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A NEW GENERALIZATION OF TRIBONACCI HYBRINOMIALS

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Abstract. In this study, we describe the (p, q, r)-tribonacci hybrinomial sequence which is established the generalization of the tribonacci sequence. Also, the (p, q, r)-tribonacci polynomial sequence are studied in the same parallel with the (p, q, r) - tribonacci hybrinomial. Special cases of the (p, q, r)-tribonacci hybrinomial sequence are obtained. We present Binet-like formula, and also generating function of these sequences. In addition we give some identities such as Catalan identity, Cassini identity, d'Ocagne identity for these polynomial sequences.

1. Introduction

Let p and q be non–zero integers such that $D = p^2 - 4q \neq 0$. U_n and V_n are

$$U_n = U_n(p, q) = pU_{n-1} - qU_{n-2} \ , \ V_n = V_n(p, q) = pV_{n-1} - qV_{n-2} \tag{1.1}$$

for $n \geqslant 2$ with initial values $U_0 = 0, U_1 = 1, V_0 = 2$ and $V_1 = p$. The sequences U_n and V_n are named the first Lucas sequences and companion Lucas sequence with parameters p and q respectively. The characteristic equation of U_n and V_n is $x^2 - px + q = 0$ and hence the roots of it are $x_1 = \frac{p+\sqrt{D}}{2}$ and $x_2 = \frac{p-\sqrt{D}}{2}$. So their Binet formulas are hence $U_n = \frac{x_1^n - x_2^n}{x_1 - x_2}$ and $V_n = x_1^n + x_2^n$ for $n \geqslant 0$. The generating functions of U_n and V_n are $U(x) = \frac{x}{1-px+qx^2}$ and $V(x) = \frac{2-px}{1-px+qx^2}$. Fibonacci, Lucas, Pell and Pell–Lucas numbers can be derived. Indeed for $p = 1$ and $q = -1$, the numbers $U_n = U_n(1, -1)$ are named the Fibonacci numbers, while the numbers $V_n = V_n(1, -1)$ are named the Lucas numbers. Similarly, for $p = 2$ and $q = -1$,

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the numbers $U_n = U_n(2,-1)$ are named the Pell numbers, while the numbers $V_n = V_n(2,-1)$ are called the Pell–Lucas numbers (for further details see [**4, 11**]).

After Fibonacci numbers, there are numerous studies on tribonacci numbers, which are another research topic. A few studies on the subject are (see [**22, 13, 9, 23**]). Recurrence relation of the Tribonacci numbers is

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}$$

for $n \geqslant 4$, with beginning conditions $T_1 = 1, T_2 = 1, T_3 = 2$. A few tribonacci numbers are $1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \cdots$. The $n-th$ tribonacci number can be given by Binet formula

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

where α, β, γ are three roots of the characteristic polynomial $P(x) = x^3 - x^2 - x - 1$. The generating functions for the Tribonacci sequence are

$$\sum_{n=0}^{\infty} T_n x^n = \frac{x}{1-x-x^2-x^3} \text{ (see [22])}.$$

Polynomials can be defined by Fibonacci-like recurrence relations are called Fibonacci polynomials. Later there have been different mathematicians studying Fibonacci polynomials, such as P. F. Byrd, M. Bicknell-Johnson, Hoggatt and Bicknell (see [**2**]) considered tribonacci polynomials $T_n(x)$ are defined by the recurrence relation

$$T_{n+2}(x) = x^2 T_{n+1}(x) + x T_n(x) + T_{n-1}(x) \tag{1.2}$$

for $n \geqslant 1$ with initial conditions $T_0(x) = 0, T_1(x) = 1, T_2(x) = x^2$. Also continuation of the terms are $T_3(x) = x^4 + x,\ T_4(x) = x^6 + 2x^3 + 1,\ T_5(x) = x^8 + 3x^5 + 3x^2,\ T_6(x) = x^{10} + 4x^7 + 6x^4 + 2x,\ T_7(x) = x^{12} + 5x^9 + 10x^6 + 7x^3 + 1,\ T_8(x) = x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2, \cdots$.

In ([**12**]), in another generalization is generalized tribonacci polynomials which are defined by recurrence relation

$$T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x)$$

for $n \geqslant 4$ where b_2, c_1, c_4 positive integers and others parameters are nonnegative integers with initial conditions

$$T_1(x) = a,\ T_2(x) = b_2 x^2 + b_1 x + b_0,\ T_3(x) = c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$

In [**15**], the (p,q,r)-tribonacci polynomials which is a generalization that includes other generalizations. For example $T_n(x)$ were extended to the (p,q,r)-tribonacci polynomials

$$TP_n(x) = p(x) TP_{n-1}(x) + q(x) TP_{n-2}(x) + r(x) TP_{n-3}(x)$$

for $n \geqslant 3$, with initial values $TP_0(x) = 0,\ TP_1(x) = F_{p,q,1}(x)$, and $TP_2(x) = F_{p,q,2}(x)$ where $p(x)$, $q(x)$, $r(x)$ are non-zero polynomials with real coefficients and $F_{p,q,n}(x)$ are the $(p,q)-$Fibonacci polynomials (see [**6**]) which is defined for $n \geqslant 2$ by

$$F_{p,q,n}(x) = p(x) F_{p,q,n-1}(x) + q(x) F_{p,q,n-2}(x)$$

with $F_{p,q,0}(x) = 0,\ F_{p,q,1}(x) = 1$. A few terms are

$TP_0(x) = 0,\ TP_1(x) = 1,\ TP_2(x) = p(x),\ TP_3(x) = p^2(x) + q(x),$

$TP_4(x) = p^3(x) + 2p(x)q(x) + r(x),$

$TP_5(x) = p^4(x) + 3p^2(x)q(x) + q^2(x) + 2p(x)r(x),\ \cdots.$

In [**8**], Özdemir characterized the hybrid numbers as a generalization of complex, hyperbolic and dual numbers.

