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ON A (k, μ)-PARACONTACT MANIFOLDS SATISFYING CERTAIN CONDITIONS ON QUASI-CONFORMAL CURVATURE TENSOR

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Abstract. In the present paper, we have studied the curvature tensors of (k,μ)-Paracontact manifold satisfying the conditions $\widetilde{C}(X,Y)\cdot P=0$, $\widetilde{C}(X,Y)\cdot R=0$, $\widetilde{C}(X,Y)\cdot \widetilde{Z}=0$ and $\widetilde{C}(X,Y)\cdot S=0$. According these cases, (k,μ)-Paracontact manifolds have been characterized.

1. Introduction

After being introduced by Kaneyuki and Williams [**7**], a systematic study of paracontact metric manifolds and their subclasses was carried out by Zamkovoy [**13**]. Subsequently, many geometers have studied paracontact metric manifolds and obtained various important properties of these manifolds. Paracontact metric manifols have been studied from different points of view. The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. In [**5**], the author introduced the class of paracontact metric manifolds for which the characteristic vektor field ξ belongs to the $(k,\mu)-$nullity condition for some real constant k and μ. Such manifolds are known as $(k,\mu)-$paracontact metric manifolds. The class of $(k,\mu)-$paracontact metric manifolds contains para-Sasakian manifolds.

C. Özgür and U.C. De researced some certain curvature conditions satisfying quasi-conformal curvature tensor in Kenmotsu manifolds [**8**]. K. Yano and S. Sawaki introduced the notion of quasi-conformal curvature tensor which is generalization of conformal curvature tensor [**11**]. It plays an important role in differential

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geometry as well as in theory of relativity. M. Atçeken studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor [**2**]. U.C. De, J.B. Jun and A.K. Gazi searched Sasakian manifolds with quasi-conformal curvature tensor [**6**]. A. Hosseinzadeh and A. Taleshian produced conformal and quasi-conformal curvature tensors of an $N(k)-$quasi Einstein manifold [**9**].

Motivated by the studies of the above authors, in this paper we classify (k, μ)-paracontact manifolds, which satify the curvature conditions $\widetilde{C}(X,Y)\cdot P = 0$, $\widetilde{C}(X,Y)\cdot R = 0$, $\widetilde{C}(X,Y)\cdot \widetilde{Z} = 0$ and $\widetilde{C}(X,Y)\cdot S = 0$, where $\widetilde{C}$, P, R, $\widetilde{Z}$ and S denote the quasi conformal, projective, Riemannian, concircular and Ricci tensors of manifold, respectively.

2. Preliminaries

A contact manifold is a $C^{\infty} - (2n+1)$ dimensional manifold M^{2n+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. Given such a form η, it is well known that there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for every vector field X on M^{2n+1}. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type $(1,1)$ such that

$$\phi^2 X = X - \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X) \tag{2.2}$$

for all vector fields X,Y on M. Then the structure (ϕ, ξ, η, g) on M is called a paracontact metric structure and the manifold equipped with such a structure is called a almost paracontact metric manifold [**13**].

We now define a $(1,1)$ tensor field h by $h = \frac{1}{2}L_{\xi}\phi$, where L denotes the Lie derivative. Then h is symmetric and satisfies the conditions

$$h\phi = -\phi h, \qquad h\xi = 0, \qquad Tr.h = Tr.\phi h = 0. \tag{2.3}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\widetilde{\nabla}_X \xi = -\phi X + \phi h X \tag{2.4}$$

for any $X \in \chi(M)$[**13**]. For a para-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$, if ξ is a killing vector field or equivalently, $h = 0$, then it is called a K-paracontact manifold.

A para-contact metric structure (ϕ, ξ, η, g) is normal, that is, satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

which is equivalent to

$$(\widetilde{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X$$

for all $X, Y \in \chi(M)$ [**13**]. If an almost paracontact metric manifold is normal, then it called paracontact metric manifold. Any para-Sasakian manifold is K-paracontact, and the converse is true when $n = 1$, that is, for 3-dimensional spaces.

