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THE MINIMUM NONSPLIT DOMINATION ENERGY OF A GRAPH

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Abstract. For a graph G, a subset D of $V(G)$ is called a *nonsplit dominating set* if the induced graph $\langle V - D\rangle$ is connected. The *nonsplit domination number* $\gamma_{ns}(G)$ is the minimum cardinality of a *nonsplit domination set.* In this paper, we introduce the minimum nonsplit dominating energy $E_{ns}(G)$ of a graph G and computed minimum nonsplit dominating energies of some standard graphs. Upper and lower bounds for $E_{ns}(G)$ are established.

1. Introduction

In this paper, by a graph $G(V, E)$ we mean a simple connected graph, that is nonempty, finite, having no loops, no multiple and directed edges. Let n and m be the number of vertices and edges, respectively, of G. The degree of a vertex v in a graph G, denoted by $deg(v)$, is the number of vertices adjacent to v. For any vertex v of a graph G, the open neighborhood of v is the set $N(v) = \{u \in V/uv \in E(G)\}$. For graph theoretic terminology we refer to [6].

Let W_n denote a wheel graph with $n + 1$ vertices ($n \geqslant 3$), which is formed by connecting a single vertex to all vertices of a cycle of length n and F_n be the friendship graph with 2n+1 vertices and 3n edges. Let C_n and P_n be the cycle and path with n vertices.

A set of vertices S is said to *dominate* the graph G, if for each $v \notin S$, there is a vertex $u \in S$ with v adjacent to u. The minimum cardinality of any *dominating set* is called the *domination number* of G and is denoted by $\gamma(G)$.

The concept of nonsplit domination was introduced by V. R. Kulli and B. Janakiram [7]. A *dominating set* D of a graph $G = (V, E)$ is a *nonsplit dominating set* if the

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induced graph $\langle V - D\rangle$ is *connected*. The *nonsplit domination number* $\gamma_{ns}(G)$ is the minimum cardinality of a *nonsplit domination set*.

The concept, energy of a graph introduced by I. Gutman [4] in the year 1978. It originates from chemistry to estimate the total π electron energy of a molecule. In chemistry, the conjugated hydrocarbons can be represented by a graph called molecular graph. Here every carbon atom is represented by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. The eigenvalues of the molecular graph represent the energy level of the electron in the molecule. An interesting quantity in Huckel theory is the sum of the energies of all the electrons in a molecule, the so called π electron energy of a molecule. The concept of dominating energy has been studied by [13].

Let $A(G) = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix $A(G)$, assumed in non-increasing order, are the eigenvalues of the graph G. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ for $r \leqslant n$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, \ldots, m_r$, respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ m_1 & m_2 & \ldots & m_n \end{pmatrix}$$

The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e. $E(G) = \sum_{i=1} |\lambda_i|$. For more details on the mathematical aspects of the theory of graph energy we refer to [5, 9, 10]. Recently C. Adiga et al.[1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Motivated by this paper, we introduce minimum nonsplit dominating energy, denoted by $E_{ns}(G)$, of a graph G, and computed minimum nonsplit dominating energies of some standard graphs. Upper and lower bounds for $E_{ns}(G)$ are established.

2. The Minimum Nonsplit Dominating Energy of Graphs

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots v_n\}$ and edge set E. A subset D of $V(G)$ is called a nonsplit dominating set if the induced graph $\langle V - D\rangle$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ of G is the minimum cardinality of a nonsplit dominating set. Any nonsplit dominating set with minimum cardinality is called a MNS set. Let D be a MNS set of a graph G. The MNS matrix of G is the $n \times n$ matrix defined by $A_{ns}(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ 1 & i = j,\ v_i \in D. \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{ns}(G)$ is denoted by

$$f_n(G, \lambda) = det(\lambda I - A_{ns}(G)).$$

The MNS eigenvalues of the graph G are the eigenvalues of $A_{ns}(G)$. Since $A_{ns}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant, \ldots, \geqslant \lambda_n$. The MNS energy of G is defined as $E_{ns}(G) = \sum_{i=1} |\lambda_i|$.

Let G be the graph in fig:1 with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and let its MNS set be $D_1 = \{v_1, v_4\}$

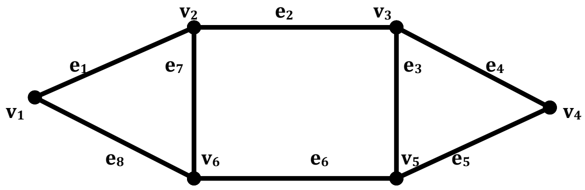

FIGURE 1. Example for the nonsplit domination energy of a graph

$$A_{ns}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic equation $f_n(G, \lambda) = \lambda^6 - 2\lambda^5 - 7\lambda^4 + 8\lambda^3 + 12\lambda^2$. The Spectrum of

$$A_{ns}(G) = \begin{pmatrix} 3 & 2 & 0 & -1 & -2 \\ 1 & 1 & 2 & 1 & 1 \end{pmatrix}$$

