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ON A CLASS OF KANNAN TYPE SELFMAPPS OF ORBITALLY-BOUNDEDLY COMPACT METRIC SPACES

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Abstract. Let (X,d) be a metric space. Let T be a selfmap of X satisfying the following property:

$$d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx)+d(y,Ty)\}, \quad \text{for all } x,y \in X. \qquad (K(\tfrac{1}{2}))$$

$(K(\frac{1}{2}))$ is not strong enough to ensure fixed points for T even if T is continuous and X is compact. So a natural question rises: what kind of (mild) condition has to be added to garantee fixed point ?. In fact here we investigate the mildest conditions which allow T to be a Picard operator. To study this problem, we introduce the orbitally-boundedly compact metric spaces and prove in them that the following assertions are equivalent: (A) T is a Picard operator. (B) For all $x,u \in X$ satisfying $\lim_{k\to\infty} T^{n_k}x = u$, then we have $\lim_{k\to\infty} T(T^{n_k}x) = u$. (C) T is asymptotically regular on X. (D) The sequence $(T^n(x))_n$ is Cauchy, for all $x \in X$. As applications, we improve some recent results (concerning Kannan-contractive maps). We give general results in arbitrary complete metric spaces. Also, we provide examples to support our investigations.

1. Introduction Definitions and Preliminaries

In all this paper, we let (X,d) be a metric space and $T : X \to X$ a selfmapping. We set $\mathrm{Fix}\,(T) := \{x \in X : Tx = x\}$. ($\mathrm{Fix}\,(T)$ is the fixed point set of T).

For all $n \in \mathbb{N}$, we set $T^{n+1} := T \circ T^n$, $T^0 = I_X$ (the identity map of X) and $T^1 := T$.

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Definition 1.1. Let (X,d) be a metric space and T be a selfmapping on X. Then the orbit of T at $x \in X$ is defined as $O_T(x) = \{x, Tx, T^2x, T^3x, \dots\}$.

We recall the following notions from [**15**] due to I. A. Rus.

Definition 1.2. Let (X,d) be a metric space and $T : X \to X$ a selfmapping.

We say that T is weakly Picard operator (WPO) if the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of T.

If the operator T is WPO and $\mathrm{Fix}\,(T) = \{x\}$ (for some $x \in X$), then T is said to be a Picard operator (PO).

In 1968, R. Kannan [**11**] established the following fixed point result.

Theorem 1.1 ([**11**]). *Let (X,d) be a complete metric space and T be a selfmapping of X satisfying*

$$d(Tx,Ty) \leqslant \lambda\,\{d(x,Tx) + d(y,Ty)\}, \quad \forall x,y \in X, \qquad (K(\lambda))$$

where $\lambda \in [0, \frac{1}{2})$. Then T is a Picard operator.

We observe that in Theorem 1.1, the map T is not supposed to be continuous.

We recall that in 1975, Subrahmanyam [**16**] proved that a metric space (X,d) is complete if and only if every Kannan mapping has a unique fixed point in X.

We point out that the value $\frac{1}{2}$ can not be taken in Theorem 1.1. Indeed, consider the following example:

Consider the space $X := \{-1, 1\}$ equipped with the usual distance and define the selfmapping T by setting $T(-1) = 1$ and $T(1) = -1$. Then we have

$$|T(1) - T(-1)| = 2 \leqslant \frac{1}{2}\,(|1 - T(1)| + |-1 - T(-1)|) = 2.$$

This example shows that there exists a continuous on a compact metric space (X,d) satisfying the condition $(K(\frac{1}{2}))$, that is :

$$d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx) + d(y,Ty)\}, \qquad \text{for all } x,y \in X. \qquad (K(\tfrac{1}{2}))$$

but without having fixed point.

We denote $\mathcal{K}(\frac{1}{2})$ the set of selfmaps T of X satisfying the condition $(K(\frac{1}{2}))$.

The example above says that the condition $(K(\frac{1}{2}))$ is not strong enough to garantee the existence of fixed points.

The aim of this paper is to investigate the mildest supplementary conditions upon a selfmap $T \in \mathcal{K}(\frac{1}{2})$ to be a Picard operator.

Let S be a selfmapping of X satisfying the following condition:

$$d(Sx,Sy) < \frac{1}{2}\{d(x,Sx) + d(y,Sy)\}, \qquad \text{for all } x \neq y \in X. \qquad (K-S)$$

Such a map will be called a Kannan contractive selfmap of X.

