BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(3)(2021), 569-582
DOI: 10.7251/BIMVI2103569S

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

Q-RUNG PICTURE FUZZY MATRICES

I. Silambarasan

Abstract. The q-rung picture fuzzy matrix model, superior to picture and spherical fuzzy matrix models, broaden the space of uncertain and vague information due to its outstanding feature of vast depiction space of admissible triplets. In this paper, we introduce q-rung picture fuzzy matrices (q-RPFMs) and discuss several properties. Then some algebraic operations, such as max-min, min-max, complement, algebraic sum, algebraic product are defined and investigated their desirable properties. Further, scalar multiplication (nA) and exponentiation (A^n) operations of a q-RPFM A using algebraic operations are constructed, and their desirable properties are studied. Finally, we define a new operation(@) on q-rung picture fuzzy matrices and discuss distributive laws in the case where the operations of $\oplus_q, \otimes_q, \wedge_q$ and $\vee_q$ are combined each other.

1. Introduction

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Pal et al. [**11**] and simultaneously Im et al. [**3**] to generalize the concept of Thomason's [**27**] fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle$ with $\zeta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ and $0 \leqslant \zeta_{a_{ij}} + \delta_{a_{ij}} \leqslant 1$. Since the presence of IFM, a few analysts have significantly added to the improvement of Fuzzy matrix, IFM hypothesis and its applications [**4, 5, 7, 9, 8, 10, 11, 13, 14, 15, 12, 16, 17**]. Now and again a portion of their participation degrees are better than 1. In such a circumstance, to accomplish a sensible result IFM falls flat. In this way, managing such circumstance [**18, 21**] in 2020, established the concept of Pythagorean

2010 *Mathematics Subject Classification.* 03E72, 08A72, 15B15.
Key words and phrases. Intuitionistic fuzzy matrix, Pythagorean fuzzy matrix, q-rung Picture fuzzy matrix. Algebraic sum, Algebraic product, Scalar multiplication, Exponentiation operations.
Communicated by Andrzej WALENDZIAK, Siedlce, Poland.

fuzzy matrices (PyFM) by assigning membership degree say $\zeta_{a_{ij}}$ along with non-membership degree say $\delta_{a_{ij}}$ with condition that $0 \leqslant \zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leqslant 1$. After the introduction of PFM, Fermatean fuzzy matrix, Generalized orthopair fuzzy matrix theory, we have developed [**19, 20, 21, 22, 26**].

Cuong [**1**] introduced picture fuzzy set (PFS) model in 2013 as direct extension of intuitonistic fuzzy sets, which may be adequate in situations when human opinions are of types: yes, abstain, no, refusal. Picture fuzzy set gives three degrees to an element named; degree of positive membership $\mu : X \to [0,1]$, degree of neutral/abstinence membership $\eta : X \to [0,1]$, and degree of negative membership $\delta : X \to [0,1]$, under the condition $0 \leqslant \zeta(x) + \eta(x) + \delta(x) \leqslant 1$, where $\phi(x) = 1 - (\zeta(x) + \eta(x) + \delta(x))$ is the degree of refusal membership. Dogra and Pal [**2**] construction of picture fuzzy matrices (PFM) is of exceptional reputation but decision makers are some how restricted in assigning values due to the condition on $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$. In [**23, 24**], some algebraic operations of Picture fuzzy set and matrices are defined and their desirable properties are proved. Sometimes sum of their membership degrees are superior then 1. In such situation, to attain reasonable outcome PFM fails. To describe this situation, we take an example, for provision and in contradiction of the membership degrees. The alternatives are $0.2, 0.6$ and 0.6 respectively. This gratifies the situation that their sum is superior then 1 and PFM fails to deal such type of data. Dealing with such kind of circumstances, we proposed new structure by defining spherical fuzzy matrices (SFMs) which enlarge the space of membership degrees $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ somehow bigger than that of picture fuzzy matrices. In SFM, membership degrees are satisfying the condition $0 \leqslant \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leqslant 1$ [**25**].

Li et al. [**6**] studied the conception of -rung picture fuzzy set (q-RPFS) with constraint $\zeta^q + \eta^q + \delta^q \leqslant 1$ in 2018. The proposed q-rung picture fuzzy matrix not only express the degree of neutral membership, but also relax the constraint of Picture and spherical fuzzy matrices. The lax constraint of q-RPFM $0 \leqslant \zeta^q_{a_{ij}} + \eta^q_{a_{ij}} + \delta^q_{a_{ij}} \leqslant 1$ $(q \geqslant 1)$ can be considered as its eminent characteristic. As q increases, the representation space of acceptable triplets membership grade increases. Thus q-RPFMs can provide more stronger model ability, more flexibility, freedom and sophistication for a system modeler in representing their understanding of concept. For instance, if a decision maker provides the degree of positive, neutral and negative memberships as 0.7, 0.5 and 0.6, respectively. It is readily seen that $0.7 + 0.5 + 0.6 \leqslant 1$ and $(0.72)^2 + (0.52)^2 + (0.62) \geqslant 1$. Thus this situation can neither be illustrated by PFM nor by SFM. However it is appropriate to apply q-RPFM since $(0.7)^q + (0.5)^q + (0.6)^q \leqslant 1$ for sufficiently large q. Thus q-RPFM model is considerably close to human nature than that of earlier concepts. Moreover, the concept of interdependence of membership functions ζ, η and δ makes this notion more close to real world situations.

The part of this paper is as follows. In section 2, Preliminaries. In section 3, q-rung picture fuzzy matrices and its algebraic operations are defined and their desirable properties are developed. In section 4, we define a new operation(@) on q-rung picture fuzzy matrices and their algebraic properties are investigated. We write the conclusion of the paper in the last Section 5.

2. Preliminaries

In this section, some basic concepts related to the intuitionistic fuzzy matrix (IFM), Pythagorean fuzzy matrix (PFM), Picture fuzzy matrix (PFM), Spherical fuzzy matrix (SFM) have been given.

Definition 2.1. ([**11**]) An Intuitionistic fuzzy matrix (IFM) of order $m \times n$ is defined as $A = \left(\left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ where $\zeta_{a_{ij}} \in [0,1]$ and $\delta_{a_{ij}} \in [0,1]$ are the membership and non-membership values of the ij^{th} element in A satisfying the condition

$$0 \leqslant \zeta_{a_{ij}} + \delta_{a_{ij}} \leqslant 1$$

for all i, j.

Definition 2.2. ([**18, 21**]) A Pythagorean fuzzy matrix (PyFM) of order $m \times n$ is defined as $A = \left(\left\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ where $\zeta_{a_{ij}} \in [0,1]$ and $\delta_{a_{ij}} \in [0,1]$ are the membership and non-membership values of the ij^{th} element in A satisfying the condition

$$0 \leqslant \zeta_{a_{ij}}^2 + \delta_{a_{ij}}^2 \leqslant 1$$

for all i, j.