The set H of hybrid numbers z, has the form

$$H = \{z = a + bi + c\varepsilon + dh;\ a, b, c, d \in \mathbb{R}\} \tag{1.3}$$

where i, ε, h are operators such that $i^2 = -1,\ \varepsilon^2 = 0,\ ih = -hi = \varepsilon + i$. For more information about these operators see [**8**]. The conjugate of hybrid number z is defined by

$$\overline{z} = \overline{a + bi + c\varepsilon + dh} = a - bi - c\varepsilon - dh.$$

The real number $C(z) = z\overline{z} = \overline{z}z = a^2 + b^2 - 2bc - d^2$ is called the character of the hybrid number z and the real number $\sqrt{|C(z)|}$ will be called the norm of the hybrid number z and it will be denoted by $\|z\|$ (See [**8**]).

In [**18**], Fibonacci hybrit sequence is defined first time. Jacobsthal and Jacobsthal-Lucas hybrid numbers are introduced by Liana and Wloch [**19**] and also they investigated some of their properties. In [**17, 14**], the authors have refered the different properties of the Horadam hybrid numbers. In [**1**], modified $k-$Pell hybrid sequence is defined and also some identities are obtained. In addition Polatlı in [**10**], defined hybrid numbers with Fibonacci and Lucas hybrid number coefficients. In [**16**], Soykan and Taşdemir introduced the generalized Tetranacci hybrid numbers and presented some combinatorial properties of these hybrid numbers.

In [**20**], Fibonacci and Lucas hybrinomials, which can be investigated that the Fibonacci and Lucas hybrid numbers. The Fibonacci and Lucas hybrinomials are defined by

$$FH_n(x) = F_n(x) + iF_{n+1}(x) + \varepsilon F_{n+2}(x) + hF_{n+3}(x)$$

for $n \geqslant 0$ and

$$LH_n(x) = L_n(x) + iL_{n+1}(x) + \varepsilon L_{n+2}(x) + hL_{n+3}(x)$$

for $n \geqslant 0$ where $F_n(x)$ is the $n - th$ Fibonacci polynomial, and $L_n(x)$ is the $n - th$ Lucas polynomial. In [**3**], the author generalized the recurrence relations of the second type of hybrinomials. In [**7**], Wloch introduced Pell hybrinomial sequences and presented results about the sequence. Also, the same authors A. Szynal-Liana and I. Włoch considered generalized Fibonacci-Pell hybrinomials in [**21**]. In this paper, we introduced the $(p, q, r)-$tribonacci polynomials and $(p, q, r)-$tribonacci hybrinomials. We obtained Binet-like formula, generating function for these polynomial sequences. Also, we obtained some identities and summation formulas about the $(p, q, r)-$tribonacci hybrinomial sequence.

2. Main Results of the $(p,q,r)-$Tribonacci Polynomials and $(p,q,r)-$Tribonacci Hybrinomials

In this section we define (p,q,r) - tribonacci hybrinomials. We prove some theorems and give some identities for these sequences.

Definition 2.1. The $(p,q,r)-$tribonacci hybrinomial sequence denoted by $\{HT_n(x)\}$ is defined by

$$HT_n(x) = TP_n(x) + TP_{n+1}(x)i + TP_{n+2}(x)\varepsilon + TP_{n+3}(x)h, \tag{2.1}$$

where $\{TP_n(x)\}$ is the $(p,q,r)-$tribonacci polynomial sequence. Thus the initial values of the $(p,q,r)-$tribonacci hybrinomial sequence are:

$$\begin{aligned}
HT_0(x) &= 0 + i + p(x)\varepsilon + (p^2(x)+q(x))h\\
HT_1(x) &= 1 + p(x)i + (p^2(x)+q(x))\varepsilon + (p^3(x)+2p(x)q(x)+r(x))h\\
HT_2(x) &= p(x) + (p^2(x)+q(x))i + (p^3(x)+2p(x)q(x)+r(x))\varepsilon\\
&\quad +(p^4(x)+3p^2(x)q(x)+q^2(x)+2p(x)r(x))h\\
HT_3(x) &= (p^2(x)+q(x)) + (p^3(x)+2p(x)q(x)+r(x))i\\
&\quad +(p^4(x)+3p^2(x)q(x)+q^2(x)+2p(x)r(x))\varepsilon\\
&\quad +(p^5(x)+4p^3(x)q(x)+3p(x)q^2(x)+3p^2(x)r(x)+2q(x)r(x))h.
\end{aligned}$$

Now we will give some special cases of the $(p,q,r)-$tribonacci hybrinomial sequence. Explicit examples involving Tribonacci, Padovan, Narayana, third order Jacobsthal polynomials are stated to highlight the results.

Corollary 2.1. *In (2.1) for special choices of $p(x), q(x)$ and $r(x)$ and initial conditions, the following cases can be obtained.*

(1) *If $p(x) = x^2$, $q(x) = x$, $r(x) = 1$ are selected, Tribonacci polynomials (1.2) are got with $T_0(x) = 0$, $T_1(x) = 1$, $T_2(x) = x^2$ initial conditions.*
(2) *If $p(x) = 0$, $q(x) = x$, $r(x) = 1$ are selected, Padovan polynomials*

$$P_n(x) = xP_{n-2}(x) + P_{n-3}(x)$$

with $P_0(x) = 0$, $P_1(x) = 1$, $P_2(x) = 0$ initial conditions.
(3) *If $p(x) = x$, $q(x) = 0$, $r(x) = 1$ are selected, Narayana polynomials*

$$N_n(x) = xN_{n-1}(x) + N_{n-3}(x)$$

with $N_0(x) = 0$, $N_1(x) = 1$, $N_2(x) = x^2$ initial conditions.
(4) *If $p(x) = 1$, $q(x) = x$, $r(x) = 2x^2$ are selected, third order Jacobsthal polynomials*

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) + 2x^2J_{n-3}(x)$$

with $J_0(x) = 0$, $J_1(x) = 1$, $J_2(x) = 1$ initial conditions.