Any para-Sasakian manifold satisfies

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.5}$$

for all $X, Y \in \chi(M)$, but this is not a sufficient condition for a para-contact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true [**4**].

A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0, 2)$ is of the from

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions on M. If $b = 0$, then the manifold is also called Einstein [**10**].

A paracontact metric manifold is said to be a $(k,\mu)-$paracontact manifold if the curvature tensor R satisfies

$$\widetilde{R}(X,Y)\xi = k\,[\eta(Y)X - \eta(X)Y] + \mu\,[\eta(Y)hX - \eta(X)hY] \tag{2.6}$$

for all $X, Y \in \chi(M)$, where k and μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ [**14**].

In particular, if $\mu = 0$, then the paracontact metric $(k,\mu)-$manifold is called paracontact metric $N(k)$-mani- fold. Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{2.7}$$

for all $X, Y \in \chi(M)$. Though the geometric behavior of paracontact metric (k,μ)-spaces is different according as $k < -1$, or $k > -1$, but there are also some common results for $k < -1$ and $k > -1$.

Lemma 2.1 ([**5**]). *There does not exist any paracontact $(k,\mu)-$manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$*

In a paracontact metric $(k,\mu)-$manifold $(M^{2n+1}\phi,\xi,\eta,g)$, $n > 1$, the following relation hold:

$$h^2 = (k+1)\phi^2, \text{ for } k \neq -1, \tag{2.8}$$

$$(\widetilde{\nabla}_X\phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \tag{2.9}$$

$$\begin{aligned} S(X,Y) &= [2(1-n)+n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$

$$\begin{aligned} QY &= [2(1-n)+n\mu]Y + [2(n-1)+\mu]hY \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(Y)\xi, \end{aligned} \tag{2.12}$$

$$Q\xi = 2nk\xi, \tag{2.13}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi \tag{2.14}$$

for any vector fields X, Y on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1}, g), respectively [**5**].

The concept of quasi-conformal curvature tensor was defined by K. Yano and S. Sawaki [**11**]. Quasi-conformal curvature tensor of a $(2n+1)$-dimensional Riemanian manifold is defined as

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{\tau}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{2.15}$$

where a and b are arbitrary scalars, and r is the scalar curvature of the manifold, Q, S and r denote the Ricci operator, Ricci tensor and scalar curvature of manifold, respectively [**5**].

Let (M, g) be an $(2n+1)$-dimensional Riemanian manifold. Then the concircular curvature tensor $\widetilde{Z}$ and the projective tensor P are

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.16}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{2.17}$$

for all $X, Y, Z \in \chi(M)$, where r is the scalar curvature of M and Q is the Ricci operator given by $g(QX, Y) = S(X, Y)$ [**10**].

3. An $(k, \mu)-$ Paracontact Metric Manifold Satisfying Certain Conditions On The Quasi-Conformal Curvature Tensor

In this section, we will give the main results for this paper.

Let M be a $(2n+1)-$dimensional $(k, \mu)-$paracontact metric manifold and we denote the Riemannian curvature tensor of R, from (2.6), we have for later

$$R(\xi, Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.1}$$

In the same way, choosing $X = \xi$ in (2.15) and (2.6), we have

$$\begin{aligned}\widetilde{C}(\xi,Y)Z &= (ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)(g(Y,Z)\xi - \eta(Z)Y) \\ &\quad + a\mu(g(hY,Z)\xi - \eta(Z)hY) + b(S(Y,Z)\xi - \eta(Z)QY)\end{aligned} \tag{3.2}$$

In (3.2), choosing $Z = \xi$ and using (2.11), we obtain

$$\begin{aligned}\widetilde{C}(\xi,Y)\xi &= (ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)(\eta(Y)\xi - Y) \\ &\quad - a\mu hY + b(2nk\eta(Y)\xi - QY).\end{aligned} \tag{3.3}$$