Definition 2.3. ([**2, 23**]) A Picture fuzzy matrix (PFM) A of the form, $A = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition

$$0 \leqslant \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leqslant 1$$

for all i, j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

Definition 2.4. [**25**] A spherical fuzzy matrix (SFM) A of the form, $A = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition

$$0 \leqslant \zeta_{a_{ij}}^2 + \eta_{a_{ij}}^2 + \delta_{a_{ij}}^2 \leqslant 1$$

for all i, j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

3. q-rung picture fuzzy matrices and their basic operations

In this section, q-rung picture fuzzy matrix and their algebraic operations are defined. Then some algebraic properties, such as idempotency, commutativity, associativity, absorption law, distributivity and De Morgan's laws over complement are proved.

Now, we are going to define Algebraic operations of q-rung picture fuzzy matrices by restricting the measure of positive, neutral and negative membership but keeping their sum in the interval $[0,1]$.

Definition 3.1. A q-rung picture fuzzy matrix (q-RPFM) A of the form, $A = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition

$$0 \leqslant \zeta^q_{a_{ij}} + \eta^q_{a_{ij}} + \delta^q_{a_{ij}} \leqslant 1$$

for all i, j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

Example 3.1. $\mathbf{A} = \begin{bmatrix} (0.7, 0.5, 0.6) & (0.2, 0.4, 0.2) \\ (0.3, 0.4, 0.2) & (0.4, 0.4, 0.2) \end{bmatrix}$ is not a PFM, SFM, but it A is a q-RPFM.

Each element in an PFM is expressed by an ordered pair $\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle$ with $\zeta_{a_{ij}}, \eta_{a_{ij}}$ and $\delta_{a_{ij}} \in [0,1]$ and $0 \leqslant \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leqslant 1$. It was clearly seen that $0.7 + 0.5 + 0.6 > 1$, and thus it could not be described by PFM. To describe such evaluation in this paper we have proposed q-rung picture fuzzy matrix (q-RPFM)and its algebraic operations. Each element in an q-RPFM is expressed by an ordered pair $\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle$ with $\zeta_{a_{ij}}, \eta_{a_{ij}}$ and $\delta_{a_{ij}} \in [0,1]$ and $0 \leqslant \zeta^q_{a_{ij}} + \eta^q_{a_{ij}} + \delta^q_{a_{ij}} \leqslant 1$. Also, we can get $(0.7)^q + (0.5)^q + (0.6)^q \leqslant 1$ $(q \geqslant 1)$, which is good enough to apply the q-RPFM to control it.

Definition 3.2. The q-rung picture fuzzy matrices A and B of the form, $A = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ and $B = \left(\left\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right)$. Then

- $A < B$ iff $\forall i, j, \zeta_{a_{ij}} \leqslant \zeta_{b_{ij}}, \eta_{a_{ij}} \leqslant \eta_{b_{ij}}$ or $\eta_{a_{ij}} \geqslant \eta_{b_{ij}}, \delta_{a_{ij}} \geqslant \delta_{b_{ij}}$
- $A^C = \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}} \right\rangle\right)$
- $A \vee_q B = \left(\left\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \right\rangle\right)$
- $A \wedge_q B = \left(\left\langle \min\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \right\rangle\right)$
- $A \oplus_q B = \left(\left\langle \sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} - \zeta^q_{a_{ij}} \zeta^q_{b_{ij}}}, \eta_{a_{ij}} \eta_{b_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \right\rangle\right)$
- $A \otimes_q B = \left(\left\langle \zeta_{a_{ij}} \zeta_{b_{ij}}, \sqrt[q]{\eta^q_{a_{ij}} + \eta^q_{b_{ij}} - \eta^q_{a_{ij}} \eta^q_{b_{ij}}}, \sqrt[q]{\delta^q_{a_{ij}} + \delta^q_{b_{ij}} - \delta^q_{a_{ij}} \delta^q_{b_{ij}}} \right\rangle\right)$.

Definition 3.3. The scalar multiplication operation over q-RPFM A and is defined by

$$nA = \left(\left\langle \sqrt[q]{1 - [1 - \zeta^q_{a_{ij}}]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right\rangle\right)$$

Definition 3.4. The exponentiation operation over q-RPFM A and is defined by

$$A^n = \left(\left\langle [\zeta_{a_{ij}}]^n, \sqrt[q]{1 - [1 - \eta^q_{a_{ij}}]^n}, \sqrt[q]{1 - [1 - \delta^q_{a_{ij}}]^n} \right\rangle\right).$$

Let $S_{m \times n}$ denote the set of all the q-rung picture fuzzy matrices. The following theorem relation between algebraic sum, and algebraic product of q-RPFMs.

Theorem 3.1. *For A, B q-rung picture fuzzy matrices, then $A \otimes_q B \leqslant A \oplus_q B$.*

PROOF. Let $A \oplus_q B = \left(\left\langle \sqrt[q]{\zeta_{a_{ij}}^q + \zeta_{b_{ij}}^q - \zeta_{a_{ij}}^q \zeta_{b_{ij}}^q}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$ and

$$A \otimes_q B = \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[q]{\eta_{a_{ij}}^q + \eta_{b_{ij}}^q - \eta_{a_{ij}}^q \eta_{b_{ij}}^q}, \sqrt[q]{\delta_{a_{ij}}^q + \delta_{b_{ij}}^q - \delta_{a_{ij}}^q \delta_{b_{ij}}^q} \right\rangle\right)$$

Assume that

$$\zeta_{a_{ij}}\zeta_{b_{ij}} \leqslant \sqrt[q]{\zeta_{a_{ij}}^q + \zeta_{b_{ij}}^q - \zeta_{a_{ij}}^q \zeta_{b_{ij}}^q}$$

i.e $$\zeta_{a_{ij}}\zeta_{b_{ij}} - \sqrt[q]{\zeta_{a_{ij}}^q + \zeta_{b_{ij}}^q - \zeta_{a_{ij}}^q \zeta_{b_{ij}}^q} \geqslant 0$$

i.e $$\zeta_{a_{ij}}^q(1 - \zeta_{b_{ij}}^q) + \zeta_{b_{ij}}^q(1 - \zeta_{a_{ij}}^q) \geqslant 0$$

which is true as $0 \leqslant \zeta_{a_{ij}}^q \leqslant 1$ and $0 \leqslant \zeta_{b_{ij}}^q \leqslant 1$ and

$$\eta_{a_{ij}}\eta_{b_{ij}} \leqslant \sqrt[q]{\eta_{a_{ij}}^q + \eta_{b_{ij}}^q - \eta_{a_{ij}}^q \eta_{b_{ij}}^q}$$

i.e $$\eta_{a_{ij}}\eta_{b_{ij}} - \sqrt[q]{\eta_{a_{ij}}^q + \eta_{b_{ij}}^q - \eta_{a_{ij}}^q \eta_{b_{ij}}^q} \geqslant 0$$