We denote $\mathcal{K}_s$ the set of selfmaps S of X satisfying the condition $((K-S))$. It is clear that $\mathcal{K}_s \subset \mathcal{K}(\frac{1}{2})$ and that, in general, this inclusion is stricte.

In 1978, B. Fisher [**7**] and M. S. Khan [**14**] simultaneously proved two fixed point results related to Kannan contractive type mappings. They proved that a continuous mapping on a compact metric space (X,d) has a unique fixed point if T satisfies

$$d(Tx,Ty) < \frac{1}{2}\{d(x,Ty)+d(y,Tx)\}$$

or

$$d(Tx,Ty) < (d(x,Tx)d(y,Ty))^{\frac{1}{2}}$$

for all $x, y \in X$ with $x \neq y$ respectively.

In 1980, Chen and Yeh [**3**] extended the above two results in a more general way.

In 2017, J. Górnicki [**10**] has made some new contributions to selfmappings of a metric space (X,d) satisfying the condition:

$$d(Tx,Ty) < k\{d(x,Tx)+d(y,Ty)\}, \quad \text{for all } x \neq y \in X, \qquad (K-G)$$

where $k \in [0,1]$. Also, in [**10**] one can find a new proof of Theorem 1.1 and a new proof to the following (known) result.

THEOREM 1.2 ([**10**]). *Let (X,d) be a compact metric space and $T : X \to X$ be a continuous mapping satisfying*

$$d(Tx,Ty) < \frac{1}{2}\{d(x,Tx)+d(y,Ty)\}$$

for all $x, y \in X$ with $x \neq y$. Then T is a Picard operator.

It is natural to ask whether a similar result exists for selfmaps T in the class $\mathcal{K}(\frac{1}{2})$. To start replying, we recall the following result due to Kannan (see [**12**] and [**13**]).

THEOREM 1.3 ([**12, 13**]). *Let (X,d) be a compact metric space and let T be a continuous mapping of X into itself such that*

(i) $d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx)+d(y,Ty)\}$, *for all $x, y \in X$.*

(ii) *For every closed subset F of X which contains more than one element and is mapped into itself by T there exists $x \in F$ such that*

$$d(x,Tx) < \sup_{y\in F} d(y,Ty).$$

Then T has a unique fixed point u in X.

If in addition to the hypothesis (i) and (ii), we have

(iii) $d(Tx,u) < d(x,u)$ *if $x \neq u$, where u is the unique fixed point of T, then T is a Picard operator.*

In Theorem 1.3, we observe that a selfmap $T \in \mathcal{K}(\frac{1}{2})$ is far from being a Picard operator, even for continuous selfmapppings.

The aim of this paper is to investigate necesary and sufficient conditions on a selfmap $T \in \mathcal{K}(\frac{1}{2})$ in order that T becomes a Picard operator. Before stating our results we need some definitions.

We notice that other results refining and completing results of Theorem 1.3 are established in [**18**].

We start by presenting some weak versions of compactness for a metric space X and recalling some related known results for mappings in the class $\mathcal{K}_s$. As observed before, these results will also be valid for mappings in the clas $\mathcal{K}(\frac{1}{2})$.

DEFINITION 1.3. ([**6**]) A metric space (X, d) is said to be boundedly compact if every bounded sequence in X has a convergent subsequence.

It is clear from definition that every compact metric space is boundedly compact, but a boundedly compact metric space need not be compact, for example, the set of real numbers $\mathbb{R}$ with usual metric is not compact but boundedly compact.

Suppose that the metric space (X, d) is boundedly compact. Then it is easy to see that for any subset Y of X, we have: Y is compact, if and only if, Y is closed and bounded. In particular, every (non trivial) finite dimensional (real or complexe) normed linear space is boundedly compact (but certainly not compact).

In 2017, H. Garai, T. Senapati and L.K. Dey (see [**8**] and [**9**]) have proved the following result.

THEOREM 1.4 ([**8, 9**]). *Let (X, d) be a boundedly compact metric space and $T : X \to X$ be an orbitally continuous mapping such that*

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in X$ with $x \neq y$. Then T will be a Picard operator.

We recall the following definition of orbitally continuous maps wich was introduced by Ćirić in [**4**].