In same way from (2.6) and (2.16), we get

$$\widetilde{Z}(X,Y)\xi = (k - \frac{r}{2n(2n+1)})(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{3.4}$$

from which

$$\widetilde{Z}(\xi,Y)\xi = (k - \frac{r}{2n(2n+1)})(\eta(Y)\xi - Y) - \mu hY. \tag{3.5}$$

From (2.6) and (2.17), we have

$$P(X,Y)\xi = \mu(\eta(Y)hX - \eta(X)hY). \tag{3.6}$$

Choosing $X = \xi$ in (3.6), we obtain

$$P(\xi,Y)\xi = -\mu hY. \tag{3.7}$$

THEOREM 3.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $\widetilde{C}(X,Y)\cdot\widetilde{Z} = 0$ if and only if M is an η-Einstein manifold.*

PROOF. Suppose that $\widetilde{C}(X,Y)\cdot\widetilde{Z} = 0$. This implies that

$$\begin{aligned}(\widetilde{C}(X,Y)\widetilde{Z})(U,W)Z &= \widetilde{C}(X,Y)\widetilde{Z}(U,W)Z - \widetilde{Z}(\widetilde{C}(X,Y)U,W)Z\\ &\quad -\widetilde{Z}(U,\widetilde{C}(X,Y)W)Z - \widetilde{Z}(U,W)\widetilde{C}(X,Y)Z = 0,\end{aligned} \tag{3.8}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X = Z = \xi$ in (3.8), making use of (3.2) and (3.4) we have for $A = [ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)]$ and $B = k - \frac{r}{2n(2n+1)}$,

$$\begin{aligned}(\widetilde{C}(\xi,Y)\widetilde{Z})(U,W)\xi &= \widetilde{C}(\xi,Y)(B(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW)\\ &\quad -\widetilde{Z}(A(g(Y,U)\xi - \eta(U)Y) + a\mu(g(hY,U)\xi - \eta(U)hY)\\ &\quad +b(S(Y,U)\xi - \eta(U)QY,W)\xi - \widetilde{Z}(U,A(g(Y,W)\xi\\ &\quad -\eta(W)Y) + a\mu(g(hY,W)\xi - \eta(W)hY) + b(S(Y,W)\xi\\ &\quad -\eta(W)QY)\xi - \widetilde{Z}(U,W)(A(\eta(Y)\xi - Y) - a\mu hY\\ &\quad +b(2nk\eta(Y)\xi - QY)) = 0.\end{aligned} \tag{3.9}$$

Taking into account (3.2), (3.4), (3.5) and inner product both sides of (3.9) by $Z \in \chi(M)$, we obtain

$$\begin{aligned}&Ag(\widetilde{Z}(U,W)Y,Z) + a\mu g(\widetilde{Z}(U,W)hY,Z) + bg(\widetilde{Z}(U,W)QY,Z)\\ &+A\mu(\eta(W)\eta(Z)g(Y,hU) - \eta(U)\eta(Z)g(Y,hW))\\ &+a\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U) - \eta(U)\eta(Z)g(Y,W))\\ &+b\mu(\eta(W)\eta(Z)S(Y,hU) - \eta(U)\eta(Z)S(Y,hW))\\ &+AB(g(Y,U)g(W,Z) - g(Y,W)g(U,Z))\\ &+A\mu(g(Y,U)g(hW,Z) - g(Y,W)g(hU,Z))\\ &+a\mu B(g(hY,U)g(W,Z) - g(hY,W)g(U,Z))\\ &+a\mu^2(g(hY,U)g(hW,Z) - g(hY,W)g(hU,Z))\\ &+Bb(S(Y,U)g(W,Z) - S(Y,W)g(U,Z))\\ &+\mu b(S(Y,U)g(hW,Z) - S(Y,W)g(hU,Z)) = 0.\end{aligned} \tag{3.10}$$

