,\, b \vee a) > 0$ and $A(b \vee a,\, a \vee b) > 0$. Hence by antisymmetry property of A, $a \vee b = b \vee a$. Put $x = a \vee b$. Then $a = a \wedge (a \vee b) = a \wedge x$. Therefore $a \leqslant x$ and hence $a \vee x = x$. Also, $b = b \wedge (b \vee a) = b \wedge x$. Therefore $b \leqslant x$ and hence $b \vee x = x$. Hence $a \vee x = b \vee x$. Therefore $A(a \vee x,\, b \vee x) > 0$ and $A(b \vee x,\, a \vee x) > 0$.

($\Leftarrow$) Suppose $A(a \vee x,\, b \vee x) > 0$ and $A(b \vee x,\, a \vee x) > 0$. Then $a \vee x = b \vee x$. Let $a \vee x = b \vee x = t$. Then $a \wedge t = a \wedge (a \vee x) = a$ and $b \wedge t = b \wedge (b \vee x) = b$.

Now, $a \wedge b = a \wedge b \wedge t = b \wedge a \wedge t = b \wedge a$. Therefore $A(a \wedge b,\, b \wedge a) > 0$ and $A(b \wedge a,\, a \wedge b) > 0$ and hence $a \sim_A b$. □

LEMMA 3.4. *Let M_A be a maximal set in a GADFL (R, A) and $x \in R$ be such that $x \sim_A a$ for all $a \in M_A$. Then $x \in M_A$.*

PROOF. Sinc $a \sim_A x$ for all $a \in M_A$ and M_A is a compatible set, we get $M_A \cup \{x\}$ is a compatible set and hence, by maximality of M_A, $x \in M_A$. □

Now we prove every maximal set M_A in a GADFL (R, A) is distributive fuzzy lattice.

THEOREM 3.1. *If M_A is a maximal set in a GADFL (R, A) then (M_A, A) is a distributive fuzzy lattice under the induced operations $\vee$ and $\wedge$.*

PROOF. Let M_A be a maximal set in a GADFL (R, A) and $a, b \in M_A$. Then for any $c \in M_A$, $A(\,(a \wedge b) \wedge c,\, c \wedge (a \wedge b)) = A(a \wedge b \wedge c,\, c \wedge (a \wedge b)) = A(a \wedge c \wedge b,\, c \wedge (a \wedge b)) = A(c \wedge a \wedge b,\, c \wedge (a \wedge b)) = A(c \wedge (a \wedge b),\, c \wedge (a \wedge b)) = 1$. Hence $A((a \wedge b) \wedge c,\, c \wedge (a \wedge b)) > 0$. Similarly, $A(c \wedge (a \wedge b),\, (a \wedge b) \wedge c) > 0$. Therefore $a \wedge b \sim_A c$ for all $c \in M_A$, and hence by Lemma 3.4, we get $a \wedge b \in M_A$.

Again,

$$\begin{aligned}
A(c \wedge (a \vee b), (a \vee b) \wedge c) &= A((c \wedge a) \vee (c \wedge b), (a \vee b) \wedge c)) \\
&= A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c)) \\
&= A(\{(a \wedge c) \vee b\} \wedge \{(a \wedge c) \vee c\}, (a \vee b) \wedge c)) \\
&= A(\{b \vee (a \wedge c)\} \wedge c, (a \vee b) \wedge c)) \\
&= A(\{(b \vee a) \wedge (b \vee c)\} \wedge c, (a \vee b) \wedge c)) \\
&= A(\{(a \vee b) \wedge (c \vee b)\} \wedge c, (a \vee b) \wedge c)) \\
&= A((a \vee b) \wedge (c \vee b) \wedge c, (a \vee b) \wedge c)) \\
&= A((a \vee b) \wedge c \wedge c, (a \vee b) \wedge c) \\
&= A((a \vee b) \wedge c), (a \vee b) \wedge c)) \\
&= 1.
\end{aligned}$$