i.e $$\eta_{a_{ij}}^q(1 - \eta_{b_{ij}}^q) + \eta_{b_{ij}}^q(1 - \eta_{a_{ij}}^q) \geqslant 0$$

which is true as $0 \leqslant \eta_{a_{ij}}^q \leqslant 1$ and $0 \leqslant \eta_{b_{ij}}^q \leqslant 1$ and

$$\delta_{a_{ij}}\delta_{b_{ij}} \leqslant \sqrt[q]{\delta_{a_{ij}}^q + \delta_{b_{ij}}^q - \delta_{a_{ij}}^q \delta_{b_{ij}}^q}$$

i.e $$\delta_{a_{ij}}\delta_{b_{ij}} - \sqrt[q]{\delta_{a_{ij}}^q + \delta_{b_{ij}}^q - \delta_{a_{ij}}^q \delta_{b_{ij}}^q} \geqslant 0$$

i.e $$\delta_{a_{ij}}^q(1 - \delta_{b_{ij}}^q) + \delta_{b_{ij}}^q(1 - \delta_{a_{ij}}^q) \geqslant 0$$

which is true as $0 \leqslant \delta_{a_{ij}}^q \leqslant 1$ and $0 \leqslant \delta_{b_{ij}}^q \leqslant 1$. Hence $A \otimes_q B \leqslant A \oplus_q B$. □

THEOREM 3.2. *For any q-rung picture fuzzy matrix A, then*

(i) $A \oplus_q A \geqslant A$,

(ii) $A \otimes_q A \leqslant A$.

PROOF. (i) Let $A \oplus_q A = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \oplus_q (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$

$$A \oplus_q A = \left(\left\langle \sqrt[q]{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^q}, (\eta_{a_{ij}})^q, (\delta_{a_{ij}})^q \right\rangle\right)$$

$$\sqrt[q]{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^q} = \sqrt[q]{\zeta_{a_{ij}} + \zeta_{a_{ij}}(1 - \zeta_{a_{ij}})} \geqslant \zeta_{a_{ij}} \text{ for all } i,j$$

and $(\eta_{a_{ij}})^q \leqslant \eta_{a_{ij}}$ for all i,j and $(\delta_{a_{ij}})^q \leqslant \delta_{a_{ij}}$ for all i,j. Hence $A \oplus_q A \geqslant A$.
Similarly, we can prove that (ii) $A \otimes_q A \leqslant A$. □

THEOREM 3.3. *For A, B and C q-rung picture fuzzy matrices, then*

(i) $A \oplus_q B = B \oplus_q A$,

(ii) $A \otimes_q B = B \otimes_q B$,

(iii) $(A \oplus_q B) \oplus_q C = A \oplus_q (B \oplus_q C)$,

(iv) $(A \otimes_q B) \otimes_q C = A \otimes_q (B \otimes_q C)$.

PROOF. (i) $A \oplus_q B = \left(\left\langle \sqrt[q]{\zeta_{a_{ij}}^q + \zeta_{b_{ij}}^q - \zeta_{a_{ij}}^q \zeta_{b_{ij}}^q}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$

$$= \left(\left\langle \sqrt[q]{\zeta_{b_{ij}}^q + \zeta_{a_{ij}}^q - \zeta_{b_{ij}}^q \zeta_{a_{ij}}^q}, \eta_{b_{ij}}\eta_{a_{ij}}, \delta_{b_{ij}}\delta_{a_{ij}} \right\rangle\right)$$

$$= B \oplus_q A.$$

(ii) $A \otimes_q B = \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[q]{\eta^q_{a_{ij}} + \eta^q_{b_{ij}} - \eta^q_{a_{ij}}\eta^q_{b_{ij}}}, \sqrt[q]{\delta^q_{a_{ij}} + \delta^q_{b_{ij}} - \delta^q_{a_{ij}}\delta^q_{b_{ij}}}\right\rangle\right)$

$$= \left(\left\langle \zeta_{b_{ij}}\zeta_{a_{ij}}, \sqrt[q]{\eta^q_{b_{ij}} + \eta^q_{a_{ij}} - \eta^q_{b_{ij}}\eta^q_{a_{ij}}}, \sqrt[q]{\delta^q_{b_{ij}} + \delta^q_{a_{ij}} - \delta^q_{b_{ij}}\delta^q_{a_{ij}}}\right\rangle\right)$$

$$= B \otimes_q A.$$

(iii) $(A \oplus_q B)\oplus_q$ C

$$= \left(\left\langle\left(\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\right) \oplus_q \left(\zeta_{c_{ij}}, \eta_{c_{ij}}, \delta_{c_{ij}}\right)\right\rangle\right)$$

$$= \Big[\sqrt[q]{\left(\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}}\right)^q + \zeta^q_{c_{ij}} - \left(\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}}\right)^q \zeta^q_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Big]$$

$$= \Big[\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} + \zeta^q_{c_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}\zeta^q_{c_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{c_{ij}} - \zeta^q_{b_{ij}}\zeta^q_{c_{ij}} + \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}\zeta^q_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Big]$$

$$= \Big[\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} + \zeta^q_{c_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{c_{ij}} - \zeta^q_{b_{ij}}\zeta^q_{c_{ij}} + \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}\zeta^q_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Big]$$

$A \oplus_q (B \oplus_q C)$

$$= \Big[\sqrt[q]{\zeta^q_{a_{ij}} + \left(\sqrt[q]{\zeta^q_{b_{ij}} + \zeta^q_{c_{ij}} - \zeta^q_{b_{ij}}\zeta^q_{c_{ij}}}\right)^q - \zeta^q_{a_{ij}}\left(\sqrt[q]{\zeta^q_{b_{ij}} + \zeta^q_{c_{ij}} - \zeta^q_{b_{ij}}\zeta^q_{c_{ij}}}\right)^q},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Big]$$

$$= \Big[\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} + \zeta^q_{c_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{c_{ij}} - \zeta^q_{b_{ij}}\zeta^q_{c_{ij}} + \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}\zeta^q_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Big].$$

Hence $(A \oplus_q B) \oplus_q C = A \oplus_q (B \oplus_q C)$.

Similarly, we can prove that (iv)$(A \otimes_q B) \otimes_q C = A \otimes_q (B \otimes_q C)$. □

THEOREM 3.4. *For A, B q-rung picture fuzzy matrices, then*

(i) $A \oplus_q (A \otimes_q B) \geqslant A$,

(ii) $A \otimes_q (A \oplus_q B) \leqslant A$.