Using (2.1), (2.12) and (2.16) and choosing $W = Y = e_i, \xi$ in (3.10), $1 \leqslant i \leqslant n$, for orthonormal basis of $\chi(M)$, we arrive

$$\begin{aligned}
&(A + b[2(1-n) + n\mu] + kb)S(U,Z) + (a\mu + b[2(n-1) + \mu] \\
&-\mu b)S(U,hZ) + (bk[2(n-1) + n(2k-\mu)] - 2nkA \\
&-kbr + a\mu^2(1+k))g(U,Z) + (Ak - AB + b\mu[2(n-1) \\
&+n(2k-\mu)] - \mu br + a\mu B - 2nA\mu)g(U,hZ) \\
&-(bk[2(n-1) + n(2k-\mu)] - A\mu^2(1+k)(2n+1) \\
&-2nb\mu(1+k)[2(n-1)+\mu])\eta(U)\eta(Z) = 0. \qquad (3.11)
\end{aligned}$$

Setting (2.8) and replacing hZ of Z in (3.11), we get

$$\begin{aligned}
&(A + b[2(1-n) + n\mu] + kb)S(U,hZ) + (1+k)(a\mu \\
&+b[2(n-1) + \mu] - \mu b)S(U,Z) - 2nk(1+k)(a\mu \\
&+b[2(n-1) + \mu] - \mu b)\eta(U)\eta(Z) + (bk[2(n-1) \\
&+n(2k-\mu)] - 2nkA - kbr + a\mu^2(1+k))g(U,hZ) \\
&+(1+k)(Ak - AB + b\mu[2(n-1) + n(2k-\mu)] \\
&-\mu br + a\mu B - 2nA\mu)g(U,Z) - (1+k)(Ak - AB \\
&+b\mu[2(n-1) + n(2k-\mu)] - \mu br + a\mu B \\
&-2nA\mu)\eta(U)\eta(Z) = 0. \qquad (3.12)
\end{aligned}$$

From (3.11), (3.12) and also using (2.10), for the sake of brevity we set

$$\begin{aligned}
c &= (A + b[2(1-n) + n\mu] + kb), \\
d &= (a\mu + b[2(n-1) + \mu] - \mu b), \\
e &= (bk[2(n-1) + n(2k-\mu)] - 2nkA - kbr + a\mu^2(1+k)), \\
f &= (Ak - AB + b\mu[2(n-1) + n(2k-\mu)] - \mu br + a\mu B - 2nA\mu), \\
t &= -(bk[2(n-1) + n(2k-\mu)] - A\mu^2(1+k)(2n+1) \\
&\quad -2nb\mu(1+k)[2(n-1)+\mu])
\end{aligned}$$

and

$$\begin{aligned}
E &= (fd(1+k) - ec)[2(n-1) + \mu] + (fc - ed)[2(1-n) + n\mu], \\
D &= (c^2 - d^2(1+k))[2(n-1) + \mu] + (fc - de), \\
F &= (fc - de)[2(n-1) + n(2k-\mu)] \\
&\quad -(ct + 2nkd^2(1+k) + fd(1+k))[2(n-1) + \mu],
\end{aligned}$$

that is,

$$DS(U,Z) = Eg(U,Z) + F\eta(U)\eta(Z).$$

Thus, M is an $\eta-$Einstein manifold. The converse is obvious. □

THEOREM 3.2. *Let* $M^{2n+1}(\phi,\xi,\eta,g)$ *be a* (k,μ)*-paracontact space. Then* $\widetilde{C}(X,Y)\cdot P = 0$ *if and only if* M *is an* $\eta-$*Einstein manifold.*