Hence $A(c \wedge (a \vee b),\, (a \vee b) \wedge c) > 0$. Similarly, $A((a \vee b) \wedge c,\, c \wedge (a \vee b)) > 0$. Therefore $a \vee b \sim_A c$ for all $c \in M_A$ and hence by lemma 3.4, we get $a \vee b \in M_A$. Thus (M_A, A) is a sub GADFL of (R, A). Since M_A maximal, it is maximal compatible set and hence for any $a, b \in M_A$, $a \sim_A b$. Hence $A(a \wedge b,\, b \wedge a) > 0$ and $A(b \wedge a,\, a \wedge b) > 0$ for all $a, b \in M_A$. Therefore (M_A, A) is distributive fuzzy lattice. □

PROPOSITION 3.1. *Let M_A be a maximal set. Then M_A is an initial segment in the fuzzy poset (R, A). That is for any $x \in R$ and $a \in M_A$, $A(x, a) > 0$ implies $x \in M_A$.*

PROOF. Let $x \in R$, $a \in M_A$ and $A(x, a) > 0$. Then for any $b \in M_A$, $A(x \wedge b,\, b \wedge x) = A(x \wedge a \wedge b,\, b \wedge x)$ $[A(x, a) > 0 \Rightarrow x \wedge a = x]$ $= A(x \wedge b \wedge a,\, b \wedge x) = A(b \wedge x \wedge a,\, b \wedge x) = A(b \wedge x,\, b \wedge x) = 1$. Hence $A(x \wedge b,\, b \wedge x) > 0$. Similarly, $A(b \wedge x,\, x \wedge b) > 0$. Therefore $x \sim_A b$ and hence by lemma 3.4, $x \in M_A$. □

COROLLARY 3.1. *Let M_A be a maximal set in a GADFL (R, A) and $a \in M_A$. Then, for any $x \in R$, $x\wedge\, a \in M_A$.*

PROOF. Since $(x \wedge a) \vee a = a \Rightarrow x \wedge a \leqslant a \Rightarrow A(x \wedge a, a) > 0$. Hence by proposition 3.1, $x \wedge a \in M_A$. □

LEMMA 3.5. *Let M_A be a maximal set in a GADFL (R, A) and $x \in R$ be M_A- amicable. Then there exists an element $a \in M_A$ with the following properties:*

(1) $A(x, a \wedge x) > 0$
(2) $b \in R$, $A(x, b \wedge x) > 0 \Rightarrow A(a, b \wedge a) > 0$.

PROOF. (1) Since x is M_A- amicable, then there exists $c \in M_A$ such that $A(x, c\wedge x) > 0$. Since $A(c\wedge x, x) > 0$, by antisymmetry property of A, we have $x = c\wedge x$. Write $a = x \wedge c$. Then by Corollary 3.1 $a \in M_A$ and $A(x, a \wedge x) = A(x, x \wedge c \wedge x)$ $= A(x, c \wedge x \wedge x) = A(x, c \wedge x) = A(x, x) = 1$. Hence $A(x, a \wedge x) > 0$.

(2) If $b \in R$ and $A(x, b\wedge x) > 0$. Since $A(b\wedge x, x) > 0$. Then by antisymmetry property of A, we have $x = b \wedge x$. Now,

$$A(a, b \wedge a) = A(a, b \wedge x \wedge c) = A(a, x \wedge c) = A(a, a) = 1.$$

Hence $A(a, b \wedge a) > 0$. □

If $b \in M_A$ in (2) of the above lemma, since M_A is a maximal set $a \sim_A b$ and hence $A(a \wedge b, b \wedge a) > 0$ and $A(b \wedge a, a \wedge b) > 0$ which implies

$a \wedge b = b \wedge a$ ....($\star$).

By (2) of the above lemma $A(a, b\wedge a) > 0$ and we know that $A(b\wedge a, a) > 0$. Hence

$a = b \wedge a$ .... ($\star\star$)

From ($\star$) and ($\star\star$) $a = b \wedge a = a \wedge b$ and hence $A(a, b) > 0$.

Hence we have the following corollary.

COROLLARY 3.2. *Let M_A be a maximal set in a GADFL (R, A) and $x \in R$ is M_A- amicable, then there is a smallest element a of M_A with the property $A(x, a \wedge x) > 0$.*

We denote the element a of M_A in the above corollary by x^M. That is if M_A is a maximal set and $x \in R$ is M_A- amicable, then there is a smallest element x^M of M_A with the property $A(x, x^M \wedge x) > 0$.