Hence the MNS eigen values are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = -1, \lambda_6 = -2$. Therefore the MSN energy of G is $E_{ns}(G) = 8$. Suppose if we take the MNS set of G as $D_2 = \{v_5, v_6\}$. Then

$$A_{ns}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic equation $f_n(G, \lambda) = \lambda^6 - 2\lambda^5 - 7\lambda^4 + 8\lambda^3 + 13\lambda^2 - 3$. The Spectrum of

$$A_{ns}(G) = \begin{pmatrix} -1 & -1.732 & -0.675 & 0.461 & 1.732 & 3.214 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence the MNS eigen values are

$$\lambda_1 = -1, \lambda_2 = -1.732, \lambda_3 = -0.675, \lambda_4 = 0.461, \lambda_5 = 1.732, \lambda_6 = 3.214.$$

Therefore the MSN energy of G is $E_{ns}(G) \simeq 8.814$. This illustrates the fact that the MNS energy of the graph G depends on the choice of the MNS set.

3. Properties of minimum nonsplit dominating eigenvalues

Theorem 3.1. *Let G be the graph of order n. Let $f_n(G, \lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n$ be the characteristic polynomial of MNS matrix of a graph G and D be the minimum nonsplit dominating set of G Then*

(i) $c_1 = -|D|$.

(ii) $c_2 = \binom{|D|}{2}$-$|E(G)|$.

Proof. (i) Since the sum of diagonal elements of $A_{ns}(G)$ is equal to $|D|$, the sum of determinants of all 1×1 principal submatrices of $A_{ns}(G)$ is the trace of $A_{ns}(G)$, which evidently is equal to $|D|$. Thus, $(-1)^1 c_1 = |D|$.

(ii) c_2 is equal to the sum of determinants of all 2×2 principal submatrices of $A_{ns}(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1 \leqslant i<j \leqslant n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \leqslant i<j \leqslant n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \leqslant i<j \leqslant n} a_{ii}a_{jj} - \sum_{1 \leqslant i<j \leqslant n} a_{ij}^2 \\ &= \binom{|D|}{2}\text{-}|E(G)| \end{aligned}$$

□

Theorem 3.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda n$ be the eigen values of $A_{ns}(G)$ and D be the minimum nonsplit dominating set of G. Then*

(i) $\sum_{i=1}^{i=n} \lambda_i = |D|$.

(ii) $\sum_{i=1}^{i=n} \lambda_i^2 = |D| + 2|E(G)|$.

Proof. (i) Since the sum of the eigenvalues of $A_{ns}(G)$ is the trace of $A_{ns}(G)$, it follows that

$$\sum_{i=1}^{i=n} \lambda_i = \sum_{i=1}^{i=n} a_{ii} = |D|.$$

(ii) Similarly the sum of squares of the eigenvalues of $A_{ns}(G)$ is the trace of $(A_{ns}(G))^2$. Then

$$\begin{aligned} \sum_{i=1}^{i=n} \lambda_i^2 &= \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{i=n} a_{ii}^2 + \sum_{i \neq j}^{i=n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{i=n} a_{ii}^2 + 2\sum_{i<j}^{i=n} a_{ij}^2 \\ &= |D| + 2|E(G)|. \end{aligned}$$

□

Theorem 3.3. *Let G be a graph of order n and let $\lambda_1(G)$ be the largest eigenvalue of $A_{ns}(G)$. Then $\lambda_1(G) \geqslant \frac{2|E(G)|+D}{n}$ where D is the nonsplit domination number.*

Proof. Let G be a graph of order n and let λ_1 be the largest nonspit eigenvalue of $A_{ns}(G)$. Then $\lambda_1 = max_{X \neq 0} = \{\frac{X^t A_{ns}(G) X}{X^t X}\}$, where X is any nonzero vector and X^t is its transpose and $A_{ns}(G)$ is a nonsplit dominating matrix. If we take $X = J = \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}$.

Then $\lambda_1(G) \geqslant \frac{J^t A_{ns}(G) J}{J^t J} = \frac{2|E(G)|+k}{n}$, where D is the nonsplit domination number. □

For example: Consider the graph $G = C_4$, the largest eigen value will be $\lambda_1(G) = 2.61$ and $\frac{2|E(G)|+k}{n} = \frac{2*4+2}{4} = 2.5$.