PROOF. Suppose that $\widetilde{C}(X,Y)\cdot P=0$. Then we have

$$\begin{aligned}(\widetilde{C}(X,Y)P)(U,W)Z &= \widetilde{C}(X,Y)P(U,W)Z-P(\widetilde{C}(X,Y)U,W)Z\\ &\quad -P(U,\widetilde{C}(X,Y)W)Z-P(U,W)\widetilde{C}(X,Y)Z=0,\end{aligned}\tag{3.13}$$

for any $X,Y,U,W,Z\in\chi(M)$. Using $X=Z=\xi$ in (3.13) and using (3.2), (3.6) for $A=[ak+2nkb-\frac{r}{(2n+1)}(\frac{a}{2n}+2b)]$, we obtain

$$\begin{aligned}(\widetilde{C}(\xi,Y)P)(U,W)\xi &= \widetilde{C}(\xi,Y)(\mu(\eta(W)hU-\eta(U)hW)-P(A(g(Y,U)\xi\\ &\quad -\eta(U)Y)+a\mu(g(hY,U)\xi-\eta(U)hY)\\ &\quad +b(S(Y,U)\xi-\eta(U)QY),W)\xi-P(U,A(g(Y,W)\xi\\ &\quad -\eta(W)Y)+a\mu(g(hY,W)\xi-\eta(W)hY)\\ &\quad +b(S(Y,W)\xi-\eta(W)QY))\xi\\ &\quad -P(U,W)(A(\eta(Y)\xi-Y)-a\mu hY\\ &\quad +b(2nk\eta(Y)\xi-QY))=0.\end{aligned}\tag{3.14}$$

Taking into account that (3.2), (3.6), (3.7) and inner product both sides of (3.14) by $Z\in\chi(M)$, we get

$$\begin{aligned}&Ag(P(U,W)Y,Z)+a\mu g(P(U,W)hY,Z)+bg(P(U,W)QY,Z)\\ &+A\mu(\eta(W)\eta(Z)g(Y,hU)-\eta(U)\eta(Z)g(Y,hW)\\ &+a\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U)-\eta(U)\eta(Z)g(Y,W))\\ &+b\mu(\eta(W)\eta(Z)S(Y,hU)-\eta(U)\eta(Z)S(Y,hW))\\ &+A\mu(g(Y,U)g(hW,Z)-g(Y,W)g(hU,Z))\\ &+a\mu^2(1+k)(g(hY,U)g(hW,Z)-g(hY,W)g(hU,Z))\\ &+b\mu(S(Y,U)g(hY,W)-S(Y,W)g(hY,U))=0.\end{aligned}\tag{3.15}$$

Using (2.1), (2.17) and choosing $W=Y=e_i$, ξ, for orthonormal basis of $\chi(M)$ in (3.15), $1\leqslant i\leqslant n$, we get

$$\begin{aligned}&(A+b[2(1-n)+n\mu]+\frac{A}{2n}+\frac{b}{2n}[2(1-n)+n\mu])S(U,Z)\\ &+(a\mu+b[2(n-1)+\mu]+\frac{a\mu}{2n}+\frac{b}{2n}[2(n-1)+\mu]+b\mu)S(U,hZ)\\ &+(a\mu^2(1+k)-\frac{Ar}{2n}-a\mu(1+k)[2(n-1)+\mu]-\frac{br}{2n}[2(1-n)+n\mu]\\ &-b[2(n-1)+\mu](1+k))g(U,Z)\\ &+(b\mu[2(n-1)+n(2k-\mu)]-2nA\mu-b\mu r)g(U,hZ)\\ &+(-a\mu^2(1+k)(2n+1)-2n\mu b(1+k)[2(n-1)+\mu])\eta(U)\eta(Z)=0.\end{aligned}\tag{3.16}$$