LEMMA 3.6. *Let M_A be a maximal set in a GADFL (R, A), $x \in R$ be M_A- amicable and $a \in R$ such that $A(a, x\wedge a) > 0$. Then a is M_A- amicable and $A(a^M, x^M) > 0$.*

PROOF. Suppose $x \in R$ is M_A- amicable, then there exists $x^M \in M_A$ such that $A(x, x^M \wedge x) > 0$. To show that a is M_A- amicable.

$A(a, x^M \wedge a) = A(a, x^M \wedge x \wedge a) = A(a, x \wedge a) = A(a, a) = 1$
$\Rightarrow A(a, x^M \wedge a) > 0$ for $x^M \in M_A$.

Hence a is M_A- amicable, as a is M_A- amicable then there exists a smallest element a^M such that $A(a, a^M \wedge a) > 0$ and hence $A(a^M, x^M) > 0$ by Corollary 3.2 as a^M is smaller than x^M. □

COROLLARY 3.3. *Let M_A be a maximal set in a GADFL (R, A) and $x \in R$ be M_A- amicable. Then x^M is the largest element of M_A with the property $A(x^M, x \wedge x^M) > 0$.*

PROOF. Suppose $x \in R$ is M_A- amicable. There exists $c \in M_A$ such that $A(x, c \wedge x) > 0$. Now let $b = x \wedge c$, since $x \wedge c \leqslant c$ then $b = x \wedge c \in M_A$. Hence by Lemma 3.5(1), $A(x, b \wedge x) > 0$. On the other hand by Corollary 3.2, we have $A(x, x^M \wedge x) > 0$ for $x^M \in M_A$. Since both x^M, $b \in M_A \Rightarrow b \sim_A x^M$.

Claim: $A(b, x^M) > 0$.

$$A(b \wedge x^M, b) = A(x^M \wedge b, b) = A(x^M \wedge x \wedge c, b) = A(x \wedge c, b) = A(b, b) = 1.$$

Hence $A(b \wedge x^M, b) > 0$. Similarly, $A(b, b\wedge x^M) > 0$. Then by antisymmetry property of A we have

$$b \wedge x^M = b \Rightarrow b \leqslant x^M \Rightarrow A(b, x^M) > 0 \quad ... \quad (\star)$$

Hence x^M is the largest. To show $A(x^M, x \wedge x^M) > 0$. Since $A(x, b \wedge x) > 0$ and $A(x, x^M \wedge x) > 0$ for b, $x^M \in M$. Then by Corollary 3.2, x^M is the smallest.

$$\Rightarrow x^M \leqslant b \Rightarrow A(x^M, b) > 0 \quad ... \quad (\star\star)$$

From $(\star)$, $(\star\star)$ and antisymmetry property of A, we have $x^M = b = x \wedge c$. Now,

$$A(x^M, x \wedge x^M) = A(x^M, c \wedge x \wedge x^M) = A(x^M, x \wedge c \wedge x^M)$$
$$= A(x^M, x^M \wedge x^M) = A(x^M, x^M) = 1.$$

Hence $A(x^M, x \wedge x^M) > 0$. □

COROLLARY 3.4. *Let M_A be a maximal set in a GADFL (R, A) and $x \in R$ be M_A- amicable. Then for any $a \in R$, $A(x, a \wedge x) > 0$ and $A(a, x \wedge a) > 0$ if and only if a is M_A- amicable and $A(x^M, a^M) > 0$ and $A(a^M, x^M) > 0$.*

Now we prove the following.

THEOREM 3.2. *Let M_A be a maximal set in a GADFL (R, A) and $x \in R$ be M_A- amicable. Then x^M is the unique element of M_A such that*

$$A(x^M, x\wedge x^M) > 0 \text{ and } A(x, x^M \wedge x) > 0.$$

PROOF. Let $x \in R$ be M_A- amicable. Then there exists $x^M \in M_A$ such that $A(x, x^M \wedge x) > 0$ and $A(x^M, x\wedge x^M) > 0$ by Corollary 3.2 and Corollary 3.3. Let $b \in M_A$ such that $A(x, b \wedge x) > 0$ and $A(b, x \wedge b) > 0$.

Claim: $b = x^M$.