PROOF. (i) Let $A \oplus_q (A \otimes_q B)$

$$= \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}}\right\rangle\right) \oplus \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[q]{\eta^q_{a_{ij}} + \eta^q_{b_{ij}} - \eta^q_{a_{ij}}\eta^q_{b_{ij}}}, \sqrt[q]{\delta^q_{a_{ij}} + \delta^q_{b_{ij}} - \delta^q_{a_{ij}}\delta^q_{b_{ij}}}\right\rangle\right)$$

$$= \Big[\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{a_{ij}}\zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}[\zeta^q_{a_{ij}}\zeta^q_{b_{ij}}]}, \eta_{a_{ij}}[\sqrt[q]{\eta^q_{a_{ij}} + \eta^q_{b_{ij}} - \eta^q_{a_{ij}}\eta^q_{b_{ij}}}],$$
$$\delta_{a_{ij}}[\sqrt[q]{\delta^q_{a_{ij}} + \delta^q_{b_{ij}} - \delta^q_{a_{ij}}\delta^q_{b_{ij}}}]\Big]$$

$$= \Big[\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}[1 - \zeta^q_{a_{ij}}]}, \eta_{a_{ij}}\left(\sqrt[q]{1 - [1 - \eta^q_{a_{ij}}][1 - \eta^q_{b_{ij}}]}\right),$$

$$\delta_{a_{ij}} \left(\sqrt[q]{1 - [1 - \delta^q_{a_{ij}}][1 - \delta^q_{b_{ij}}]} \right) \Big] \geqslant A.$$

Hence $A \oplus_q (A \otimes_q B) \geqslant A$.

Similarly, we can prove that (ii) $A \otimes_q (A \oplus_q B) \leqslant A$. □

The following theorem is obvious.

THEOREM 3.5. *For A, B q-rung picture fuzzy matrices, then*

(i) $A \vee_q B = B \vee_q AE$,
(ii) $A \wedge_q B = B \wedge_q A$.

THEOREM 3.6. *For A, B q-rung picture fuzzy matrices, then*

(i) $A \oplus_q (B \vee_q C) = (A \oplus_q B) \vee_q (A \oplus_q C)$,
(ii) $A \otimes_q (B \vee_q C) = (A \otimes_q B) \vee_q (A \otimes_q C)$,
(iii) $A \oplus_q (B \wedge_q C) = (A \oplus_q B) \wedge_q (A \oplus_q C)$,
(iv) $A \otimes_q (B \wedge_q C) = (A \otimes_q B) \wedge_q (A \otimes_q C)$.

PROOF. In the following, we shall prove (i), and $(ii) - (iv)$ can be proved analogously.

(i) $A \oplus_q (B \vee_q C) =$

$$\Big[\sqrt[q]{\zeta^q_{a_{ij}} + \max\left(\zeta^q_{b_{ij}}, \zeta^q_{c_{ij}}\right) - \zeta^q_{a_{ij}} . \max\left(\zeta^q_{b_{ij}}, \zeta^q_{c_{ij}}\right)},$$

$$\eta_{a_{ij}} . \max\left(\eta_{b_{ij}}, \eta_{c_{ij}}\right), \delta_{a_{ij}} . \max\left(\delta_{b_{ij}}, \delta_{c_{ij}}\right) \Big]$$

$$= \Big[\sqrt[q]{\max\left(\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}}, \zeta_{a^q_{ij}} + \zeta^q_{c_{ij}}\right) - \max\left(\zeta^q_{a_{ij}}\zeta^q_{b_{ij}}, \zeta^q_{a_{ij}}\zeta^q_{c_{ij}}\right)},$$

$$\min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right) \Big]$$

$$= \Big[\sqrt[q]{\max\left(\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}, \zeta^q_{a_{ij}} + \zeta^q_{c_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{c_{ij}}\right)},$$

$$\min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right) \Big]$$

$$= (A \oplus_q B) \vee_q (A \oplus_q C).$$ □

THEOREM 3.7. *For A, B q-rung picture fuzzy matrices, then*

(i) $(A \wedge_q B) \oplus_q (A \vee_q B) = A \oplus_q B$,
(ii) $(A \wedge_q B) \otimes_q (A \vee_q B) = A \otimes_q B$,
(iii) $(A \oplus_q B) \wedge_q (A \otimes_q B) = A \otimes_q B$,
(iv) $(A \oplus_q B) \vee_q (A \otimes_q B) = A \oplus_q B$.

PROOF. In the following, we shall prove (i), and $(ii) - (iv)$ can be proved analogously.

(i) $(A \wedge_q B) \oplus_q (A \vee_q B)$

$$= \left[\sqrt[q]{\min\left(\zeta^q_{a_{ij}}, \zeta^q_{b_{ij}}\right) + \max\left(\zeta^q_{a_{ij}}, \zeta^q_{b_{ij}}\right) - \min\left(\zeta^q_{a_{ij}}, \zeta^q_{b_{ij}}\right) . \max\left(\zeta^q_{a_{ij}}, \zeta^q_{b_{ij}}\right)}, \right.$$

$$\max\left(\eta_{a_{ij}},\eta_{b_{ij}}\right).\min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right), \qquad \max\left(\delta_{a_{ij}},\delta_{b_{ij}}\right).\min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right)\Bigg]$$

$$= \left(\left\langle \sqrt[q]{\zeta_{a_{ij}}^q+\zeta_{b_{ij}}^q-\zeta_{a_{ij}}^q\zeta_{b_{ij}}^q}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\right\rangle\right)$$

$$= A \oplus_q B. \qquad \square$$

In the following theorems, the operator complement obey th De Morgan's laws for the operation $\oplus, \otimes, \vee_q, \wedge_q$.

Theorem 3.8. *For A, B q-rung picture fuzzy matrices, then*
(i) $(A \oplus_q B)^C = A^C \otimes_q B^C$,
(ii) $(A \otimes_q B)^C = A^C \oplus_q B^C$,
(iii) $(A \oplus_q B)^C \leqslant A^C \oplus_q B^C$,
(iv) $(A \otimes_q B)^C \geqslant A^C \otimes_q B^C$.

Proof. We shall prove $(iii), (iv)$, and $(i), (ii)$ are straightfforward.

(iii) $(A \oplus_q B)^C = \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[q]{\eta_{a_{ij}}^q+\eta_{b_{ij}}^q-\eta_{a_{ij}}^q\eta_{b_{ij}}^q}, \sqrt[q]{\zeta_{a_{ij}}^q+\zeta_{b_{ij}}^q-\zeta_{a_{ij}}^q\zeta_{b_{ij}}^q}\right\rangle\right)$.

$$A^C \oplus_q B^C = \left(\left\langle \sqrt[q]{\delta_{a_{ij}}^q+\delta_{b_{ij}}^q-\delta_{a_{ij}}^q\delta_{b_{ij}}^q}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}}\right\rangle\right).$$

Since $\delta_{a_{ij}}\delta_{b_{ij}} \leqslant \sqrt[q]{\delta_{a_{ij}}^q+\delta_{b_{ij}}^q-\delta_{a_{ij}}^q\delta_{b_{ij}}^q}$, we have $\sqrt[q]{\eta_{a_{ij}}^q+\eta_{b_{ij}}^q-\eta_{a_{ij}}^q\eta_{b_{ij}}^q} \geqslant \eta_{a_{ij}}\eta_{b_{ij}}$ and $\sqrt[q]{\zeta_{a_{ij}}^q+\zeta_{b_{ij}}^q-\zeta_{a_{ij}}^q\zeta_{b_{ij}}^q} \geqslant \zeta_{a_{ij}}\zeta_{b_{ij}}$. Hence $(A \oplus_q B)^C \leqslant A^C \oplus_q B^C$.