DEFINITION 1.4. ([**4**]) Let (X, d) be a metric space and T be a selfmapping on X. A selfmapping T of X is said to be T-orbitally continuous on X, if for all $x, u \in X$ satisfying $\lim_{k\to\infty} T^{n_k}x = u$, then we have $\lim_{k\to\infty} T(T^{n_k}x) = Tu$.

The following concept was introduced in [**8**] and [**9**]

DEFINITION 1.5. ([**9**]) Let (X, d) be a metric space and T be a selfmapping on X, then X is said to be T-orbitally compact if every sequence in $O_T(x)$ has a convergent subsequence for all $x \in X$.

Now we introduce the following concept.

DEFINITION 1.6. Let (X, d) be a metric space and T be a selfmapping on X, then X is said to be T-orbitally-boundedly compact if every bounded sequence in $O_T(x)$ has a convergent subsequence for all $x \in X$.

It is clear that if X is T-orbitally compact then it is T-orbitally-boundedly compact but the converse is not true (see example below).

EXAMPLE 1.1. Let $X = [0, \infty)$ be a metric space with respect to usual metric on $\mathbb{R}$. Define the mapping T on X by

$$Tx = 2x, \text{ for all } x \in X.$$

Then clearly X is T-orbitally-boundedly compact but, X is not T-orbitally compact.

Moreover, it is easy to see that every compact metric space is T-orbitally compact and that every T-orbitally compact metric space is complete, but the converses are not true. (See the example below).

Also note that boundedly compactness and T-orbitally-boundedly compactness are totally independent. Even T-orbitally compactness of X does not give the guaranty to be complete. To show this, we consider the following examples.

EXAMPLE 1.2. Let $X = [0, 1)$ endowed with the usual metric. Define $T : X \to X$ by $Tx = \frac{x}{3}$. Then it is easy to see that X is T-orbitally-boundedly compact but
(i) X is not compact.
(ii) X is not boundedly compact.
(iii) X is not complete.

In 1966, the concept of asymptotically regular mappings in metric spaces was introduced by Browder and Petryshyn (see [**2**]).

DEFINITION 1.7. ([**2**]) Let T be a selfmapping of a metric space (X, d). Then, T is said to be asymptotically regular at a point x in X, if $\lim_{n\to\infty} d(T^n x, T^n Tx) = 0$, where $T^n x$ denotes the n-th iterate of T at x.

T is said to be asymptotically regular on X if it is asymptotically regular at any point x in X.

We known that the condition $(K(\frac{1}{2}))$ is not strong enough to ensure fixed points for T even if T is continuous and X is compact. So a natural question rises: what kind of (mild) condition has to be added to garantee fixed point ? The aim of this paper is to study this problem.

This paper is organized as follows: The first section is devoted to some preliminaries, definitions and recalls of some related results.

In the second section, we expose the results of our study. we start by introducing the concept of an orbitally-boundedly compact metric space (X, d). Consider a selfmap T of X satisfying the condition $(K(\frac{1}{2}))$ on X. Then in the first main result (see Theorem 2.1), we prove that the following assertions are equivalent:
(A) T is a Picard operator.
(B) For all $x, u \in X$, if $\lim_{k\to\infty} T^{n_k} x = u$, then the sequence $(T(T^{n_k} x))_k$ converges to u as k tends to $+\infty$.
(C) T is asymptotically regular on X.
(D) T is a Cauchy operator (see definition 2.1).

As applications, (see 2.1 and 2.2) we improve some recent results (concerning Kannan-contractive maps) obtained in [**10**], [**8**] and [**9**].

Suppose that (X,d) be a complete metric space $T : X \to X$ satisfies $(K(\frac{1}{2}))$. We prove in Theorem 2.2, that (A), (C) and (D) are still equivalent. Suppose in addition that T satisfies $(K(\frac{1}{2}))$ and the following condition:

(E) For any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that $d(T^i x, T^j x) < \epsilon + \delta$ implies $d(T^{i+1}x, T^{j+1}x) < \epsilon$ for any $i, j \in \mathbb{N} \cup \{0\}$.

Then, we prove in Theorem 2.3 that T is a Picard operator. The condition (E) is a variant of the condition (D), given by T. Suzuki, in the paper [**17**].

In section 3, we provide examples to support our investigations.

In section 4, we provide some concluding remarks.

2. The results

We start by giving the following definition.