Now let's find the recurrence relation we most need to be able to prove in theorems.

LEMMA 2.1. *Let* $\{HT_n(x)\}$ *is the* (p,q,r)*-tribonacci hybrinomial sequence and* $n \geqslant 3$ *be an integer. Then the recurrence relation of* $\{HT_n(x)\}$ *is*

$$HT_n(x) = p(x)HT_{n-1}(x) + q(x)HT_{n-2}(x) + r(x)HT_{n-3}(x).$$

PROOF. By (2.1), we get

$$\begin{aligned} & HT_n(x) - xHT_{n-1}(x) - HT_{n-3}(x) \\ = \; & (TP_n(x) - p(x)TP_{n-1}(x) - q(x)TP_{n-2}(x) - r(x)TP_{n-3}(x)) \\ & +(TP_{n+1}(x) - p(x)TP_n(x) - q(x)TP_{n-1}(x) - r(x)TP_{n-2}(x))i \\ & +(TP_{n+2}(x) - p(x)TP_{n+1}(x) - q(x)TP_n(x) - r(x)TP_{n-1}(x))\varepsilon \\ & +(TP_{n+3}(x) - p(x)TP_{n+2}(x) - q(x)TP_{n+1}(x) - r(x)TP_n(x))h. \end{aligned}$$

Since $\{TP_n(x)\}$ is the $(p,q,r)-$tribonacci polynomial sequence, consequently the right side of the above equality is equal to zero. Therefore we conclude that

$$HT_n(x) - p(x)HT_{n-1}(x) - q(x)HT_{n-2}(x) - r(x)HT_{n-3}(x) = 0.$$

Thus the proof is completed. □

THEOREM 2.1. *Let* $\{HT_n(x)\}$ *is the* $(p,q,r)-$*tribonacci hybrinomial sequence. The generating function for* $\{HT_n(x)\}$ *is*

$$\sum_{n=0}^{\infty} HT_n(x)t^n = \frac{\left\{ \begin{array}{c} HT_0(x) + (HT_1(x) - p(x)HT_0(x))\,t \\ +\,(HT_2(x) - p(x)HT_1(x) - q(x)HT_0(x))\,t^2 \end{array} \right\}}{1 - p(x)t - q(x)t^2 - r(x)t^3}.$$

PROOF. Suppose that the generating function for the $(p,q,r)-$tribonacci hybrinomials, $\{TP_n(x)\}$ has the following formal power series

$$g(t) = \sum_{n=0}^{\infty} HT_n(x)t^n = HT_0(x) + HT_1(x)t + HT_2(x)t^2 + HT_3(x)t^3 + \cdots.$$

Then we have

$$\begin{aligned} p(x)tg(t) &= p(x)HT_0(x)t + p(x)HT_1(x)t^2 + p(x)HT_2(x)t^3 + p(x)HT_3(x)t^4 + \cdots \\ q(x)t^2g(t) &= q(x)HT_0(x)t^2 + q(x)HT_1(x)t^3 + q(x)HT_2(x)t^4 + q(x)HT_3(x)t^5 + \cdots \\ r(x)t^3g(t) &= r(x)HT_0(x)t^3 + r(x)HT_1(x)t^4 + r(x)HT_2(x)t^5 + r(x)HT_3(x)t^6 + \cdots \end{aligned}$$

Therefore we get

$$\begin{aligned} & g(t) - p(x)tg(t) - q(x)t^2g(t) - r(x)t^3g(t) = \\ & \big(HT_0(x) + HT_1(x)t + HT_2(x)t^2 + HT_3(x)t^3 + \cdots\big) \\ & - \big(p(x)HT_0(x)t + p(x)HT_1(x)t^2 + p(x)HT_2(x)t^3 + p(x)HT_3(x)t^4 + \cdots\big) \\ & - \big(q(x)HT_0(x)t^2 + q(x)HT_1(x)t^3 + q(x)HT_2(x)t^4 + q(x)HT_3(x)t^5 + \cdots\big) \\ & - \Big(r(x)HT_0^)(x)t^3 + r(x)HT_1^)(x)t^4 + r(x)HT_2(x)t^5 + r(x)HT_3^)(x)t^6 + \cdots\Big) \\ & = HT_0(x) + (HT_1(x) - p(x)HT_0(x))t + (HT_2(x) - p(x)HT_1(x) - q(x)HT_0(x))t^2 \\ & +(HT_3(x) - p(x)HT_2(x) - q(x)HT_1(x) - r(x)HT_0(x))t^3 + \cdots \\ & +(HT_m(x) - p(x)HT_{m-1}(x) - q(x)HT_{m-2}(x) - r(x)HT_{m-3}(x))t^n + \cdots \end{aligned}$$

By Lemma 2.1 we find that

$$g(t)(1-p(x)t-q(x)t^2-r(x)t^3) = HT_0(x)+(HT_1(x)-p(x)HT_0(x))t + (HT_2(x)-p(x)HT_1(x)-q(x)HT_0(x))t^2.$$