Replacing hZ of Z in (3.16) and making use of (2.8), we get

$$\begin{aligned}
&(A+b[2(1-n)+n\mu]+\frac{A}{2n}+\frac{b}{2n}[2(1-n)+n\mu])S(U,hZ)\\
&+(1+k)(a\mu+b[2(n-1)+\mu]+\frac{a\mu}{2n}+\frac{b}{2n}[2(n-1)+\mu]+b\mu)S(U,Z)\\
&-2nk(1+k)(a\mu+b[2(n-1)+\mu]+\frac{a\mu}{2n}+\frac{b}{2n}[2(n-1)+\mu]+b\mu)\eta(U)\eta(Z)\\
&+(a\mu^2(1+k)-\frac{Ar}{2n}-a\mu(1+k)[2(n-1)+\mu]-\frac{br}{2n}[2(1-n)+n\mu]\\
&-b[2(n-1)+\mu](1+k))g(U,hZ)+(1+k)(b\mu[2(n-1)+n(2k-\mu)]\\
&-2nA\mu-b\mu r)g(U,Z)-(1+k)(b\mu[2(n-1)+n(2k-\mu)]\\
&-2nA\mu-b\mu r)\eta(U)\eta(Z)=0. \qquad (3.17)
\end{aligned}$$

From (3.16), (3.17) and using (2.10), for the sake of brevity we put

$$\begin{aligned}
c &= (A+b[2(1-n)+n\mu]+\frac{A}{2n}+\frac{b}{2n}[2(1-n)+n\mu]),\\
d &= (1+k)(a\mu+b[2(n-1)+\mu]+\frac{a\mu}{2n}+\frac{b}{2n}[2(n-1)+\mu]+b\mu),\\
e &= (a\mu^2(1+k)-\frac{Ar}{2n}-a\mu(1+k)[2(n-1)+\mu]\\
&\quad -\frac{br}{2n}[2(1-n)+n\mu]-b[2(n-1)+\mu](1+k)),\\
f &= (b\mu[2(n-1)+n(2k-\mu)]-2nA\mu-b\mu r),\\
s &= (-a\mu^2(1+k)(2n+1)-2n\mu b(1+k)[2(n-1)+\mu])
\end{aligned}$$

and

$$\begin{aligned}
E &= (fd(1+k)-ec)[2(n-1)+\mu]+(fc-ed)[2(1-n)+n\mu],\\
D &= (c^2-d^2(1+k))[2(n-1)+\mu]+(fc-de),\\
F &= (fc-de)[2(n-1)+n(2k-\mu)]-(cs+2nkd^2(1+k)+fd(1+k))[2(n-1)+\mu]
\end{aligned}$$

we infer

$$DS(U,Z)=Eg(U,Z)+F\eta(U)\eta(Z),$$

So, M is an η-Einstein manifold. The converse is obvious. This completes of the proof. □

THEOREM 3.3. *Let* $M^{2n+1}(\phi,\xi,\eta,g)$ *be a* (k,μ)*-paracontact space. Then* $\widetilde{C}(X,Y)\cdot R=0$ *if and only if* M *is an* $\eta-$*Einstein manifold.*

PROOF. Suppose that $\widetilde{C}(X,Y)\cdot R=0$. This means that

$$\begin{aligned}
(\widetilde{C}(X,Y)R)(U,W)Z &= \widetilde{C}(X,Y)R(U,W)Z-R(\widetilde{C}(X,Y)U,W)Z\\
&\quad -R(U,\widetilde{C}(X,Y)W)Z-R(U,W)\widetilde{C}(X,Y)Z=0, \qquad (3.18)
\end{aligned}$$

for any $X, Y, U, W, Z \in \chi(M)$. Setting $X = Z = \xi$ in (3.18) and making use of (2.6), (3.2), (3.3), for $A = [ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)]$, we obtain

$$\begin{aligned}(\widetilde{C}(\xi, Y)R)(U, W)\xi \;=\;& \widetilde{C}(\xi, Y)(k(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU \\ & -\eta(U)hW) - R(A(g(Y, U)\xi - \eta(U)Y) \\ & +a\mu(g(hY, U)\xi - \eta(U)hY + b(S(Y, U) \\ & -\eta(U)QY), W)\xi - R(U, A(g(Y, W)\xi - \eta(W)Y) \\ & +a\mu(g(hY, W)\xi - \eta(W)hY) + b(S(Y, W) \\ & -\eta(W)QY)\xi - R(U, W)(A(\eta(Y)\xi - Y) \\ & -a\mu hY + b(2nk\eta(Y)\xi - QY)) = 0. \end{aligned} \tag{3.19}$$