Definition 2.1. Let (X,d) be a metric space and $T : X \to X$ a selfmapping. We say that T is a Cauchy operator, if the sequence $(T^n(x))_{n\in\mathbb{N}}$ is Cauchy, for all $x \in X$.

Let (X,d) be a metric space and let T be a selfmap of X.

It is clear that if T is a Picard operator then T is a Cauchy operator.

It is easy to see that if T is Picard operator then T is orbitally continuous on X.

Now, we have all ingredients to state our first main result.

Theorem 2.1. *Let (X,d) be a metric space. Let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx) + d(y,Ty)\}, \quad \textit{for all } x,y \in X. \qquad (K(\tfrac{1}{2}))$$

We suppose that the space X is T-orbitally-boundedly compact.

Then the following assertions are equivalent:

(A) *T is a Picard operator.*

(B) *For all $x,u \in X$, if $\lim_{k\to\infty} T^{n_k}x = u$, then we have $\lim_{k\to\infty} T(T^{n_k}x) = u$.*

(C) *T is asymptotically regular on X.*

(D) *T is a Cauchy operator.*

Proof. (A) $\Longrightarrow$ (B). Since T is a Picard operator, there exists a unique fixed point $z \in X$ such that $z = \lim_{n\to+\infty} T^n x$ for all $x \in X$.

Let $x,u \in X$ and let $(T^{n_k}x)_k$ be a subsequence of $(T^n x)_n$ such that $u = \lim_{k\to+\infty} T^{n_k}x$. Then we infer that $u = z$. Moreover, the sequence $(T(T^{n_k}x))_k = (T^{n_k+1}x)_k$ is also a subsequence of $(T^n x)_n$, therefore we have

$$\lim_{k\to+\infty} T(T^{n_k}x) = u.$$

Thus we have proved that T satisfies the condition (B).

(B) $\Longrightarrow$ (C). Let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. We set $\tau_n = d(x_n, x_{n+1})$ for each

$n \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned} \tau_n &= d(T^n x_0, T^{n+1} x_0) \\ &= d(T(T^{n-1} x_0), T(T^n x_0) \\ &\leqslant \frac{1}{2}\{d(T^{n-1} x_0, T^n x_0) + d(T^n x_0, T^{n+1} x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_n) \\ &\Longrightarrow \tau_n \leqslant \tau_{n-1}. \end{aligned}$$

This shows that (τ_n) is a non-increasing sequence of nonegative real numbers, so it must be a convergent sequence. Let us denote $\tau := \lim_{n\to+\infty} \tau_n$. We must show that $\tau = 0$.

From the inequalities above, we know that for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &= d(T(T^{n-1} x_0), T(T^{m-1} x_0) \\ &\leqslant \frac{1}{2}\{d(T^{n-1} x_0, T^n x_0) + d(T^{m-1} x_0, T^m x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_{m-1}) \\ &\Longrightarrow d(x_n, x_m) \leqslant \tau_0. \end{aligned}$$

Therefore, (x_n) is a bounded sequence in X. Since X is is T-orbitally-boundedly compact, the sequence (x_n) has a convergent subsequence, say (x_{n_k}) which converges to some $z \in X$. Since T is satisfying the condition (B), we infer that $\lim_{k\to+\infty} T(T^{n_k} x) = z$. Therefore, by the continuity of the distance function, we obtain

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = d(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} x_{n_k+1}) = d(z, z) = 0.$$

Hence, the subsequence $(\tau_{n_k})_k$ converges to zero. Since the whole sequence (τ_n) converges to τ, we must have $\tau = 0$. So we have proved that T is asymptotically regular on X.

(C) $\Longrightarrow$ (D). Let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. As before, we denote $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption (C), we know that $\lim_{n\to\infty} \tau_n = 0$.

Now, we show that the sequence $(x_n = T^n(x_0))$ is a Cauchy sequence. To this end, we observe that for all integers $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &= d(T(T^{n-1} x_0), T(T^{m-1} x_0) \\ &\leqslant \frac{1}{2}\{d(T^{n-1} x_0, T^n x_0) + d(T^{m-1} x_0, T^m x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_{m-1}) \\ &\Longrightarrow \lim_{n,\, m\to\infty} d(x_n, x_m) = 0. \end{aligned}$$

This says that (x_n) is a Cauchy sequence. hence, we have proved that (C) implies (D).