So, we get

$$\sum_{n=0}^{\infty} HT_n(x)t^n = \frac{\left\{\begin{array}{c} HT_0(x)+(HT_1(x)-p(x)HT_0(x))\,t \\ +\,(HT_2(x)-p(x)HT_1(x)-q(x)HT_0(x))\,t^2 \end{array}\right\}}{1-p(x)t-q(x)t^2-r(x)t^3}.$$

□

Lemma 2.2. *Let $\{TP_n(x)\}$ is the $(p,q,r)-$tribonacci polynomial sequence and $n \geqslant 0$ be an integer. The Binet-Like formula for $\{TP_n(x)\}$ is*

$$TP_n(x) = v_1\alpha^n + v_2\beta^n + v_3\gamma^n,$$

where

$$v_1 = \frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)},\ v_2 = \frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)},\ v_3 = \frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)},$$

and α, β, γ are the roots of the characteristic equation $t^3-p(x)t^2-q(x)t-r(x)=0$.

Proof. We know that the recurrence relation

$$TP_n(x) = p(x)TP_{n-1}(x)+q(x)TP_{n-2}(x)+r(x)TP_{n-3}(x)$$

has the characteristic equation $f(t) = t^3-p(x)t^2-q(x)t-r(x) = 0$. For an arbitrary value of x we know that this equation has three distinct roots α, β, γ. Hence $\frac{1}{\alpha}$, $\frac{1}{\beta}, \frac{1}{\gamma}$ are the roots of $h(t) = f(\frac{1}{t}) = 1-p(x)t-q(x)t^2-r(x)t^3$. We have

$$h(x) = 1-p(x)t-q(x)t^2-r(x)t^3 = (1-\alpha t)(1-\beta t)(1-\gamma t).$$

With the help of the generating function of the $(p,q,r)-$tribonacci polynomials we have

$$\begin{aligned} G(t) &= \frac{t}{1-p(x)t-q(x)t^2-r(x)t^3} \\ &= \frac{A}{1-\alpha t}+\frac{B}{1-\beta t}+\frac{C}{1-\gamma t} \\ &= A\sum_{j=0}^{\infty}(\alpha t)^j + B\sum_{j=0}^{\infty}(\beta t)^j + C\sum_{j=0}^{\infty}(\gamma t)^j. \end{aligned} \tag{2.2}$$

In this way, we know from [**5**] that the generating function for the (p,q,r)-tribonacci polynomial sequence is $G(t) = \frac{t}{1-p(x)t-q(x)t^2-r(x)t^3}$. So,

$$\begin{aligned} G(t) &= \frac{t}{1-p(x)t-q(x)t^2-r(x)t^3} \\ &= \frac{A(1-\beta t)(1-\gamma t)+B(1-\alpha t)(1-\gamma t)+C(1-\alpha t)(1-\beta t)}{(1-\alpha t)(1-\beta t)(1-\gamma t)}. \end{aligned}$$

For this reason, by comparison of the left and right sides of this equality we obtain,

$$t = A(1-\beta t)(1-\gamma t) + B(1-\alpha t)(1-\gamma t) + C(1-\alpha t)(1-\beta t).$$

So, we derive

$$A = \frac{p(x)-(\beta+\gamma)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly we get

$$B = \frac{p(x)-(\alpha+\gamma)}{(\beta-\alpha)(\beta-\gamma)},\ C = \frac{p(x)-(\alpha+\beta)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus by (2.2), we have

$$\begin{aligned} G(t) &= \sum_{j=0}^{\infty} \tfrac{(p(x)-(\beta+\gamma))\alpha^j}{(\alpha-\beta)(\alpha-\gamma)} t^j + \sum_{j=0}^{\infty} \tfrac{(p(x)-(\alpha+\gamma))\beta^j}{(\beta-\alpha)(\beta-\gamma)} t^j + \sum_{j=0}^{\infty} \tfrac{(p(x)-(\alpha+\beta))\gamma^j}{(\gamma-\alpha)(\gamma-\beta)} t^j \\ &= \sum_{j=0}^{\infty} \left\{ \tfrac{(p(x)-(\beta+\gamma))\alpha^j}{(\alpha-\beta)(\alpha-\gamma)} + \tfrac{(p(x)-(\alpha+\gamma))\beta^j}{(\beta-\alpha)(\beta-\gamma)} + \tfrac{(p(x)-(\alpha+\beta))\gamma^j}{(\gamma-\alpha)(\gamma-\beta)} \right\} t^j. \end{aligned}$$

As a result, we obtain

$$TP_{,j}(x) = \left\{ \frac{(p(x)-(\beta+\gamma))\alpha^j}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(p(x)-(\alpha+\gamma))\beta^j}{(\beta-\alpha)(\beta-\gamma)} + \frac{(p(x)-(\alpha+\beta))\gamma^j}{(\gamma-\alpha)(\gamma-\beta)} \right\}.$$

We get

$$TP_n(x) = \frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n.$$

Thus the proof is completed. □

THEOREM 2.2. *Let $\{HT_n(x)\}$ is the $(p,q,r)-$tribonacci hybrinomial sequence and $n \geqslant 0$ be an integer. The Binet-Like formula for $\{HT_n(x)\}$ is*

$$HT_n(x) = t_1\alpha^n + t_2\beta^n + t_3\gamma^n$$

where

$$\begin{aligned} t_1 &= \frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}, \\ t_2 &= \frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}, \\ t_3 &= \frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