(iv) $(A \otimes_q B)^C = \left(\left\langle \sqrt[q]{\delta_{a_{ij}}^q+\delta_{b_{ij}}^q-\delta_{a_{ij}}^q\delta_{b_{ij}}^q}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}}\right\rangle\right)$.

$$A^C \otimes_q B^C = \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[q]{\eta_{a_{ij}}^q+\eta_{b_{ij}}^q-\eta_{a_{ij}}^q\eta_{b_{ij}}^q}, \sqrt[q]{\zeta_{a_{ij}}^q+\zeta_{b_{ij}}^q-\zeta_{a_{ij}}^q\zeta_{b_{ij}}^q}\right\rangle\right).$$

Since $\sqrt[q]{\delta_{a_{ij}}^q+\delta_{b_{ij}}^q-\delta_{a_{ij}}^q\delta_{b_{ij}}^q} \geqslant \delta_{a_{ij}}\delta_{b_{ij}}$, we have $\eta_{a_{ij}}\eta_{b_{ij}} \leqslant \sqrt[q]{\eta_{a_{ij}}^q+\eta_{b_{ij}}^q-\eta_{a_{ij}}^q\eta_{b_{ij}}^q}$ and $\zeta_{a_{ij}}\zeta_{b_{ij}} \leqslant \sqrt[q]{\zeta_{a_{ij}}^q+\zeta_{b_{ij}}^q-\zeta_{a_{ij}}^q\zeta_{b_{ij}}^q}$. Hence $(A \otimes_q B)^C \geqslant A^C \otimes_q B^C$. $\square$

Theorem 3.9. *For A, B q-rung picture fuzzy matrices, then*
(i) $(A^C)^C = A$,
(ii) $(A \vee_q B)^C = A^C \wedge_q B^C$,
(iii) $(A \wedge_q B)^C = A^C \vee_q B^C$.

Proof. (i) is obvious.
We shall prove (ii) only,

$$A \vee_q B = \left(\left\langle \max\left(\zeta_{a_{ij}},\zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right)\right\rangle\right)$$

$$(A \vee_q B)^C = \left(\left\langle \min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}},\zeta_{b_{ij}}\right)\right\rangle\right)$$

It implies $A^C = \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}}\right\rangle\right)$ and $B^C = \left(\left\langle \delta_{b_{ij}}, \eta_{b_{ij}}, \zeta_{b_{ij}}\right\rangle\right)$. from here it follows

$$A^C \wedge_q B^C = \left(\left\langle \min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}},\zeta_{b_{ij}}\right)\right\rangle\right).$$

Hence $(A \vee_q B)^C = A^C \wedge_q B^C$.

Similarly, we can prove that (iii) $(A \wedge_q B)^C = A^C \vee_q B^C$. $\square$

Based on the Definition 3.2, definition 3.3 and Definition 3.4., we shall next prove the algebraic properties of q-rung picture fuzzy matrices under the operations of scalar multiplication and exponentiation.

THEOREM 3.10. *For A, B q-rung picture fuzzy matrices, then $n > 0$,*

(i) $n(A \oplus_q B) = nA \oplus_q nB$, $n > 0$,

(ii) $n_1 A \oplus_q n_2 A = (n_1 + n_2)A$, $n_1, n_2 > 0$,

(iii) $(A \otimes_q B)^n = A^n \otimes_q B^n$, $n > 0$,

(iv) $A_1^n \otimes_q A_2^n = A^{(n_1+n_2)}$, $n_1, n_2 > 0$.

PROOF. For the two q-RPFMs A and B, and $n, n_1, n_2 > 0$, according to definition, we can obtain

(i) Let $n(A \oplus_q B)$

$$= n\left(\left\langle \sqrt[q]{\zeta_{a_{ij}}^q + \zeta_{b_{ij}}^q - \zeta_{a_{ij}}^q \zeta_{b_{ij}}^q}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$$

$$= \left(\left\langle \sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q]^n [1 - \zeta_{a_{ij}}^q]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$

$$= \left(\left\langle \sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q + \zeta_{b_{ij}}^q - \zeta_{a_{ij}}^q \zeta_{b_{ij}}^q]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$

$nA \oplus_q nB$

$$= \left(\left\langle \left(\sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right) \oplus_q \left(\sqrt[q]{1 - [1 - \zeta_{b_{ij}}^q]^n}, [\eta_{b_{ij}}]^n, [\delta_{b_{ij}}]^n \right) \right\rangle\right)$$

$$= \Big[\sqrt[q]{\left(1 - [1 - \zeta_{a_{ij}}^q]^n + 1 - [1 - \zeta_{a_{ij}}^q]^n\right) - \left(1 - [1 - \zeta_{a_{ij}}^q]^n\right)\left(1 - [1 - \zeta_{b_{ij}}^q]^n\right)},$$
$$[\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \Big]$$

$$= \left(\left\langle \sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q]^n [1 - \zeta_{b_{ij}}^q]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$

$$= \left(\left\langle \sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q + \zeta_{b_{ij}}^q - \zeta_{a_{ij}}^q \zeta_{b_{ij}}^q]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$

$= n(A \oplus_q B)$.

(ii) $n_1 A \oplus_q n_2 B$

$$= \Big[\left(\sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q]^{n_1}}, [\eta_{a_{ij}}]^{n_1}, [\delta_{a_{ij}}]^{n_1} \right) \oplus_q$$
$$\left(\sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q]^{n_2}}, [\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_2} \right) \Big]$$

$$= \Big[\sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q]^{n_1} + 1 - [1 - \zeta_{a_{ij}}^q]^{n_2} - \left(1 - [1 - \zeta_{a_{ij}}^q]^{n_1}\right)\left(1 - [1 - \zeta_{a_{ij}}^q]^{n_2}\right)},$$
$$[\eta_{a_{ij}}]^{n_1}[\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_1}[\delta_{a_{ij}}]^{n_2} \Big]$$

$$= \left(\left\langle \sqrt[q]{1 - [1 - \zeta_{a_{ij}}^q]^{n_1+n_2}}, [\eta_{a_{ij}}]^{n_1+n_2}, [\delta_{a_{ij}}]^{n_1+n_2} \right\rangle\right)$$

$= (n_1 + n_2)A$.