Inner product both sides of (3.19) by $Z \in \chi(M)$ and using of (2.6), (3.1) and (3.3) we get

$$\begin{aligned}& Ag(R(U, W)Y, Z) + a\mu g(R(U, W)hY, Z) + bg(R(U, W)QY, Z) \\ & +A\mu(\eta(W)\eta(Z)g(Y, hU) - \eta(U)\eta(Z)g(Y, hW)) \\ & +a\mu^2(1 + k)(\eta(W)\eta(Z)g(Y, U) - \eta(U)\eta(Z)g(Y, W)) \\ & +b\mu(\eta(W)\eta(Z)S(Y, hU) - \eta(U)\eta(Z)S(Y, hW)) \\ & +Ak(g(Y, U)g(W, Z) - g(Y, W)g(U, Z)) \\ & +A\mu(g(Y, U)g(hW, Z) - g(Y, W)g(hU, Z)) \\ & +bk(g(Y, U)S(W, Z) - g(Y, W)S(U, Z)) \\ & +b\mu(g(hW, Z)S(Y, U) - g(hU, Z)S(Y, W)) \\ & +a\mu^2(g(hY, U)g(hW, Z) - g(hY, W)g(hU, Z)) \\ & +a\mu k(g(hY, U)g(W, Z) - g(hY, W)g(U, Z)) = 0. \end{aligned} \tag{3.20}$$

Making use of (2.8), (2.12) and choosing $W = Y = e_i, \xi,\ 1 \leqslant i \leqslant n$, for orthonormal basis of $\chi(M)$ in (3.20), we have

$$\begin{aligned}& (A + b[2(1 - n) + n\mu] + bk)S(U, Z) + (a\mu + b[2(n - 1) + \mu] + b\mu)S(U, hZ) \\ & +(bk[2(n - 1) + n(2k - \mu)] - 2nkA + a\mu^2(1 + k) - bkr)g(U, Z) \\ & +(-bk[2(n - 1) + n(2k - \mu)] - a\mu^2(1 + k)(2n + 1) \\ & -2nb\mu(1 + k)[2(n - 1) + \mu])\eta(U)\eta(Z) \\ & +(b\mu[2(n - 1) + n(2k - \mu)] - \mu br + a\mu k - 2nA\mu)g(U, hZ) = 0. \end{aligned} \tag{3.21}$$

Replacing hZ of Z in (3.21) and taking into account (2.8), we get

$$\begin{aligned}&(A+b[2(1-n)+n\mu]+bk)S(U,hZ)+(1+k)(a\mu+b[2(n-1)+\mu]\\&+b\mu)S(U,Z)-(1+k)(a\mu+b[2(n-1)+\mu]+b\mu)\eta(U)\eta(Z)\\&+(bk[2(n-1)+n(2k-\mu)]-2nkA+a\mu^2(1+k)-bkr)g(U,hZ)\\&+(1+k)(b\mu[2(n-1)+n(2k-\mu)]-\mu br+a\mu k-2nA\mu)g(U,Z)\\&-(1+k)(b\mu[2(n-1)+n(2k-\mu)]-\mu br+a\mu k-2nA\mu)\eta(U)\eta(Z)=0.\end{aligned}\tag{3.22}$$

From (3.21), (3.22) and by using (2.10), for the sake of brevity we set

$$\begin{aligned}c&=(A+b[2(1-n)+n\mu]+bk),\\d&=(a\mu+b[2(n-1)+\mu]+b\mu),\\e&=(bk[2(n-1)+n(2k-\mu)]-2nkA+a\mu^2(1+k)-bkr),\\f&=(-bk[2(n-1)+n(2k-\mu)]-a\mu^2(1+k)(2n+1)-2nb\mu(1+k)[2(n-1)+\mu]),\\t&=(b\mu[2(n-1)+n(2k-\mu)]-\mu br+a\mu k-2nA\mu)\end{aligned}$$

and

$$\begin{aligned}E&=(td(1+k)-ec)[2(n-1)+\mu]+(tc-ed)[2(1-n)+n\mu],\\D&=(c^2-d^2(1+k))[2(n-1)+\mu]+(tc-de),\\F&=(tc-de)[2(n-1)+n(2k-\mu)]-(cf+2nkd^2(1+k)+td(1+k))[2(n-1)+\mu]\end{aligned}$$