Now, $b \in M_A$, $A(x, b \wedge x) > 0 \Rightarrow A(x^M, b) > 0$ by Corollary 3.2 and $b \in M_A$, $A(b, x \wedge b) > 0$ implies $A(b, x^M) > 0$ by Corollary 3.3. Hence $A(x^M, b) > 0$ and $A(b, x^M) > 0$ implies $b = x^M$. □

DEFINITION 3.4. Let (R, A) be a GADFL. Then (R, A) is said to be associative if the operation $\vee$ in R is associative.

If M_A is a maximal set in a GADFL (R, A), then we denote the set of all M_A- amicable elements of R by $M_A(R)$.

THEOREM 3.3. *Let M_A be a maximal set in a GADFL (R, A). Then $(M_A(R), A)$ is a sub GADFL of (R, A). More over, if (R, A) is associative then for any $x, y \in R$, we have*

$$A((x \vee y)^M,\ x^M \vee y^M) > 0 \text{ and } A(x^M \vee y^M,\ (x \vee y)^M) > 0 \text{ and}$$
$$A((x \wedge y)^M,\ x^M \wedge y^M) > 0 \text{ and } A(x^M \wedge y^M,\ (x \wedge y)^M) > 0.$$

PROOF. Let $x, y \in M_A(R)$.

WTS: $x \vee y,\ x \wedge y \in M_A(R)$ Now, $x, y \in M_A(R) \Rightarrow x, y$ are M_A- amicable elements of R. Then there exist $x^M,\ y^M \in M$ such that $A(x, x^M \wedge x) > 0$ and $A(y, y^M \wedge y) > 0$.

Claim: (i) $A(x \vee y,\ (x^M \vee y^M) \wedge (x \vee y)) > 0$.

(ii) $A(x \wedge y,\ (x^M \wedge y^M) \wedge (x \wedge y)) > 0$.

(i). $A(x \vee y,\ (x^M \vee y^M) \wedge (x \vee y))$
$= A(x \vee y,\ \{(x^M \vee y^M) \wedge x\} \vee \{(x^M \vee y^M) \wedge y\})$
$= A(x \vee y,\ \{(x^M \vee y^M) \wedge x^M \wedge x\} \vee \{(x^M \vee y^M) \wedge y^M \wedge y\})$
$= A(x \vee y,\ \{x^M \wedge x\} \vee \{(x^M \vee y^M)\} \wedge y^M \wedge y)$
$= A(x \vee y,\ x \vee \{(x^M \vee y^M)\} \wedge y^M \wedge y)$
$= A(x \vee y,\ x \vee \{(y^M \vee x^M)\} \wedge y^M \wedge y)$
$= A(x \vee y,\ x \vee\ (y^M \wedge y))$
$= A(x \vee y,\ x \vee y)$
$= 1.$

Hence $A(x \vee y,\ (x^M \vee y^M) \wedge (x \vee y)) > 0$.

(ii)

$$\begin{aligned} A(x \wedge y, (x^M \wedge y^M) \wedge (x \wedge y)) &= A(x \wedge y, x^M \wedge y^M \wedge x \wedge y) \\ &= A(x \wedge y, x^M \wedge x \wedge y^M \wedge y) \\ &= A(x \wedge y, x \wedge y) \\ &= 1 \end{aligned}$$

Hence $A(x \wedge y,\ (x^M \wedge y^M) \wedge x \wedge y) > 0$. Therefore $(M_A(R), A)$ is a sub GADFL of (R, A).

Suppose (R, A) is associative and $x, y \in M_A(R)$. Then,
$A(x^M \vee y^M,\ (x \vee y) \wedge (x^M \vee y^M))$
$= A(x^M \vee y^M,\ \{(x \vee y) \wedge x^M\} \vee \{(x \vee y) \wedge y^M\})$
$= A(x^M \vee y^M,\ \{(x \vee y) \wedge x \wedge x^M\} \vee \{(x \vee y) \wedge y^M\})$
$= A(x^M \vee y^M,\ \{x \wedge x^M\} \vee \{(x \vee y) \wedge y^M\})$
$= A(x^M \vee y^M,\ x^M \vee \{(x \vee y) \wedge y^M\})$
$= A(x^M \vee y^M,\ \{x^M \vee (x \vee y)\} \wedge (x^M \vee y^M))$
$= A(x^M \vee y^M,\ \{(x^M \vee x) \vee y\} \wedge (x^M \vee y^M))$
$= A(x^M \vee y^M,\ (x^M \vee y) \wedge (x^M \vee y^M))$
$= A(x^M \vee y^M,\ x^M \vee (y \wedge y^M))$
$= A(x^M \vee y^M,\ x^M \vee y^M)$
$= 1.$