(iii) $$(A \otimes_q B)^n = \Big[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[q]{1 - [1 - \eta_{a_{ij}}^q + \eta_{b_{ij}}^q - \eta_{a_{ij}}^q \eta_{b_{ij}}^q]^n},$$
$$\sqrt[q]{1 - [1 - \delta_{a_{ij}}^q + \delta_{b_{ij}}^q - \delta_{a_{ij}}^q \delta_{b_{ij}}^q]^n} \Big]$$

$$= \Big[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[q]{1 - [1 - \eta_{a_{ij}}^q]^n [1 - \eta_{b_{ij}}^q]^n, 1 - [1 - \delta_{a_{ij}}^q]^n [1 - \delta_{b_{ij}}^q]^n} \Big]$$

$$A^n \otimes_q B^n = \Big[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n,$$

$\sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n+1-[1-\eta^q_{b_{ij}}]^n-\left(1-[1-\eta^q_{a_{ij}}]^n\right)\left(1-[1-\eta^q_{b_{ij}}]^n\right)},$

$\sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n+1-[1-\delta^q_{b_{ij}}]^n-\left(1-[1-\delta^q_{a_{ij}}]^n\right)\left(1-[1-\delta^q_{b_{ij}}]^n\right)}\Big]$

$=\left(\left\langle\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n[1-\eta^q_{b_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n[1-\delta^q_{b_{ij}}]^n}\right\rangle\right)$

$=(A\otimes_q B)^n.$

(iv) $A^{n_1}\otimes_q A^{n_2}=\Big[\left(\zeta_{a_{ij}}\right)^{n_1+n_2},$

$\sqrt[q]{1-[1-\eta^q_{a_{ij}}]^{n_1}+1-[1-\eta^q_{a_{ij}}]^{n_2}-\left(1-[1-\eta^q_{a_{ij}}]^{n_1}\right)\left(1-[1-\eta^q_{a_{ij}}]^{n-2}\right)},$

$\sqrt[q]{1-[1-\delta^q_{a_{ij}}]^{n_1}+1-[1-\delta^q_{a_{ij}}]^{n-2}-\left(1-[1-\delta^q_{a_{ij}}]^{n_1}\right)\left(1-[1-\delta^q_{a_{ij}}]^{n_2}\right)}\ \Big]$

$=\left(\left\langle\left(\zeta_{a_{ij}}\right)^{n_1+n_2}, \sqrt[q]{1-[1-\eta^q_{a_{ij}}]^{n_1+n_2}}, \sqrt[q]{1-[1-\delta^q_{a_{ij}}]^{n_1+n_2}}\right\rangle\right)$

$=A^{(n_1+n_2)}.$

Hence proved. □

THEOREM 3.11. *For A,B q-rung picture fuzzy matrices, then $n>0$,*

(i) $nA\leqslant nB$,

(ii) $A^n\leqslant B^n$.

PROOF. (i) Let $A\leqslant B$. Then $\zeta_{a_{ij}}\leqslant\zeta_{b_{ij}}$ and $\eta_{a_{ij}}\geqslant\eta_{b_{ij}}$ and $\delta_{a_{ij}}\geqslant\delta_{b_{ij}}$ for all i,j. Further on $\sqrt[q]{1-[1-\zeta^q_{a_{ij}}]^n}\leqslant\sqrt[q]{1-[1-\zeta^q_{b_{ij}}]^n}$, $[\eta_{a_{ij}}]^n\geqslant[\eta_{b_{ij}}]^n$ and $[\delta_{a_{ij}}]^n\geqslant[\delta_{b_{ij}}]^n$. for all i,j.

(ii) Also, we have $[\zeta_{a_{ij}}]^n\geqslant[\zeta_{b_{ij}}]^n$ and $\sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n}\leqslant\sqrt[q]{1-[1-\eta^q_{b_{ij}}]^n}$ and $\sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n}\leqslant\sqrt[q]{1-[1-\delta^q_{b_{ij}}]^n}$, for all i,j. □

THEOREM 3.12. *For A,B q-rung picture fuzzy matrices, for $n>0$, then:*

(i) $n(A\wedge_q B)=nA\wedge_q nB$,

(ii) $n(A\vee_q B)=nA\vee_q nB$.

PROOF. (i) $n(A\wedge_q B)$

$=\Big[\sqrt[q]{1-[1-\min\left(\zeta^q_{a_{ij}},\zeta^q_{b_{ij}}\right)]^n}, \max\left([\eta_{a_{ij}}]^n,[\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n,[\delta_{b_{ij}}]^n\right)\Big]$

$=\Big[\sqrt[q]{1-[\max\left(1-\zeta^q_{a_{ij}},1-\zeta^q_{b_{ij}}\right)]^n}, \max\left([\eta_{a_{ij}}]^n,[\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n,[\delta_{b_{ij}}]^n\right)\Big]$

$=\Big[\sqrt[q]{1-\left(\max\left([1-\zeta^q_{a_{ij}}]^n,[1-\zeta^q_{b_{ij}}]^n\right)\right)}, \max\left([\eta_{a_{ij}}]^n,[\eta_{b_{ij}}]^n\right),$
$\max\left([\delta_{a_{ij}}]^n,[\delta_{b_{ij}}]^n\right)\Big]$

$=\Big[\max\left(\sqrt[q]{1-[1-\zeta^q_{a_{ij}}]^n}, \sqrt[q]{1-[1-\zeta^q_{b_{ij}}]^n}\right),$
$\max\left([\eta_{a_{ij}}]^n,[\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n,[\delta_{b_{ij}}]^n\right)\Big]$

$=nA\wedge_q nB$. Hence $n(A\wedge_q B)=nA\wedge_q nB$,

Similarly, we can prove that (ii) $n(A\vee_q B)=nA\vee_q nB$. □

THEOREM 3.13. *For A, B q-rung picture fuzzy matrices, for $n > 0$ then:*
(i) $(A \wedge_q B)^n = A^n \wedge_q B^n$,
(ii) $(A \vee_q B)^n = A^n \vee_q B^n$.