Theorem 3.4. *Let G_1 and G_2 be two graphs with n vertices. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of $A_{ns}(G_1)$ and $\lambda_1^{'}, \lambda_2^{'}, \ldots, \lambda_n^{'}$ are the eigen values of $A_{ns}(G_2)$. Then $\sum\limits_{i=1}^{i=n} \lambda_i \lambda_i^{'} \leqslant \sqrt{(2|E(G_1)| + |D_1|)(2|E(G_2)| + |D_2|}$. Where $A_{ns}(G_i)$ is the minimum nonsplit dominating matrix of $G_i, i = 1, 2$ and D_1, D_2 are the minimium nonsplit dominating sets of G_1 and G_2 respectively.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of $A_{ns}(G_1)$ and $\lambda_1^{'}, \lambda_2^{'}, \ldots, \lambda_{n'}$ are the eigen values of $A_{ns}(G_2)$. Then by Cauchy-Schwartiz inequality

$$(\sum_{i=1}^{i=n} a_i b_i)^2 \leqslant (\sum_{i=1}^{i=n} a_i^2)(\sum_{i=1}^{i=n} b_i^2).$$

If $a_i = \lambda_i$ and $b_i = \lambda_i^{'}$, then

$$(\sum_{i=1}^{i=n} \lambda_i \lambda_i^{'})^2 \leqslant (\sum_{i=1}^{i=n} \lambda_i^2)(\sum_{i=1}^{i=n} (\lambda_i^{'})^2).$$

$$(\sum_{i=1}^{i=n} \lambda_i \lambda_i^{'})^2 \leqslant (2|E(G_1)| + |D_1|)(2|E(G_2)| + |D_2|).$$

$$(\sum_{i=1}^{i=n} \lambda_i \lambda_i^{'}) \leqslant \sqrt{(2|E(G_1)| + |D_1|)(2|E(G_2)| + |D_2|)}.$$ □

For example: Consider the graph $G_1 = C_4$ and $G_2 = K_4$, we have eigen values of G_1 are $0.38196, 0.61803, -1.61803, 2.61803$ and the eigen values of G_2 are $-1, -1, -0.6055, 3.30$ so that $\sum\limits_{i=1}^{i=n} \lambda_i \lambda_i^{'} = 8.619$ and

$$\sqrt{(2|E(G_1)| + |D_1|)(2|E(G_2)| + |D_2|} = 9.48.$$

4. Bounds for minimum nonsplit dominating energy of a graph G

Similar to Milovanović [12] sharp bounds for energy of a graph, bounds for minimum nonsplit dominating energy of a graph is given in the following theorem.

Theorem 4.1. *Let G be a graph with n vertices and D be a nonsplit dominating set of G. Let $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n| \geqslant 0$ be a non-increasing order of eigen values of $A_{ns}(G)$, then $E_{ns}(G) \geqslant \frac{|D|+2|E(G)|+n|\lambda_1||\lambda_n|}{(|\lambda_1|+|\lambda_n|)}$.*

Proof. Let a_i, b_i, r, and R be real numbers satisfying $ra_i \leqslant b_i \leqslant Ra_i$, then the following equality holds [12].

$$\sum_{i=1}^{i=n} b_i^2 + rR \sum_{i=1}^{i=n} a_i \leqslant (r+R) \sum_{i=1}^{i=n} a_i b_i.$$

put $b_i = |\lambda_i|, a_i = 1, r = |\lambda_n|, R = |\lambda_1|$, then

$$\sum_{i=1}^{i=n} |\lambda_i|^2 + |\lambda_1||\lambda n| \sum_{i=1}^{i=n} 1 \leqslant (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{i=n} |\lambda_i|.$$

$$|D| + 2|E(G)| + n|\lambda_1||\lambda_n| \leqslant (|\lambda_1| + |\lambda_n|)E_{ns}(G)$$

$$E_{ns}(G) \geqslant \frac{|D|+2|E(G)|+n|\lambda_1||\lambda_n|}{(|\lambda_1|+|\lambda_n|)}. \qquad \square$$

For example: Consider the graph $G = C_4$,$|\lambda_1| = 0.38$ and $|\lambda_2| = 2.61, E_{ns}(G) = 5.23$. Thus $5.23 \geqslant \frac{|D|+2|E(G)|+n|\lambda_1||\lambda_n|}{(|\lambda_1|+|\lambda_n|)} = \frac{(2+2*4+4*0.38*2.61)}{(0.38+2.61)} = 4.67$

Bapat and S.Pati [2] proved that if the graph energy is a rational number then it is an even integer, similar result for minimum nonsplit dominating energy is given in the following theorem.