PROOF. By (2.1), we have

$$HT_n(x) = TP_n(x) + TP_{n+1}(x)i + TP_{n+2}(x)\varepsilon + TP_{n+3}(x)h.$$

By Lemma 2.2 we get,

$$TP_n(x) = \frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n.$$

So we get

$$HT_n(x) = \left(\tfrac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \tfrac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \tfrac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n \right)$$

$+\left(\frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1}+\frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1}+\frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}\right)i$

$+\left(\frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+2}+\frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+2}+\frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+2}\right)\varepsilon$

$+\left(\frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+3}+\frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+3}+\frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+3}\right)h.$

Consequently by some computations we have

$HT_n^{(p,q,r)}(x)=\left(\frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^n+\left(\frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^n$

$+\left(\frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^n.$ □

COROLLARY 2.2. *Let $\{HT_n(x)\}$ is the (p,q,r)-tribonacci hybrinomial sequence. The exponential generating function for $\{HT_n(x)\}$ is*

$$\sum_{n=0}^{\infty}HT_n(x)\frac{t^n}{n!}=t_1e^{\alpha t}+t_2e^{\beta t}+t_3e^{\gamma t}$$

where

$$t_1=\frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)},\quad t_2=\frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)},$$
$$t_3=\frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}.$$

PROOF. By using the Binet-like formula for the $(p,q,r)-$tribonacci hybrinomial sequence, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}HT_n(x)\frac{t^n}{n!}\\
=\ &\sum_{n=0}^{\infty}\left[\begin{array}{l}\left(\frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^n\\ +\left(\frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^n\end{array}\right.\\
&\left.+\left(\frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^n\right]\frac{t^n}{n!}\\
=\ &\left(\frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}\\
&+\left(\frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!}\\
&+\left(\frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)\sum_{n=0}^{\infty}\frac{(\gamma t)^n}{n!}.
\end{aligned}$$

So, we get the result. □

3. Special Identities

LEMMA 3.1. *Let $\{TP_n(x)\}$ is the $(p,q,r)-$tribonacci polynomial sequences and $\{HT_n(x)\}$ is the $(p,q,r)-$tribonacci hybrinomial sequence respectively. Then*

(1) *The sum of* $\{TP_n(x)\}$ *is*

$$\sum_{j=0}^{n} TP_{,j}(x) = \frac{TP_{n+1}(x) + (q(x)+r(x))TP_n(x) + r(x)TP_{n-1}(x) - 1}{p(x)+q(x)+r(x)-1}$$

for $j \geqslant 0$ *be an integer and for every positive integer* n.

(2) *The sum of* $\{HT_n(x)\}$ *is*

$$\sum_{j=0}^{n} HT_j(x) = \frac{\left\{ \begin{array}{c} HT_{n+1}(x) + (q(x)+r(x))HT_n(x) \\ +r(x)HT_{n-1}(x) - (1+i+\varepsilon+h) \end{array} \right\}}{p(x)+q(x)+r(x)-1} - \varepsilon - (1+p(x))h$$

for $j \geqslant 0$ *be an integer and for every positive integer* n.

Proof. (1) By the $(p,q,r)-$tribonacci polynomial sequences we know that

$$TP_n(x) - p(x)TP_{n-1}(x) = q(x)TP_{n-2}(x) + r(x)TP_{n-3}(x).$$

Thus, we have

$$\begin{array}{rcl} TP_3(x) - p(x)TP_2(x) & = & q(x)TP_1(x) + r(x)TP_0(x), \\ TP_4(x) - p(x)TP_3(x) & = & q(x)TP_2(x) + r(x)TP_1(x), \\ TP_5(x) - p(x)TP_4(x) & = & q(x)TP_3(x) + r(x)TP_2(x), \\ & \vdots & \\ TP_n(x) - p(x)TP_{n-1}(x) & = & q(x)TP_{n-2}(x) + r(x)TP_{n-3}(x), \\ TP_{n+1}(x) - p(x)TP_n(x) & = & q(x)TP_{n-1}(x) + r(x)TP_{n-2}(x). \end{array}$$

As a result of some calculations, we get

$$\begin{gathered} (p(x)+q(x)+r(x)-1)\sum_{j=0}^{n} TP_j(x) = \\ r(x)TP_{n-1}(x) + (q(x)+r(x))TP_n(x) + TP_{n+1}(x) - 1. \end{gathered}$$

Consequently we obtain the result.

(2) As we know

$$\sum_{j=0}^{n} HT_j(x) = HT_0(x) + HT_1(x) + HT_2(x) + \cdots + HT_n(x).$$

So we obtain

$$\begin{aligned} \sum_{j=0}^{n} HT_j(x) = {} & (TP_0(x) + TP_1(x)i + TP_2(x)\varepsilon + TP_3(x)h) \\ & + (TP_1(x) + TP_2(x)i + TP_3(x)\varepsilon + TP_4(x)h) \\ & + \cdots + (TP_n(x) + TP_{n+1}(x)i + TP_{n+2}(x)\varepsilon + TP_{n+3}(x)h) \end{aligned}$$

$$\begin{aligned}
&= (TP_0(x)+TP_1(x)+TP_2(x)+\cdots+TP_n(x)) \\
&\quad + (TP_1(x)+TP_2(x)+\cdots+TP_{n+1}(x)+TP_0(x)-TP_0(x))\,i \\
&\quad + \left\{ \begin{array}{c} TP_2(x)+TP_3(x)+\cdots+TP_{n+2}(x)+TP_0(x) \\ +TP_1(x)-TP_0(x)-TP_1(x) \end{array} \right\} \varepsilon \\
&\quad + \left\{ \begin{array}{c} TP_3(x)+TP_4(x)+\cdots+TP_{n+3}(x) \\ +TP_0(x)+TP_1(x)+TP_2(x)-TP_0(x) \\ -TP_1(x)-TP_2(x) \end{array} \right\} h \\
&= \sum_{j=0}^{n} TP_{,j}(x) + \left(\sum_{j=0}^{n+1} TP_{,j}(x) \right) i + \left(\sum_{j=0}^{n+2} TP_{,j}(x) - 1 \right) \varepsilon \\
&\quad + \left(\sum_{j=0}^{n+3} TP_{,j}(x) - 1 - p(x) \right) h.
\end{aligned}$$