This implies,

$$DS(U,V)=Eg(U,Z)+F\eta(U)\eta(Z),$$

which verifies our assertion. The converse is obvious. □

THEOREM 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $\widetilde{C}(X,Y)\cdot S=0$ if and only if M is an $\eta-$Einstein manifold.*

PROOF. Suppose that $\widetilde{C}(X,Y)\cdot S=0$. This means that

$$S(\widetilde{C}(X,Y)U,W)+S(U,\widetilde{C}(X,Y)W)=0\tag{3.23}$$

for any $X,Y,U,W\in\chi(M)$. Setting $X=\xi$, in (3.23) and making use of (3.2), for

$$A=[ak+2nkb-\frac{r}{(2n+1)}(\frac{a}{2n}+2b)],$$

we obtain

$$\begin{aligned}&S(A(g(Y,U)\xi-\eta(U)Y)+a\mu(g(hY,U)\xi-\eta(U)hY\\&+b(S(Y,U)-\eta(U)QY),W)\xi-S(U,A(g(Y,W)\xi-\eta(W)Y)\\&+a\mu(g(hY,W)\xi-\eta(W)hY)+b(S(Y,W)\xi-\eta(W)QY)\xi=0.\end{aligned}\tag{3.24}$$

Taking $U=\xi$ in (3.24) and using (2.12), we reach at

$$\begin{aligned}&(2nkb-A-b[2(1-n)+n\mu])S(Y,W)+(-a\mu-b[2(n-1)\\&+\mu])S(Y,hW)+2nkAg(Y,W)+2nka\mu g(Y,hW)\\&+(-2nkb[2(n-1)+n(2k-\mu)])\eta(Y)\eta(W)=0.\end{aligned}\tag{3.25}$$

Replacing hW of W in (3.25) and taking into account (2.8), we get

$$\begin{aligned}&(2nkb - A - b[2(1-n) + n\mu])S(Y, hW) + (1+k)(-a\mu\\&-b[2(n-1)+\mu])S(Y,W) - (1+k)(-a\mu - b[2(n-1)\\&+\mu])\eta(Y)\eta(W) + 2nkAg(Y,hW) + 2nka\mu(1+k)g(Y,W)\\&-2nka\mu(1+k)\eta(Y)\eta(W) = 0.\end{aligned} \tag{3.26}$$

From (3.25), (3.26) and by using (2.10), for the sake of brevity we set

$$\begin{aligned}c &= (2nkb - A - b[2(1-n) + n\mu]),\\ d &= (-a\mu - b[2(n-1)+\mu]),\\ e &= (-2nkb[2(n-1) + n(2k-\mu)]),\end{aligned}$$

and

$$\begin{aligned}E &= [2nka\mu d(1+k) - Ac][2(n-1)+\mu] - 2nk(Ad - a\mu c)[2(1-n)+n\mu],\\ D &= (c^2 - d^2(1+k))[2(n-1)+\mu] - 2nk(Ad - a\mu c),\\ F &= -(ce + 2nkd^2(1+k) + 2nka\mu d(1+k))[2(n-1)+\mu]\\ &\quad +2nk(Ad - a\mu c) - [2(n-1) + n(2k-\mu)]\end{aligned}$$

then, we have

$$DS(Y,W) = Eg(Y,W) + F\eta(Y)\eta(W).$$

This tell us, M is an η-Einstein manifold. The converse is obvious. □

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