Theorem 4.2. *Let G be a graph with a minimum nonsplit dominating set D. If the minimum nonsplit dominating energy $E_{ns}(G)$ of G is a rational number, then $E_{ns}(G) \equiv \gamma_{ns}(G)|mod 2|$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the minimum nonsplit dominating eigen values of a graph G of which $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive and the rest are non-positive, then

$$\sum_{i=1}^{i=n} \lambda_i = (\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \ldots + \lambda_n).$$

$$\sum_{i=1}^{i=n} \lambda_i = (\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \ldots + \lambda_n)$$
$$+(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_1 + \lambda_2 + \ldots + \lambda_r)$$

$$\sum_{i=1}^{i=n} \lambda_i = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_1 + \lambda_2 + \ldots + \lambda_r) + (\lambda_{r+1} + \lambda_{r+2} + \ldots + \lambda_n)$$
$$= 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_1 + \lambda_2 + \ldots + \lambda_n).$$
$$= 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (|D|).$$
$$= 2|q| - |D|, \text{ where } q = \lambda_1 + \lambda_2 + \ldots + \lambda_r.$$

By the result of Fiedler on additive compounds[3], the partial sum $\lambda_1 + \lambda_2 + \ldots + \lambda_r$ is an eigenvalue of a matrix whose characteristic polynomial has integer coefficients. If $\sum_{i=1}^{i=n} |\lambda_i|$ is rational, then $\lambda_1 + \lambda_2 + \ldots + \lambda_r$ is rational and hence it must be an integer. Therefore $E_{ns}(G) \equiv \gamma_{ns}(G)|mod 2|$. $\square$

Theorem 4.3. *Let G be a connected graph of order n. Then*

$$\sqrt{2|E(G)| + \gamma_{ns}(G)} \leqslant E_{ns}(G) \leqslant \sqrt{n(2|E(G)| + \gamma_{ns}(G))}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$(\sum_{i=1}^{i=n} a_i b_i)^2 \leqslant (\sum_{i=1}^{i=n} a_i^2)(\sum_{i=1}^{i=n} b_i^2).$$

Choose $a_i = 1$ and $b_i = |\lambda_i|$ and by Theorem 3.2, we get

$$\begin{aligned}(E_{ns}(G))^2 = (\sum_{i=1}^{i=n} |\lambda_i|)^2 &\leqslant (\sum_{i=1}^{i=n} 1)(\sum_{i=1}^{i=n} \lambda_i^2).\\ &\leqslant n(2|E(G)| + |D|).\\ &\leqslant n(2|E(G)| + \gamma_{ns}(G)).\end{aligned}$$

Therefore, the upper bound is hold. For the lower bound, since

$$(\sum_{i=1}^{i=n} |\lambda_i|)^2 \geqslant (\sum_{i=1}^{i=n} \lambda_i^2)$$

it follows by Theorem 3.2 that

$$(E_{ns}(G))^2 \geqslant (\sum_{i=1}^{i=n} \lambda_i^2) = 2|E(G)| + |D| = 2|E(G)| + \gamma_{ns}(G).$$

Therefore, the lower bound is hold. □

For example: Consider the friendship graph F_2. $E_{ns}(F_2) = 4.6$,

$$\sqrt{2|E(G)| + \gamma_{ns}(G)} = 3.74 \text{ and } \sqrt{n(2|E(G)| + \gamma_{ns}(G))} = 8.36.$$

Theorem 4.4. *Let G be a graph with n vertices and let D be a minimum nonsplit dominating set. Then*

$$\sqrt{(2|E(G)| + |D|) + (n-1)n|det(A_{ns}(G))|^{\frac{2}{n}}} \leqslant E_{ns}(G) \leqslant \sqrt{n(2|E(G)| + |D|)}.$$

Proof. This proof follows the idea of McClellands bounds [11] for graphs $E(G)$. For the upper bound, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of the minimum nonsplit dominating matrix $A_{ns}(G)$. By Theorem 4.3. $(E_{ns}(G)) \leqslant \sqrt{n(2|E(G)| + |D|)}$ which is the upper bound.

For the lower bound, by using arithmetic mean and geometric mean inequality, we have

$$\begin{aligned}&\tfrac{1}{n(n-1)} \sum_{i\neq j}^{i=n} |\lambda_i||\lambda_j| \geqslant (\prod_{i\neq j}^{i=n} |\lambda_i||\lambda_j|)^{\frac{1}{n(n-1)}}.\\ &\sum_{i\neq j}^{i=n} |\lambda_i||\lambda_j| \geqslant n(n-1)(\prod_{i=1}^{i=n} |\lambda_i|^{2(n-1)})^{\frac{1}{n(n-1)}}.\\ &\sum_{i\neq j}^{i=n} |\lambda_i||\lambda_j| \geqslant n(n-1)(\prod_{i=1}^{i=n} |\lambda_i|)^{\frac{2}{n}}.\end{aligned}$$