(D) $\Longrightarrow$ (A). Suppose that T is a Cauchy operator and let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. By assumption, this sequence is a cauchy sequence. Therefore, (x_n) is a bounded sequence in X. Since X is is T-orbitally-boundedly compact, the sequence (x_n) has a convergent subsequence, say (x_{n_k}) which converges to some $z \in X$. This implies that the whole sequence (x_n) converges to the point $z = z_{x_0}$ which may depend on x_0. Also, we have

$$\begin{aligned} d(z,Tz) &\leqslant d(z,T^{n+1}x_0) + d(T^{n+1}x_0,Tz) \\ &< d(z,T^{n+1}x_0) + \frac{1}{2}\{d(T^n x_0,T^{n+1}x_0) + d(z,Tz)\} \\ \Rightarrow \frac{1}{2}d(z,Tz) &< d(z,T^{n+1}x_0) + \frac{1}{2}d(T^n x_0,T^{n+1}x_0) \to 0 \text{ as n} \to \infty. \end{aligned}$$

This implies that $z = Tz$, i.e., z is a fixed point of T.

Next, we prove the uniqueness of z. We argue by contradiction, let z^* be another (different) fixed point of T, then

$$\begin{aligned} d(z,z^*) &= d(Tz,Tz^*) \\ &< \frac{1}{2}\{d(z,Tz) + d(z^*,Tz^*)\} \\ \Longrightarrow d(z,z^*) &< 0, \end{aligned}$$

which leads to a contradiction. Hence, our assumption was wrong. Therefore, z must be the unique fixed point T.

Let $y_0 \in X$ be arbitrary but fixed and consider the iterated sequence (y_n) where $y_n = T^n y_0$ for each $n \in \mathbb{N}$. We denote $\sigma_n = d(y_n, y_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption we know that $\lim_{n\to\infty} \tau_n = 0 = \lim_{n\to\infty} \sigma_n$. Then for all integer $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n,y_n) &= d(T^n x_0, T^n y_0) \\ &= d(T(T^{n-1}x_0), T(T^{n-1}y_0) \\ &\leqslant \frac{1}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^{n-1}y_0, T^n y_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \sigma_{n-1}) \\ &\Longrightarrow d(z_{x_0}, z_{y_0}) = \lim_{n\to\infty} d(x_n, y_m) = 0. \end{aligned}$$

This says that the fixed point does not depend on the initial value x_0 for any arbitrary point, so for every $x \in X$, the iterated sequence $(T^n x)$ converges to the unique fixed point of T, i.e., T is a Picard operator. This ends the proof. □

Corollary 2.1. *Let (X,d) be a metric space. Let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) < \frac{1}{2}\{d(x,Tx) + d(y,Ty)\}, \quad \textit{for all } x,y \in X \textit{ with } x \neq y.$$

We suppose that the space X is T-orbitally-boundedly compact.

Then the following assertions are equivalent:

(A) *T is a Picard operator.*

(B) *For all $x, u \in X$, if $\lim_{k\to\infty} T^{n_k}x = u$, then $\lim_{k\to\infty} T(T^{n_k}x) = u$.*

(C) *T is asymptotically regular on X.*

(D) *T is a Cauchy operator.*

We point out that

(a) If X is boundedly compact, then X is T-orbitally-boundedly compact, for every selfmap T of X.

(b) If X is T-orbitally compact then X is T-orbitally-boundedly compact, for every selfmap T of X.

(c) We know from Example 1.2, that the converses of (a) and (b) are not true. As a consequence we see that our Corollary unifies and generalizes both Theorem 2.1 and Theorem 2.2 of [**9**] and Theorem 1.2. Indeed, we have the following general result.

Corollary 2.2. *Let (X, d) be a metric space. Let $T : X \to X$ be satisfying the following property:*

(C-1) *For all $x, u \in X$, if $\lim_{k\to\infty} T^{n_k}x = u$, then $\lim_{k\to\infty} T(T^{n_k}x) = u$.*

(C-2) $d(Tx, Ty) \leqslant \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$, *for all $x, y \in X$.*

(C-3) *The space X satisfies one of the following assumptions:*

(H 1) *X is compact, or*

(H 2) *X is boundedly compact, or*

(H 3) *X is T-orbitally compact, or*

(H 4) *X is T-orbitally-boundedly compact.*

Then T is a Picard operator.