Hence $A(x^M \vee y^M,\ (x \vee y) \wedge (x^M \vee y^M)) > 0$. Also, $A(x \vee y,\ (x^M \vee y^M) \wedge (x \vee y)) > 0$. Since by Theorem 3.2, $(x \vee y)^M$ is the unique element of M_A such that $A((x \vee y)^M,$

$(x \vee y) \wedge (x \vee y)^M) > 0$ and $A(x \vee y,\ (x \vee y)^M \wedge (x \vee y)) > 0$. Hence we have $(x \vee y)^M = x^M \vee y^M$. Therefore $A((x \vee y)^M,\ x^M \vee y^M) > 0$ and $A(x^M \vee y^M,$ $(x \vee y)^M) > 0$. Similarly,

$$\begin{aligned} A(x^M \wedge y^M, (x \wedge y) \wedge (x^M \wedge y^M)) &= A(x^M \wedge y^M, x \wedge y \wedge x^M \wedge y^M) \\ &= A(x^M \wedge y^M, x \wedge x^M \wedge y \wedge y^M) \\ &= A(x^M \wedge y^M, x^M \wedge y^M) \\ &= 1 \end{aligned}$$

Hence $A(x^M \wedge y^M,\ (x \wedge y) \wedge (x^M \wedge y^M)) > 0$. Also, $A(x \wedge y,\ (x^M \wedge y^M) \wedge (x \wedge y)) > 0$. Since by Theorem 3.2, $(x \wedge y)^M$ is the unique element of M_A such that $A((x \wedge y)^M,$ $(x \wedge y) \wedge (x \wedge y)^M) > 0$ and $A(x \wedge y,\ (x \wedge y)^M \wedge (x \wedge y)) > 0$. Hence we have $(x \wedge y)^M$ $= x^M \wedge y^M$. Therefore

$$A((x \wedge y)^M,\ x^M \wedge y^M) > 0 \text{ and } A(x^M \wedge y^M,\ (x \wedge y)^M) > 0.$$

□

Proposition 3.2. *Let M_A be a maximal set, $x, y \in R$ be M_A- amicable and $x \sim_A y$. Then $A(x^M,\ y^M) > 0$ and $A(y^M,\ x^M) > 0$ if and only if $A(x, y) > 0$ and $A(y, x) > 0$.*

Proof. ($\Rightarrow$) Suppose $A(x^M,\ y^M) > 0$ and $A(y^M,\ x^M) > 0$.

Claim: $A(x, y) > 0$ and $A(y, x) > 0$. Now,

$$\begin{aligned} A(x, x \wedge y) &= A(x, y \wedge x) \\ &= A(x, y^M \wedge y \wedge x) \ldots\ldots \lceil A(y, y^M \wedge y) > 0, A(y^M \wedge y, y) > 0 \rfloor \\ &= A(x, y \wedge y^M \wedge x) \\ &= A(x, y^M \wedge x) \\ &= A(x, x^M \wedge x) \\ &= A(x, x) \\ &= 1. \end{aligned}$$

Hence $A(x,\ x \wedge y) > 0$. Similarly $A(x \wedge y,\ x) > 0$. Hence we have $x \wedge y = x$ and then $A(x, y) > 0$. Again,

$$\begin{aligned} A(y, y \wedge x) &= A(y, x \wedge y) \\ &= A(y, x^M \wedge x \wedge y) \\ &= A(y, x \wedge x^M \wedge y) \\ &= A(y, x^M \wedge y) \\ &= A(y, y^M \wedge y) \\ &= A(y, y) \\ &= 1 \end{aligned}$$

Hence $A(y,\ y \wedge x) > 0$. Similarly $A(y \wedge x,\ y) > 0$. Then $x \wedge y = y$ and hence $A(y, x) > 0$.