Consider $(E_{ns}(G))^2 = (\sum_{i=1}^{i=n} |\lambda_i|)^2 = \sum_{i=1}^{i=n} |\lambda_i|^2 + \sum_{i\neq j}^{i=n} |\lambda_i||\lambda_j|.$

$$(E_{ns}(G))^2 \geqslant \sum_{i=1}^{i=n} |\lambda_i|^2 + n(n-1)(\prod_{i=1}^{i=n} |\lambda_i|)^{\frac{2}{n}}.$$

$$\geqslant 2|E(G)| + |D| + n(n-1)|det(A_{ns}(G))|^{\frac{2}{n}}.$$

$$E_{ns}(G) \geqslant \sqrt{2|E(G)| + |D| + n(n-1)|det(A_{ns}(G))|^{\frac{2}{n}}}$$

which is the lower bound. □

For example: Consider the graph wheel graph $G = W_5$, $E_{ns}(G) = 7.12$. $\sqrt{(2|E(G)| + |D|) + (n-1)n|det(A_{ns}(G))|^{\frac{2}{n}}} = 4.24$ and $\sqrt{n(2|E(G)| + |D|)} = 9.219$.

Theorem 4.5. *Let $G \neq K_{n-1}$ be a connected graph of order $n \geqslant 2$. Then*

$$\sqrt{n+1} \leqslant E_{ns}(G) \leqslant n\sqrt{n}.$$

Proof. Since for any graph $G \neq K_{n-1}$, $\gamma_{ns}(G) \leqslant n-2$(see[7]), it follows that by using Theorem 4.4 and using the result, $2|E(G)| \leqslant n^2 - n$, we have

$$E_{ns}(G) \leqslant \sqrt{n(2|E(G)| + \gamma_{ns}(G))} \leqslant \sqrt{n[(n^2-n) + n - 2]} \leqslant n\sqrt{n}.$$

For the lower bound, since for any connected graph, $n \geqslant 2|E(G)|$ and $\gamma_{ns}(G) \geqslant 1$ ([7]), it follows by Theorem 4.4

$$E_{ns}(G) \geqslant \sqrt{2|E(G)| + \gamma_{ns}(G)} \geqslant \sqrt{n+1}.$$

□

For example: Consider the graph $G = K_{1,4}$, $E_{ns}(G) = 7.12, \sqrt{n+1} = 2.44$, and $n\sqrt{n+1} = 11.18$.

Similar to Koolen and Moulton's [8], upper bound energy of a graph, upper bound for $E_{ns}(G)$ is given in the following theorem.

Theorem 4.6. *Let G be a connected graph of order n and $2|E(G)| + \gamma_{ns}(G) \geqslant n$. Then*

$$E_{ns}(G) \leqslant \frac{2|E(G)| + \gamma_{ns}(G)}{n} + \sqrt{(n-1)[2|E(G)| + \gamma_{ns}(G) - (\frac{2|E(G)| + \gamma_{ns}(G)}{n})^2]}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$(\sum_{i=2}^{i=n} a_i b_i)^2 \leqslant (\sum_{i=2}^{i=n} a_i^2)(\sum_{i=2}^{i=n} b_i^2).$$

Choose $a_i = 1$ and $b_i = |\lambda_i|$ and by Theorem 3.2, we get

$$(\sum_{i=2}^{i=n} |\lambda_i|)^2 \leqslant (\sum_{i=2}^{i=n} 1)(\sum_{i=2}^{i=n} \lambda_i^2).$$

$$(E_{ns}(G) - |\lambda_1|)^2 \leqslant (n-1)(2|E(G)| + \gamma_{ns}(G) - \lambda_1^2).$$

$$E_{ns}(G) \leqslant \lambda_1 + \sqrt{(n-1)(2|E(G)| + \gamma_{ns}(G) - \lambda_1^2)}.$$

From Theorem 3.3, we have $\lambda_1(G) \geqslant \frac{2|E(G)| + \gamma_{ns}(G)}{n}$. Since

$$f(x) = x + \sqrt{(n-1)(2|E(G)| + \gamma_{ns}(G) - x^2)}$$

is a decreasing function. It follows

$$x \geqslant \sqrt{\frac{2|E(G)| + \gamma_{ns}(G)}{n}}.$$

Since $2|E(G)|+\gamma_{ns}(G)\geqslant n$, we have $\sqrt{\frac{2|E(G)|+\gamma_{ns}(G)}{n}}\leqslant\frac{2|E(G)|+\gamma_{ns}(G)}{n}\leqslant\lambda_1$.

$$f(\lambda_1)\leqslant f(\tfrac{2|E(G)|+\gamma_{ns}(G)}{n}).$$
$$E_{ns}(G)\leqslant f(\lambda_1)\leqslant f(\tfrac{2|E(G)|+\gamma_{ns}(G)}{n}).$$
$$E_{ns}(G)\leqslant f(\tfrac{2|E(G)|+\gamma_{ns}(G)}{n}).$$
$$E_{ns}(G)\leqslant\tfrac{2|E(G)|+\gamma_{ns}(G)}{n}+\sqrt{(n-1)[2|E(G)|+\gamma_{ns}(G)-(\tfrac{2|E(G)|+\gamma_{ns}(G)}{n})^2]}.\quad\square$$

For example: Consider the graph $G=K_{2,2}$, $E_{ns}(G)=5.21$,
$E_{ns}(G)\leqslant$
$$\tfrac{2|E(G)|+\gamma_{ns}(G)}{n}+\sqrt{(n-1)[2|E(G)|+\gamma_{ns}(G)-(\tfrac{2|E(G)|+\gamma_{ns}(G)}{n})^2]}=5.85.$$

5. Minimum Nonsplit Dominating Energy of Some Standard Graphs

In this section, we investigate the exact values of the MNS energy of some standard graphs.