Consequently using Lemma 3.1 we get,

$$\begin{aligned}
&\sum_{j=0}^{n} HT_j(x) \\
&= \left(\frac{TP_{n+1}(x)+(q(x)+r(x))TP_n(x)+r(x)TP_{n-1}(x)-1}{p(x)+q(x)+r(x)-1} \right) \\
&\quad + \left(\frac{TP_{n+2}(x)+(q(x)+r(x))TP_{n+1}(x)+r(x)TP_n(x)-1}{p(x)+q(x)+r(x)-1} \right) i \\
&\quad + \left(\frac{TP_{n+4}(x)+(q(x)+r(x))TP_{n+3}(x)+r(x)TP_{n+2}(x)-1}{p(x)+q(x)+r(x)-1} - 1 \right) \varepsilon \\
&\quad + \left(\frac{TP_{n+5}(x)+(q(x)+r(x))TP_{n+4}(x)+r(x)TP_{n+3}(x)-1}{p(x)+q(x)+r(x)-1} - 1 - p(x) \right) h \\
&= \frac{\left\{ \begin{array}{c} HT_{n+1}(x)+(q(x)+r(x))HT_n(x) \\ +r(x)HT_{n-1}(x)-(1+i+\varepsilon+h) \end{array} \right\}}{p(x)+q(x)+r(x)-1} - \varepsilon - (1+p(x))h.
\end{aligned}$$

Thus the proof is completed. □

THEOREM 3.1. *Let $\{HT_n(x)\}$ is the $(p,q,r)-$tribonacci hybrinomial sequences and $\{TP_n(x)\}$ is the $(p,q,r)-$tribonacci polynomial sequences respectively. Then:*

(1) *The Catalan identity for $\{HT_n(x)\}$ is*

$HT_{n+r}(x)HT_{n-r}(x) - (HT_n(x))^2(x) =$
$(l_1 l_2 \alpha^n \beta^n (\alpha^r \beta^{-r} - 2)\alpha^{n+1}\beta^{n-1} + l_2 l_1 \beta^{n+1}\alpha^{n-1})$
$+(l_1 l_3 \alpha^n \gamma^n (\alpha^r \gamma^{-r} - 2) + l_3 l_1 \gamma^{n+1}\alpha^{n-1}) + (l_2 l_3 \beta^n \gamma^n (\beta^r \gamma^{-r} - 2) + l_3 l_2 \gamma^{n+1}\beta^{n-1},$

where

$$\begin{aligned} t_1 &= \frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}, \\ t_2 &= \frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}, \\ t_3 &= \frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

and for every positive integer $n,\ r$.

(2) *The Catalan identity for* $\{TP_m(x)\}$ *is*

$$\begin{aligned} & TP_{n+r}(x)TP_{n-r}(x) - TP_n^2(x) \\ = \ & v_1v_2(\alpha^r-\beta^r)^2(\alpha\beta)^{n-r} + v_1v_3(\alpha^r-\gamma^r)^2(\alpha\gamma)^{n-r} + v_2v_3(\beta^r-\gamma^r)^2(\beta\gamma)^{n-r}, \end{aligned}$$

where

$$v_1 = \frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}, v_2 = \frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}, v_3 = \frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)},$$

and for every positive integer $n,\ r$.

PROOF. (1) With the help of Binet-Like formula of the $(p,q,r)-$tribonacci hybrinomial sequence, we write

$HT_{n+r}(x)HT_{n-r}(x) - (HT_n(x))^2(x) =$

$(t_1\alpha^{n+r} + t_2\beta^{n+r} + t_3\gamma^{n+r}) \times (t_1\alpha^{n-r} + t_2\beta^{n-r} + t_3\gamma^{n-r}) - (t_1\alpha^n + t_2\beta^n + t_3\gamma^n)^2$,

where

$$\begin{aligned} t_1 &= \frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}, \\ t_2 &= \frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}, \\ t_3 &= \frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

By some calculation,

$$\begin{aligned} & HT_{n+r}(x)HT_{n-r}(x) - (HT_n(x))^2(x) \\ = \ & (t_1^2\alpha^{2n} + t_1t_2\alpha^{n+r}\beta^{n-r} + t_1t_3\alpha^{n+r}\gamma^{n-r} + t_2t_1\beta^{n+r}\alpha^{n-r} + t_2^2\beta^{2n} + \\ & t_2t_3\beta^{n+r}\gamma^{n-r} + t_3t_1\gamma^{n+r}\alpha^{n-r} \\ & + t_3t_2\gamma^{n+r}\beta^{n-r} + t_3^2\gamma^{2n}) - (t_1^2\alpha^{2n} + t_2^2\beta^{2n} + t_3^2\gamma^n + 2t_1t_2\alpha^n\beta^n + \\ & 2t_1t_3\alpha^n\gamma^n + 2t_2t_3\beta^n\gamma^n) \\ = \ & (t_1t_2\alpha^{n+r}\beta^{n-r} + t_2t_1\beta^{n+r}\alpha^{n-r} - 2t_1t_2\alpha^n\beta^n) \\ & + (t_1t_3\alpha^{n+r}\gamma^{n-r} + t_3t_1\gamma^{n+r}\alpha^{n-r} - 2t_1t_3\alpha^n\gamma^n) + (t_2t_3\beta^{n+r}\gamma^{n-r} + \\ & t_3t_2\gamma^{n+r}\beta^{n-r} - 2t_2t_3\beta^n\gamma^n) \end{aligned}$$

the proof is completed.