PROOF. (i) $(A \wedge_q B)^n$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \sqrt[q]{1 - [\max\left(1-\eta^q_{a_{ij}}, 1-\eta^q_{b_{ij}}\right)]^n}, \sqrt[q]{1 - [\max\left(1-\delta^q_{a_{ij}}, 1-\delta^q_{b_{ij}}\right)]^n} \Big]$$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \sqrt[q]{1 - \left(\min\left([1-\eta^q_{a_{ij}}]^n, [1-\eta^q_{b_{ij}}]^n\right)\right)}, \sqrt[q]{1 - \left(\min\left([1-\delta^q_{a_{ij}}]^n, [1-\delta^q_{b_{ij}}]^n\right)\right)} \Big]$$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(\sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n}, \sqrt[q]{1-[1-\eta^q_{b_{ij}}]^n}\right), \max\left(\sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{b_{ij}}]^n}\right) \Big]$$

$A^n \wedge_q B^n$

$$= \Big[\left([\zeta_{a_{ij}}]^n, \sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n}\right) \bigwedge \left([\zeta_{b_{ij}}]^n, \sqrt[q]{1-[1-\eta^q_{b_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{b_{ij}}]^n}\right) \Big]$$

$$= \Big[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(\sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n}, \sqrt[q]{1-[1-\eta^q_{b_{ij}}]^n}\right), \max\left(\sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{b_{ij}}]^n}\right) \Big]$$

$= (A \wedge_q B)^n$.
Hence $(A \wedge_q B)^n = A^n \wedge_q B^n$,
Similarly, we can prove that (ii) $(A \vee_q B)^n = A^n \vee_q B^n$. □

THEOREM 3.14. *For A, B q-rung picture fuzzy matrices, $n > 0$, then*

$$(A \oplus_q B)^n \neq A^n \oplus_q B^n.$$

PROOF. Let $(A \oplus_q B)^n$

$$= \Big[\left(\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}}\right)^n, \sqrt[q]{1-[1-\eta^q_{a_{ij}}\eta^q_{b_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{a_{ij}}\delta^q_{b_{ij}}]^n} \Big]$$

$$A^n = \left(\left\langle [\zeta_{a_{ij}}]^n, \sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n} \right\rangle\right)$$

$$B^n = \left(\left\langle [\zeta_{b_{ij}}]^n, \sqrt[q]{1-[1-\eta^q_{b_{ij}}]^n}, \sqrt[q]{1-[1-\delta^q_{b_{ij}}]^n} \right\rangle\right)$$

$A^n \oplus_q B^n$

$$= \Big[\sqrt[q]{[\zeta^n_{a_{ij}}]^q + [\zeta^n_{b_{ij}}]^q - [\zeta^n_{a_{ij}}]^q[\zeta^n_{b_{ij}}]^q}, \left(\sqrt[q]{1-[1-\eta^q_{a_{ij}}]^n}\right)^n . \left(\sqrt[q]{1-[1-\eta^q_{b_{ij}}]^n}\right)^n, \left(\sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n}\right)^n . \left(\sqrt[q]{1-[1-\delta^q_{a_{ij}}]^n}\right)^n \Big]$$

Hence $(A \oplus_q B)^n \neq A^n \oplus_q B^n$. □

4. New operation (@) on q-rung picture fuzzy matrices

In this section, we define a new operation(@) on q-rung picture fuzzy matrices and proved their algebraic properties. Further, we discuss the Distributive laws in the case where the operations of $\oplus, \otimes, \vee_q$ and $\wedge_q$ combined each other.

DEFINITION 4.1. A q-rung picture fuzzy matrices A and B of the form, $A = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ and $B = (\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \rangle)$. Then

$$A@B = \left(\left\langle \sqrt[q]{\frac{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}}}{2}}, \sqrt[q]{\frac{\eta^q_{a_{ij}} + \eta^q_{b_{ij}}}{2}}, \sqrt[q]{\frac{\delta^q_{a_{ij}} + \delta^q_{b_{ij}}}{2}} \right\rangle \right).$$

REMARK 4.1. Obviously, for every two q-rung picture fuzzy matrices A and B, then $A@B$ is a q-rung picture fuzzy matrix. Simple illustration given: For $A@B$,

$$0 \leqslant \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2} + \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2} + \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2} \text{ and}$$

$$\leqslant \frac{\zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}}}{2} + \frac{\zeta_{b_{ij}} + \eta_{b_{ij}} + \delta_{b_{ij}}}{2} \leqslant \frac{1}{2} + \frac{1}{2} = 1.$$

THEOREM 4.1. *For any q-rung picture fuzzy matrix A, then $A@A = A$.*

PROOF. $A@A = \left(\left\langle \sqrt[q]{\frac{\zeta^q_{a_{ij}} + \zeta^q_{a_{ij}}}{2}}, \sqrt[q]{\frac{\eta^q_{a_{ij}} + \eta^q_{a_{ij}}}{2}}, \sqrt[q]{\frac{\delta^q_{a_{ij}} + \delta^q_{a_{ij}}}{2}} \right\rangle \right)$

$= \left(\left\langle \left(\sqrt[q]{\frac{\zeta^q_{a_{ij}} + \zeta^q_{a_{ij}}}{2}} \right)^q, \left(\sqrt[q]{\frac{\eta^q_{a_{ij}} + \eta^q_{a_{ij}}}{2}} \right)^q, \left(\sqrt[q]{\frac{\delta^q_{a_{ij}} + \delta^q_{a_{ij}}}{2}} \right)^q \right\rangle \right)$

$= \left(\left\langle \frac{2\zeta^q_{a_{ij}}}{2}, \frac{2\eta^q_{a_{ij}}}{2}, \frac{2\delta^q_{a_{ij}}}{2} \right\rangle \right)$

$= (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$. Since $\zeta^q_{a_{ij}} \leqslant \zeta_{a_{ij}}, \eta^q_{a_{ij}} \leqslant \eta_{a_{ij}}, \delta^q_{a_{ij}} \leqslant \delta_{a_{ij}}$

$= A.$ □

REMARK 4.2. For $a, b \in [0,1]$,then $ab \leqslant \frac{a+b}{2}, \frac{a+b}{2} \leqslant a + b - ab$.

THEOREM 4.2. *For A, B q-rung picture fuzzy matrices, then*
(i) $(A \oplus_q B) \vee_q (A@B) = A \oplus_q B$,
(ii) $(A \otimes_q B) \wedge_q (A@B) = A \otimes_q B$,
(iii) $(A \oplus_q B) \wedge_q (A@B) = A@B$,
(iv) $(A \otimes_q B) \vee_q (A@B) = A@B$.

PROOF. we shall prove (i) and (iii), (ii) and (iv) can be proved analogously.
(i) $(A \oplus_q B) \vee_q (A@B)$

$$= \left[\max\left(\sqrt[q]{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}} - \zeta^q_{a_{ij}}\zeta^q_{b_{ij}}}, \sqrt[q]{\frac{\zeta^q_{a_{ij}} + \zeta^q_{b_{ij}}}{2}} \right), \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt[q]{\frac{\eta^q_{a_{ij}} + \eta^q_{b_{ij}}}{2}} \right), \right.$$

$$\min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[q]{\frac{\delta^q_{a_{ij}}+\delta^q_{b_{ij}}}{2}}\right)\Bigg]$$

$$=\left(\left\langle \sqrt[q]{\zeta^q_{a_{ij}}+\zeta^q_{b_{ij}}-\zeta^q_{a_{ij}}\zeta^q_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\right\rangle\right)$$