Theorem 5.1. *For the complete graph K_n, $n\geqslant 2$,*
$$E_{ns}(Kn)=(n-2)+\sqrt{(n^2-2n+5)}.$$

Proof. Let K_n be the complete graph with vertex set $V=\{v_1,v_2,\dots,v_n\}$, then $\gamma_{ns}(K_n)=1$. Hence the MNS set is $D=\{v_1\}$ and

$$A_{ns}(K_n)=\begin{pmatrix}1&1&1&\dots&1&1\\1&0&1&\dots&1&1\\1&1&0&\dots&1&1\\.&.&.&\dots&.&.\\.&.&.&\dots&.&.\\.&.&.&\dots&.&.\\1&1&1&\dots&1&0\end{pmatrix}$$

Characteristic polynomial is

$$f_n(G,\lambda)=\begin{vmatrix}\lambda-1&-1&-1&\dots&-1&-1\\-1&\lambda&-1&\dots&-1&-1\\-1&-1&\lambda&\dots&-1&-1\\.&.&.&\dots&.&.\\.&.&.&\dots&.&.\\.&.&.&\dots&.&.\\-1&-1&-1&\dots&-1&\lambda\end{vmatrix}=(\lambda+1)^{n-2}+(\lambda^2-(n-1)\lambda-1)$$

The MNS spectrum of K_n can be written as

$$MNSSpec(K_n)=\begin{pmatrix}-1&\frac{(n-1)+\sqrt{n^2-2n+5}}{2}&\frac{(n-1)-\sqrt{n^2-2n+5}}{2}\\n-2&1&1\end{pmatrix}.$$

Hence the MNS energy of complete graph is $E_{ns}(Kn)=(n-2)+\sqrt{(n^2-2n+5)}$. $\square$

Theorem 5.2. *For $n \geqslant 2$, the minimum nonsplit dominating energy of star graph $K_{1,n-1}$ is $(n-2)+\sqrt{4n-3}$.*

Proof. Consider the star graph $K_{1,n-1}$ with the vertex set

$$V = \{v_1, v_2, \ldots, v_{n-1}, v_0\}, d(v_0) = n-1.$$

The minimum nonsplit dominating set is $D = \{v_1, v_2, \ldots, v_{n-1}\}$. Then the minimum nonspit dominating matrix is

$$A_{ns}(K_n) = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ 1 & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Characteristic polynomial is

$$f_n(G,\lambda) = \begin{vmatrix} \lambda & -1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & \lambda-1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & \lambda-1 & 0 & \ldots & 0 & 0 \\ -1 & 0 & 0 & \lambda-1 & \ldots & 0 & 0 \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ -1 & 0 & 0 & 0 & \ldots & 0 & \lambda-1 \end{vmatrix}$$
$$= (\lambda-1)^{n-2} + (\lambda^2 - \lambda - (n-1))$$

The MNS spectrum of $K_{1,n-1}$ can be written as

$$\text{MNS } Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{1+\sqrt{1+4(n-1)}}{2} & \frac{1-\sqrt{1+4(n-1)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Hence the MNS energy of star graph is $E_{ns}(K_{1,n-1}) = (n-2)+\sqrt{4n-3}$. □

Theorem 5.3. *For $n \geqslant 3$, the minimum nonsplit domination energy of a complete bipartite graph $K_{n,n}$ is $n+1+\sqrt{n^2+2n-3}$.*

Proof. For the complete bipartite graph $K_{m,n}$ with vertex set

$$V = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots u_n\}$$

, with $\gamma_{ns}(K_{m,n}) = 2$. Let the MNS set of $K_{m,n}$ is $D = \{v_1, u_1\}$. Then

$$A_{ns}(K_{m,n}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & \dots & 1 \\ . & . & . & . & \dots & . & . & \dots & . \\ . & . & . & . & \dots & . & . & \dots & . \\ . & . & . & . & \dots & . & . & \dots & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 1 & \dots & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 1 & \dots & 0 & 0 & \dots & 0 \\ . & . & . & . & \dots & . & . & \dots & . \\ . & . & . & . & \dots & . & . & \dots & . \\ . & . & . & . & \dots & . & . & \dots & . \\ 1 & 1 & 1 & 1 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