Also, from the lines of proof of the theorem 2.1, one can establish the following general result for complete metric spaces.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leqslant \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \text{for all } x, y \in X. \qquad (K(\tfrac{1}{2}))$$

Then the following assertions are equivalent:

(A) *T is a Picard operator on X.*

(C) *T is asymptotically regular on X.*

(D) *T is a Cauchy operator on X.*

Theorem 2.2 is our second main result. It will infer that the condition (C) (resp. (D)) is the minimal condition to be added for a selfmapping T satisfying $(K(\frac{1}{2}))$ on a complete metric space (X, d) to become a Picard operator on this space.

We observe that the same conclusions of Theorem 2.2 are still valid if one replaces the condition $(K(\frac{1}{2}))$ by the condition $((K - S))$.

In connection with Theorem 2.2, we present our third main result.

THEOREM 2.3. *Let (X,d) be a complete metric space and $T : X \to X$ satisfying the following conditions:*

$(K(\frac{1}{2}))$: $d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx)+d(y,Ty)\}$. *for all* $x, y \in X$,

(E): *For any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that* $d(T^i x, T^j x) < \epsilon + \delta$ *implies* $d(T^{i+1}x, T^{j+1}x) < \epsilon$ *for any* $i, j \in \mathbb{N} \cup \{0\}$.

Then T is a Picard operator.

PROOF. According to Theorem 2.2, it is sufficient to show that T is asymptotically regular on X.

So, let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. As before, we denote $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. We know, by using the condition $(K(\frac{1}{2}))$, that the sequence (τ_n) is nonincreasing. Since (τ_n) is bounded below, it converges to a nonnegative number (say) τ. We show that $\tau = 0$. To get a contradiction, we suppose that $\tau > 0$. Then there exists an integer n_δ such that $d(x_n, x_{n+1}) = \tau_n < \tau + \delta$ for all integer $n \geqslant n_\delta$.

Let n be any integer satisfying $n \geqslant n_\delta$. By the the condition (E), we infer that $d(Tx_n, Tx_{n+1}) = \tau_{n+1} < \tau$. This implies that $0 < \tau \leqslant \tau_{n+1} < \tau$, which is absurd. Thus we have showed that $\lim_{n\to\infty} \tau_n = 0$. This ends the proof. □

We point out that Theorem 2.3 is the analog of Theorem 2.7 of [**9**].

We point out that the condition (E) is a variant of the condition named (D) in the paper [**17**] of T. Suzuki. It seems that our condition is stronger than the condition (D) introduced in [**17**] and considered in Theorem 2.7 of the paper [**9**].

We do not now whether the condition (E) is the mildest one in Theorem 2.3. We let this question as an open problem.

3. Examples

To support our first main result, we provide the following example.

EXAMPLE 3.1. Let $X = [1,2] \cup (-\infty, 0]$ and define $T : X \to X$ by

$$T(x) = \begin{cases} -2 & x = 2; \\ 0 & x \neq 2. \end{cases}$$

Now, for $x \neq 2$, we have

$$d(Tx, T2) = |Tx - T2| = |0 + 2| = 2,$$

whereas

$$\frac{1}{2}\{d(x,Tx) + d(2,T2)\} = \frac{1}{2}\{|x| + |2+2|\} = 2 + \frac{|x|}{2} \geqslant 2.$$

So,

$$d(Tx, T2) \leqslant \frac{1}{2}\{d(x,Tx) + d(2,T2)\}.$$

Again, for $x, y \in X$ with $x, y \neq 2$, we have $d(Tx,Ty) = 0$ but

$$\frac{1}{2}\{d(x,Tx) + d(y,Ty)\} = \frac{1}{2}\{|x| + |y|\} \geqslant 0.$$

Therefore, we have $d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx) + d(y,Ty)\}$.

Thus, T satisfies the condition $(K(\frac{1}{2}))$ on the metric space X which is boundedly compact, without being compact. The selfmap T is asymptotically regular on X. Indeed, for each $x_0 \in X$, we have $T^n(x_0) = 0$ for every integer $n \geqslant 2$. Hence, the condition (C) of Theorem 2.1 is satisfied, therefore T is a Picard operator with $\mathrm{Fix}\,(T) = \{0\}$.

To support our second main result, we provide the following example.