$= A \oplus_q B.$

(iii) $(A \oplus_q B) \wedge_q (A@B)$

$$=\Bigg[\min\left(\sqrt[q]{\zeta^q_{a_{ij}}+\zeta^q_{b_{ij}}-\zeta^q_{a_{ij}}\zeta^q_{b_{ij}}}, \sqrt[q]{\frac{\zeta^q_{a_{ij}}+\zeta^q_{b_{ij}}}{2}}\right), \max\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt[q]{\frac{\eta^q_{a_{ij}}+\eta^q_{b_{ij}}}{2}}\right),$$

$$\max\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[q]{\frac{\delta^q_{a_{ij}}+\delta^q_{b_{ij}}}{2}}\right)\Bigg]$$

$$=\left(\left\langle \sqrt[q]{\frac{\zeta^q_{a_{ij}}+\zeta^q_{b_{ij}}}{2}}, \sqrt[q]{\frac{\eta^q_{a_{ij}}+\eta^q_{b_{ij}}}{2}}, \sqrt[q]{\frac{\delta^q_{a_{ij}}+\delta^q_{b_{ij}}}{2}}\right\rangle\right) = A@B. \qquad \square$$

REMARK 4.3. The q-rung picture fuzzy matrix forms a semi-lattice, associativity, commutativity, idempotent under the q-rung picture fuzzy matrix operation of algebraic sum and algebraic product. The distributive law also holds for $\oplus_q, \otimes_q$ and $\wedge_q, \vee_q, @$ are combined each other.

5. Conclusion

In this paper, q-rung picture fuzzy matrices and its algebraic operations are defined. Then some properties, such as idempotent, commutativity, associativity, absorption law, distributive, De Morgan's laws over complement are proved. Finally, we have defined a new operation(@) on q-rung picture fuzzy matrices and discussed distributive laws in the case where the operations of $\oplus_q, \otimes_q, \wedge_q$ and $\vee_q$ are combined each other. This result can be applied further application of q-rung picture fuzzy matrix theory. For the development of q-rung picture fuzzy semi-lattice and its algebraic property the results of this paper would be helpful. In the future, the application of the proposed aggregating operators of q-RPFMs needs to be explored in the decision making, risk analysis and many other uncertain and fuzzy environment.

Acknowledgement. The authors would like to thank the reviewers for their valuable suggestions and comments to improve the presentation of this paper.

References

[1] B. C. Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, **30**(4)(2014), 409–420.

[2] S. Dogra and M. Pal. Picture fuzzy matrix and its application. *Soft Comput*, **24**(13)(2020), 9413–9428.

[3] Y. B. Im and E. B. Lee. The determinant of square intuitionistic fuzzy matrices. *Far East J. Math. Sci. (FJMS)*, **3**(5)(2001), 789–796.

[4] S. K. Khan and M. Pal. Some operations on intuitionistic fuzzy matrices. *Acta Cienc. Indica, Math.*, **32**(2)(2006), 515–524.

[5] S. K. Khan and A. Pal. The Generalised inverse of intuitionistic fuzzy matrices. *Journal of Physical Sciences*, **11**(2007), 62-67.
[6] L. Li, R. Zhang, J. Wang, X. Shang and K. Bai. A novel approach to multi-attribute group decision-making with q-rung picture linguistic information. *Symmetry,* **10**(3)(2018), 172.
[7] S. Mondal and M. Pal. Similarity relations, invertibility and eigenvalues of intutionistic fuzzy matrix. *Fuzzy Inf. Eng.*, **5**(4)(2013b), 431–443.
[8] T. Muthuraji, S. Sriram and P. Murugadas. Decomposition of intuitionistic fuzzy matrices. *Fuzzy Inf. Eng.*, **8**(3)(2016), 345–354.
[9] T. Muthuraji and K. Lalitha. Some algebraic structures on max-max, min-min compositions over intuitionistic fuzzy matrices. *Adv. Math., Sci. J.*, **9**(8)(2020), 5683-5691.
[10] M. Pal. Intuitionistic fuzzy determinant. *V. U. J. Phys Sci*, **7**(2001), 87-93.
[11] M. Pal, S. K. Khan and A. K. Shyamal. Intuitionistic fuzzy matrices. *Notes IFS*, **8**(2)(2002), 51–62.
[12] R. Pradhan and M. Pal. The generalized inverse of atanassovs intuitionistic fuzzy matrices. *International Journal of Computational Intelligence Systems*, **7**(6)(2014), 1083-1095.
[13] M. Pal. Fuzzy matrices with fuzzy rows and columns. *J. Intell. Fuzzy Syst.*, **30**(1)(2015a), 561-573.
[14] M. Pal. Interval valued fuzzy matrices with interval valued rows and columns. *Fuzzy Inf. Eng.*, **7**(3)(2015), 335-368.
[15] M. Pal and S. Mondal. Bipolar fuzzy matrices. *Soft Comput*, **23**(20)(2019), 9885-9897.
[16] R. Pradhan and M. Pal. Triangular fuzzy matrix norm and its properties. *J. Fuzzy Math.*, **25**(4)(2017), 823-834. ????
[17] I. Silambarasan and S. Sriram. Hamacher operations of intuitionistic fuzzy matrices. *Annals of Pure and Applied Mathematics*, **16** (1)(2018), 81–90.
[18] I. Silambarasan and S. Sriram. Algebraic operations on Pythagorean fuzzy matrices. *Mathematical Sciences International Research Journal*, **7**(2)(2018), 406–418.
[19] I. Silambarasan and S. Sriram. New operations for Pythagorean fuzzy matrices. *Indian Journal of Science and Technology*, **12**(20)(2019a), 1–7.
[20] I. Silambarasan and S. Sriram. Hamacher operations on Pythagorean fuzzy matrices. *Journal of Applied Mathematics and Compuational Mechanics*, **18**(3)(2019b), 69–78.
[21] I. Silambarasan and S. Sriram. Some operations over Pythagorean fuzzy matrices based on Hamacher operations. *Appl. Appl. Math.*, **15**(1)(2020a), 353–371.
[22] I. Silambarasan. Hamacher operations of Fermatean fuzzy matrices. *Appl. Appl. Math.* , **16**(1)(2021), 289–319.
[23] I. Silambarasan. Some algebraic properties of picture fuzzy sets. *Bull. Int. Math. Virtual Inst.*, **11**(3)(2021), 429–442.
[24] I. Silambarasan. Some algebraic structures of picture fuzzy matrices. *World Scientific News*, **150**(2020), 78–91.
[25] I. Silambarasan. (2020c). Spherical fuzzy matrices. *TWMS J. of Apl. & Eng. Math.* Accepted for publication.
[26] I. Silambarasan. (2021). Generalized orthopair fuzzy matrices. *Open Journal of Mathematical Sciences*. Accepted for publication.
[27] M. G. Thomason. Convergence of powers of fuzzy matrix. *J. Math. Anal. Appl.*, **57**(2)(1977), 476–480.

Received by editors 25.04.2021; Revised version 05.07.2021; Available online 12.07.2021.

Department of Mathematics, Sri Manakula Vinayagar Engineering College
Madagadipet, Puducherry-605107, India
E-mail address: sksimbuking@gmail.com