Characteristic polynomial is

$$f_n(G,\lambda) = \begin{vmatrix} \lambda-1 & 0 & 0 & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ . & . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & . & \dots & . & . & . & \dots & . \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ -1 & -1 & -1 & -1 & \dots & -1 & \lambda-1 & 0 & \dots & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\ . & . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & . & \dots & . & . & . & \dots & . \\ -1 & -1 & -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 \end{vmatrix}$$

$$= \lambda^{m+n-4}(\lambda^4 - 2\lambda^3 - (mn-1)\lambda^2 + (2mn-m-n)\lambda - (m-1)(n-1))$$

In Particular for $m = n$, we have

$$f_n(G,\lambda) = (\lambda)^{n-2}(\lambda)^{n-2}(\lambda^2 + (n-1)\lambda - (n-1))(\lambda^2 - (n+1)\lambda + (n-1)).$$

The MNS spectrum of $K_{m,n}$ can be written as

MNS $Spec(K_{n,n}) =$

$$\begin{pmatrix} 0 & 0 & \frac{-(n-1)+\sqrt{n^2+2n-3}}{2} & \frac{-(n-1)-\sqrt{n^2+2n-3}}{2} & \frac{(n+1)+\sqrt{n^2-2n+5}}{2} & \frac{(n+1)-\sqrt{n^2-2n+5}}{2} \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence the MNS energy of a complete bipartite graph is

$$E_{ns}(K_{n,n}) = n + 1 + \sqrt{n^2 + 2n - 3}.$$

□

Definition 5.4. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by joining the center vertices v_0 and u_0 by an edge. Let the vertex set of $S_{n,m}$ is $V(S_{n,m}) = \{v_0, v_1, \ldots, v_{n-1}, u_0, u_1, \ldots, u_{m-1}\}$ and edge set*

$$E(S_{n,m}) = \{v_0u_0, v_0u_i, u_0u_j, 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m-1\}.$$

Theorem 5.5. *For $m \geqslant 3$, the minimum nonsplit dominating energy of double star graph Sm, m is $E_{ns}(S_{m,m}) = (2m-4) + 2\sqrt{m} + 2\sqrt{m-1}$.*

Proof. For the double star graph $S_{m,m}$ with vertex set

$$V = \{v_0, v_1, \ldots, v_{n-1}, u_0, u_1, \ldots, u_{m-1}\}$$

the minimum nonsplit dominating set is

$$D = \{v_1, \ldots, v_{m-1}, u_1, \ldots, u_{m-1}\}.$$

Then

$$A_{ns}(K_{r,s}) = \begin{pmatrix} 0 & 1 & 1\ldots & 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0\ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1\ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ 1 & 0 & 0\ldots & 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0\ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0\ldots & 0 & 1 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0\ldots & 0 & 1 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ 0 & 0 & 0\ldots & 0 & 1 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

Characteristic polynomial is

$$f_n(G,\lambda) = \begin{vmatrix} \lambda & -1 & -1\ldots & -1 & -1 & 0 & 0 & \ldots & 0 \\ -1 & \lambda-1 & 0\ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 0 & \lambda-1\ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ -1 & 0 & 0\ldots & \lambda-1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 0 & 0\ldots & 0 & \lambda & -1 & -1 & \ldots & -1 \\ 0 & 0 & 0\ldots & 0 & -1 & \lambda-1 & 0 & \ldots & 0 \\ 0 & 0 & 0\ldots & 0 & -1 & 0 & \lambda-1 & \ldots & 0 \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ . & . & . & \ldots & . & . & . & \ldots & . \\ 0 & 0 & 0\ldots & 0 & -1 & 0 & 0 & \ldots & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)^{2m-4}(\lambda^2 - m)(\lambda^2 - 2\lambda + (m-2)).$$

The MNS spectrum of $S_{m,m}$ will be written as

$$\text{MNS } Spec(S_{m,m}) = \begin{pmatrix} 1 & \sqrt{m} & -\sqrt{m} & 1+\sqrt{m-1} & 1-\sqrt{m-1} \\ 2m-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence the MNS energy of double star graph is $E(S_{m,m}) = (2m-4)+2\sqrt{m}+2\sqrt{m-1}$. □

Definition 5.6. *The cocktail party graph, denoted by $K_{2\times n}$, is a graph having vertex set $V(K_{n\times 2}) = \bigcup\limits_{i=1}^{i=n}(u_i, v_i)$ and edge set $E(K_{n\times 2}) = \{u_iu_j, v_iv_j, i \neq j\} \cup \{u_iv_j, v_iu_j, 1 \leqslant i < j \leqslant n\}$.*