EXAMPLE 3.2. Let us choose $X = \mathbb{N}$ and we define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} \frac{1}{2} + |\frac{1}{x} - \frac{1}{y}|, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Clearly d is a metric on X. Note that every Cauchy sequence in X is eventually constant and hence (X, d) is a complete metric space.

We define a function $T : X \to X$ by setting

$$Tx = 3x,$$

for all $x \in X$. It is easy to verify that T is continuous and a fixed point free map.

Now, it remains to show that T satisfies the Kannan type contractive condition. In order to do this, we choose $x, y \in X$ with $x \leqslant y$, then,

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{2} + |\frac{1}{3x} - \frac{1}{3y}| \\ &= \frac{1}{2} + \frac{1}{3x} - \frac{1}{3y} \leqslant \frac{1}{2} + \frac{1}{3x}; \end{aligned}$$

whereas,

$$\begin{aligned} \frac{1}{2}\{d(x,Tx) + d(y,Ty)\} &= \frac{1}{2}\{\frac{1}{2} + |\frac{1}{x} - \frac{1}{3x}| + \frac{1}{2} + |\frac{1}{y} - \frac{1}{3y}|\} \\ &= \frac{1}{2} + \frac{1}{3x} + \frac{1}{3y} \geqslant \frac{1}{2} + \frac{1}{3x}. \end{aligned}$$

Therefore,

$$d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx) + d(y,Ty)\},$$

for all x, y in X with $x \leqslant y$.

Similarly, one can prove it for the case $x, y \in X$ with $x > y$.

Now from the above analysis, we can say that

$$d(Tx,Ty) \leqslant \frac{1}{2}\{d(x,Tx) + d(y,Ty)\},$$

for all x, y in X with $x \neq y$. Thus, T is a Kannan type contractive map but it is fixed point free. Because, here the condition (C) is not satisfied. Indeed, For $x_0 := 1$, For every positive integer n, we have

$$d(x_n, x_{n+1}) = \frac{1}{2} + \frac{2}{3^{n+1}},$$

which never tends to zero, when n tend to infinity. Thus the condition (C) is necessary and sufficient for T to be a Picard operator.

4. Concluding remarks

Let (X, d) be a metric space. Let T be a selfmap of X. We consider the following contractive conditions:

(Ed): $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$ with $x \neq y$.
(K-S): $d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$, for all $x, y \in X$ with $x \neq y$.
$(K(\frac{1}{2}))$: $d(Tx, Ty) \leqslant \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$, for all $x, y \in X$.

It is well known that each one of all these three conditions is not strong enough to ensure fixed points for T.

We point out that T. Suzuki investigated in [**17**] the weakest contractive conditions for Edelstein's mappings (i.e. mappings satisfying the condition (Ed) in order to have a fixed point in complete metric spaces. Some extensions of the results of [**17**] are established by H. Garai, T. Senapati, L.K. Dey in the paper [**8**] for boundely compact metric spaces and orbitally compact metric spaces.

We observe that the condition $(K(\frac{1}{2}))$ is weaker than the condition (K-S). We recall that sefmappings of a compact metric spaces satisfying $(K(\frac{1}{2}))$ or (K-S) need not to have fixed points. So a natural question rises: what kind of (mild) condition has to be added to garantee fixed point ?. In fact here we investigate the mildest conditions which allow T to be a Picard operator. To study this problem, we introduce the concept of orbitally-boundedly compact metric space. Using this concept, we prove the following general result.

Suppose that T is a selfmap of X satisfying the condition $(K(\frac{1}{2}))$ and suppose that X is T-orbitally-boundedly compact. Then we prove in Theorem 2.1 that the following assertions are equivalent:

(A) T is a Picard operator.
(B) For all $x, u \in X$, if $\lim_{k\to\infty} T^{n_k}x = u$, then $\lim_{k\to\infty} T(T^{n_k}x) = u$.
(C) T is asymptotically regular on X.
(D) T is a Cauchy operator (see definition 2.1).

As applications, our results bring improvements to some recent results (concerning the Kannan-contractive maps). In particular, in Theorem 2.3, we introduce the condition (E) and use it to provide the analog of Theorem 2.7 of [**9**].

In general, (see Theorem 2.3), when X is any arbitrary complete metric space we prove that (A), (C) and (D) are still equivalent.

We end the paper by giving examples to support our results.

Open problem. We raise the following question: The condition (E), introduced here, is it the mildest one in Theorem 2.3 ?

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