($\Leftarrow$) Suppose $A(x,y) > 0$ and $A(y,x) > 0$

$$\begin{aligned} A(x^M, x^M \wedge y^M) &= A(x^M, y^M \wedge x^M) \ldots\ldots \lceil x^M, y^M \in M_A \Rightarrow x^M \sim_A y^M \rfloor \\ &= A(x^M, y \wedge y^M \wedge x^M) \\ &= A(x^M, y^M \wedge y \wedge x^M) \\ &= A(x^M, y \wedge x^M) \\ &= A(x^M, x \wedge x^M) \\ &= A(x^M, y^M) \\ &= 1. \end{aligned}$$

Hence $A(x^M,\, x^M \wedge y^M) > 0$. Similarly $A(x^M \wedge y^M,\, x^M) > 0$. Then $x^M \wedge y^M = x^M$ and hence $A(x^M,\, y^M) > 0$. By similar approach as above we can obtain $A(x^M,\, y^M) > 0$. □

Now we prove that there exist an isomorphism between any two amicable sets in an associative GADFL.

THEOREM 3.4. *Let (R, A) be an associative GADFL. Let M_A be a maximal set and M'_A is an amicable set in a GADFL (R, A). Then the mapping $a \to a^{M'}$ is an isomorphism of the distributive fuzzy lattice (M_A, A) in to the distributive fuzzy lattice (M'_A, A). Further if M_A is also amicable then the above mapping is surjective.*

PROOF. Define $f : M_A \to M'_A$ by $f(a) = a^{M'}$ for all $a \in M_A$. To show f is homomorphism. Let $a, b \in M_A$, then

$$\begin{aligned} A(f(a \vee b), f(a) \vee f(b)) &= A((a \vee b)^{M'}, f(a) \vee f(b)) \\ &= A(a^{M'} \vee b^{M'}, f(a) \vee f(b)) \ldots\ldots \lceil \text{by Theorem 3.3} \rfloor \\ &= A(f(a) \vee f(b), f(a) \vee f(b)) \\ &= 1 \end{aligned}$$

Hence $A(f(a \vee b),\, f(a) \vee f(b)) > 0$. Similarly, $A(f(a) \vee f(b),\, f(a \vee b)) > 0$.

$$\begin{aligned} A(f(a \wedge b), f(a) \wedge f(b)) &= A((a \wedge b)^{M'}, f(a) \wedge f(b)) \\ &= A(a^{M'} \wedge b^{M'}, f(a) \wedge f(b)) \ldots\ldots \lceil \text{by Theorem 3.3} \rfloor \\ &= A(f(a) \wedge f(b), f(a) \wedge f(b)) \\ &= 1 \end{aligned}$$

Hence $A(f(a \wedge b),\, f(a) \wedge f(b)) > 0$. Similarly, $A(f(a) \wedge f(b),\, f(a) \wedge f(b)) > 0$. Therefore f is homomorphism. To show f is one-to-one,
Let $a, b \in M_A$ such that $A(f(a), f(b)) > 0$ and $A(f(b), f(a)) > 0$. Then $A(a^{M'},\, b^{M'}) > 0$ and $A(b^{M'},\, a^{M'}) > 0$. Hence $A(a, b) > 0$ and $A(b, a) > 0$ by proposition 3.2. Therefore f is one-to-one. Suppose M_A is also amicable. Let $x \in M'_A$. Then $x \in R$, hence there exists $x^M \in M_A$ such that $A(x, x^M \wedge x) > 0$ and $A(x^M,\, x \wedge x^M)$ > 0 by theorem 3.2. On the other hand $x^M \in M_A$ implies $x^M \in R$, then there exists $(x^M)^{M'} \in M'_A$ such that $A(x^M,\, (x^M)^{M'} \wedge x^M) > 0$ and $A((x^M)^{M'},\, x^M \wedge (x^M)^{M'})$

> 0. Therefore $A(x, (x^M)^{M'}) > 0$ and $A((x^M)^{M'}, x) > 0$ by Theorem 3.2. That is for $x^M \in R$, there exists a unique $x = (x^M)^{M'} \in M_A^{'}$ with the above property. Hence, $A(f(x^M), x) = A((x^M)^{M'}, x) > 0$ and $A(x,\ f(x^M)) = A(x,\ (x^M)^{M'}) > 0$. Therefore f is surjective. □

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