(2) It can be proved similar to (1). □

Corollary 3.1. *Let $\{HT_n(x)\}$ is the $(p,q,r)-$tribonacci hybrinomial sequence and $\{TP_n(x)\}$ is the $(p,q,r)-$tribonacci polynomial sequences respectively. Then:*

(1) *The Cassini identity for the $(p,q,r)-$tribonacci hybrinomial sequence is*
$HT_{n+1}(x)HT_{n-1}(x) - (HT_n(x))^2(x) =$
$(t_1t_2\alpha^n\beta^n(\alpha\beta-2)\alpha^{n+1}\beta^{n-1}+t_2t_1\beta^{n+1}\alpha^{n-1})+(t_1t_3\alpha^n\gamma^n(\alpha\gamma-2)+t_3t_1\gamma^{n+1}\alpha^{n-1})$
$+(t_2t_3\beta^n\gamma^n(\beta\gamma-2)+t_3t_2\gamma^{n+1}\beta^{n-1}-2t_2t_3\beta^n\gamma^n),$
where

$$\begin{aligned} t_1 &= \frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}, \\ t_2 &= \frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}, \\ t_3 &= \frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

and for every positive integer n.

(2) *The Cassini identity for the $(p,q,r)-$tribonacci polynomial sequence is*
$TP_{n+1}(x)TP_{n-1}(x) - TP_n^2(x) =$
$(v_1v_2(\alpha-\beta)^2)(\alpha\beta)^{n-1} + (v_1v_3(\alpha-\gamma)^2)(\alpha\gamma)^{n-1} + (v_2v_3(\beta-\gamma)^2)(\beta\gamma)^{n-1},$
where

$$v_1 = \frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}, v_2 = \frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}, v_3 = \frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)},$$

and for every positive integer n.

Proof. (1) It can be proved by replacing $r = 1$ in Catalan identity for the $(p,q,r)-$tribonacci hybrinomial sequence.

(2) It can be proved by replacing $r = 1$ in Catalan identity for the (p,q,r)-tribonacci polynomial sequence. □

Theorem 3.2. *Let $\{HT_n(x)\}$ is the $(p,q,r)-$tribonacci hybrinomial sequence and $\{TP_n(x)\}$ is the $(p,q,r)-$tribonacci polynomial sequences respectively . Then:*

(1) *The d'Ocagne identity for the $(p,q,r)-$tribonacci hybrinomial sequence is*

$$\begin{aligned} HT_m(x)HT_{n+1}(x) - HT_{m+1}(x)HT_n(x) &= t_1t_2\alpha^m\beta^n(\beta-\alpha) + t_2t_1\beta^m\alpha^n(\alpha-\beta) \\ &+t_2t_3\beta^m\gamma^n(\gamma-\beta) + t_3t_2\gamma^m\beta^n(\beta-\gamma) \\ &+t_1t_3\alpha^m\gamma^n(\gamma-\alpha) + t_3t_1\gamma^m\alpha^n(\alpha-\beta), \end{aligned}$$

where

$$\begin{aligned} t_1 &= \frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}, \\ t_2 &= \frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}, \\ t_3 &= \frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

and for every positive integer $m,\ n$.

(2) *The d'Ocagne identity for the* $(p,q,r)-$*tribonacci polynomial sequences is*

$$\begin{aligned} TP_m(x)TP_{n+1}(x) - TP_{m+1}(x)T_{p,q,r,n}(x) &= v_1v_2(\alpha-\beta)(\alpha^n\beta^m - \alpha^m\beta^n) \\ &\quad +v_1v_3(\alpha-\gamma)(\alpha^n\gamma^m - \alpha^m\gamma^n) \\ &\quad +v_2v_3^2(\beta-\gamma)(\beta^n\gamma^m - \beta^m\gamma^n) \end{aligned}$$

where

$$v_1 = \frac{(p(x)-(\beta+\gamma))}{(\alpha-\beta)(\alpha-\gamma)}, v_2 = \frac{(p(x)-(\alpha+\gamma))}{(\beta-\alpha)(\beta-\gamma)}, v_3 = \frac{(p(x)-(\alpha+\beta))}{(\gamma-\alpha)(\gamma-\beta)},$$

and for every positive integer $m,\ n$.

Proof. (1) Using the Binet-Like formula of the $(p,q,r)-$tribonacci hybrinomial sequence, we get

$$\begin{aligned} & HT_m(x)HT_{n+1}(x) - HT_{m+1}(x)HT_n(x) \\ = \ & (t_1\alpha^m + t_2\beta^m + t_3\gamma^m)(t_1\alpha^{n+1} + t_2\beta^{n+1} + t_3\gamma^{n+1}) \\ & -(t_1\alpha^{m+1} + t_2\beta^{m+1} + t_3\gamma^{m+1})(t_1\alpha^n + t_2\beta^n + t_3\gamma^n) \end{aligned}$$

where

$$\begin{aligned} t_1 &= \frac{(p(x)-(\beta+\gamma))(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}, \\ t_2 &= \frac{(p(x)-(\alpha+\gamma))(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}, \\ t_3 &= \frac{(p(x)-(\alpha+\beta))(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