Theorem 5.7. *The minimum nonsplit dominating energy of cocktail party graph $K_{n\times 2}, n \geqslant 3$ is*

$$(2n-3)+\sqrt{4n^2-4n+9}.$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph with vertex set $V(K_{n\times 2}) = \bigcup\limits_{i=1}^{i=n}(u_i, v_i)$. The minimum nonsplit dominating set is $D = \{u_1, v_1\}$. Then

$$A_{ns}(K_{n\times 2}) = \begin{pmatrix} 1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \dots & 1 & 1 & 1 & 1 \\ . & . & . & . & \dots & . & . & . & . \\ . & . & . & . & \dots & . & . & . & . \\ . & . & . & . & \dots & . & . & . & . \\ 1 & 1 & 1 & 1 & \dots & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 & 0 \end{pmatrix}$$

Characteristic polynomial is

$$f_n(G,\lambda) = \begin{vmatrix} \lambda-1 & 0 & -1 & -1 & \dots & -1 & -1 & -1 & -1 \\ 0 & \lambda-1 & -1 & -1 & \dots & -1 & -1 & -1 & -1 \\ -1 & -1 & \lambda & 0 & \dots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \lambda & \dots & -1 & -1 & -1 & -1 \\ . & . & . & . & \dots & . & . & . & . \\ . & . & . & . & \dots & . & . & . & . \\ . & . & . & . & \dots & . & . & . & . \\ -1 & -1 & -1 & -1 & \dots & \lambda & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & \dots & 0 & \lambda & -1 & -1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & \lambda & 0 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & 0 & \lambda \end{vmatrix}$$
$$= (\lambda)^{2n-1}(\lambda-1)(\lambda-1)^{n+2}(\lambda^2-(2n-3)\lambda-2n)$$

The MNS spectrum of $S_{m,m}$ will be written as

$$Spec(K_{n\times 2}) = \begin{pmatrix} 0 & 1 & 2 & 2n-3+\sqrt{4n^2-4n+9} & 2n-3-\sqrt{4n^2-4n+9} \\ n-1 & 1 & n+2 & 1 & 1 \end{pmatrix}$$

Hence the MNS energy of $E(S_{m,m}) = (2n-3) + \sqrt{4n^2 - 4n + 9}$. □

6. Conclusion

Formula and bounds obtained in this paper are useful for theoretical chemists, for whom this value can take on physical significance. For mathematicians, the concept leads to many interesting problems which are not necessarily identical to determining the spectrum of a graph but can provide certain helpful information about the graph. The basic properties including various upper and lower bounds for energy of a graph have been established and it can have remarkable chemical applications in the molecular orbital theory of conjugated molecules. The non split dominating energy of the graph can give the idea for the chemists to remove some carbon atoms that have the hydrogen bond with all the other carbon atoms and still they want the bonding has to be with all the other remaining carbon atoms.

References

[1] C. Adiga, A. Bayad, I. Gutman and S. A. Srinivas. The minimum covering energy of a graph. *Kragujevac J. Sci.,* **34**(2012), 39–56.

[2] R. B. Bapat and S. Pati. Energy of a graph is never an odd integer. *Bull. Kerala Math. Assoc.*, **1**(2011), 129–132.

[3] M. Fiedler. Additive compound matrices and an equality for eigenvalues of symmetric stochastic matrices. *Czech. Math. J.,*,**24**(99)(1974), 392–402.

[4] I. Gutman. The energy of a graph. *Ber. Math. Statist. Sekt. Forhungszetrum Graz*, **103**(1978), 1–22.

[5] I. Gutman. Comparative studies of graph energies. *Bul. Acad Serbe Sci. Arts*, **144**(2012), 1–17.

[6] F.Harary. *Graph Theory.* Addison-Wesley, reading Mass, 1969.

[7] V. R. Kulli and B. Janikiram. The nonsplit domination number of a graph. *Indian J. Pure Appl. Math.,* **31** (2000), 545–550.

[8] J. H. Koolen and V. Moulton. Maximal energy graphs. *Adv. Appl. Math.,*, **26**(2001), 47–52.

[9] X. Li, Y. Shi and I. Gutman. *Graph energy.* Springer, New York 2012.

[10] H. Liu. M. Lu and F. Tian. Some upper bounds for the energy of the graphs. *J. Math. Chem.,* **41**(2007), 79–85.

[11] B. J. Mcclelland. Properties of the latent roots of a matrix: The estimation of π-electron energies. *J. Chem. Phys.,*, **54**(1971), 640–643.

[12] I. Z. Milovanović, E. I. Milovanović and A. Zakić. A short note on graph energy. *MATH Commun. Math. Cpmputing. Chem*, **72**(2012), 179–182.

[13] M.R. Rajesh Khanna, B.N. Dharmendra and G. Sridhara. The minimum dominating energy of a graph. *International Journal of Pure and Applied Mathematics*, **85(4)**(2013), 707–718.

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