So, we get

$$\begin{aligned} & HT_m(x)HT_{n+1}(x) - HT_{m+1}(x)HT_n(x) \\ = \ & (t_1^2\alpha^{m+n+1} + t_1t_2\alpha^m\beta^{n+1} + t_1t_3\alpha^m\gamma^{n+1} + t_2t_1\beta^m\alpha^{n+1} + t_2^2\beta^{m+n+1} \\ & +t_2t_3\beta^m\gamma^{n+1} + t_3t_1\gamma^m\alpha^{n+1} + t_3t_2\gamma^m\beta^{n+1} + t_3^2\gamma^{m+n+1}) \\ & -(t_1^2\alpha^{m+n+1} + t_1t_2\alpha^{m+1}\beta^n + t_1t_3\alpha^{m+1}\gamma^n \\ & +t_2t_1\beta^{m+1}\alpha^n + t_2^2\beta^{m+n+1} + t_2t_3\beta^{m+1}\gamma^n \\ & +t_3t_1\gamma^{m+1}\alpha^n + t_3t_2\gamma^{m+1}\beta^n + t_3^2\gamma^{m+n+1}) \\ = \ & t_1t_2\alpha^m\beta^n(\beta-\alpha) + t_1t_3\alpha^m\gamma^n(\gamma-\alpha) + t_2t_1\beta^m\alpha^n(\alpha-\beta) \\ & +t_2t_3\beta^m\gamma^n(\gamma-\beta) + t_3t_1\gamma^m\alpha^n(\alpha-\beta) + t_3t_2\gamma^m\beta^n(\beta-\gamma). \end{aligned}$$

(2) It can be proved similar to (1) □

References

[1] P. Catarino and G. Bilgici. A Note on modified k-Pell hybrid numbers. *Konuralp J. Math.*, **8**(2)(2020), 229–233..

[2] V. E. Hoggatt Jr. and M. Bicknell. Generalized Fibonacci polynomials. *Fibonacci Quart.*, **11**(5)(1973), 457–465.

[3] C. Kızılateş. A note on Horadam hybrinomials. Available at doi:10.20944/preprints202001.0116.v1, 2020.
[4] T. Koshy. *Fibonacci and Lucas Numbers with Applications*. John Wiley and Sons, 2001.
[5] C. Kızılateş, W.-S. Du and F. Qi. Several determinantal expressions of generalized Tribonacci polynomials and sequences. *Tamkang J. Math.* **53**(2022), available online at https://doi.org/10.22541/au.159474102.22328885.
[6] G. Y. Lee and M. Asci. Some properties of the $(p;q)-$Fibonacci and $(p;q)-$Lucas polynomials. *J. Appl. Math.*, **2012**. Article ID 264842; 18 pages.
[7] M. Liana, A. Szynal-Liana and I. Włoch. On Pell hybrinomials. *Miskolc Math. Notes*, **20**(2)(2019), 1051–1062.
[8] M. Özdemir. Introduction to hybrid numbers. *Adv. Appl. Clifford Algebr.*, **28**(1)(2018), Art. 11, 32 pp., https://doi.org/10.1007/s00006-018-0833-3.
[9] S. Pethe. Some identities for Tribonacci sequences. *Fibonacci Quart.*, **26**(2)(1988), 144–151.
[10] E. Polatlı. Hybrid numbers with Fibonacci and Lucas hybrid number coefficients. preprint, 2020.
[11] P. Ribenboim. *My Numbers, My Friends, Popular.* Lectures on Number Theory, Springer-Verlag, New York, Inc. 2000.
[12] B. Rybolowicz and A. Tereszkiewicz. Generalized Tricobsthal and generalized Tribonacci polynomials. *Appl. Math. Comput.*, **325**(2018), 297–308.
[13] A. Shannon. Tribonacci numbers and Pascal's pyramid. *Fibonacci Quart.*, **15**(3)(1977), 268–275.
[14] T. D. Şentürk, G.Bilgici, A. Daşdemir and Z. Ünal. Study on Horadam hybrid numbers. *Turk.J. Math.*, **44**(4)(2020), 1212 – 1221.
[15] M. Shattuck and E. Tan. Incomplete generalized $(p;q;r)-$ Tribonacci polynomials. *Appl. Appl. Math.*, **13**(1)(2018), 1–18.
[16] Y. Soykan and E. Taşdemir. Generalized Tetranacci hybrid numbers. *Ann. Math. Sil.*, **35**(1)(2021), 113–130.
[17] A. Szynal-Liana. The Horadam hybrid numbers. *Discuss. Math., Gen. Algebra Appl.*, **38**(1)(2018), 91–98.
[18] A. Szynal-Liana and I. Włoch. The Fibonacci hybrid numbers. *Util. Math.*, **110**(2019), 3–10.
[19] A. Szynal-Liana and I. Wloch. Jacobsthal and Jacobsthal Lucas hybrid numbers. *Ann. Math. Sil.*, **33**(2019), 276–283.
[20] A. Szynal-Liana abd I. Włoch. Introduction to Fibonacci and Lucas hybrinomials. *Complex Var. Elliptic Equ.*, **65**(10)(2020), 1736–1747.
[21] A. Szynal-Liana and I. Włoch. Generalized Fibonacci-Pell hybrinomials. *Online J. Anal. Comb.*, **15**(2020), Article 14, pp. 1–12.
[22] C. C. Yalavigi. Properties of Tribonacci numbers. *Fibonacci Quart.*, **10**(3)(1972), 231–246.
[23] N. Yilmaz and N. Taskara. Tribonacci and Tribonacci-Lucas numbers via the determinants of special matrices. *Appl. Math. Sci., Ruse*, **8**(39)(2014), 1947–1955.

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