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EXISTENCE RESULTS FOR SECOND-ORDER MULTI-POINT IMPULSIVE TIME SCALE BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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Abstract. In this paper, we consider nonlinear second order m-point impulsive time scale boundary value problems on infinite intervals. We establish criteria for the existence of positive solutions to the nonlinear impulsive time scale boundary value problems on infinite intervals by using a result from the theory of fixed point index.

1. Introduction

Impulsive problems describe processes which experience a sudden change in their states at certain moments. Impulsive differential equations have been developed in modeling impulsive problems in physics, chemical technology, population dynamics, ecology, biological systems, biotechnology, industrial robotics, optimal control, economics, and so forth. For the introduction of the theory of impulsive differential equations, we refer to the books [**4, 14, 17**]. Especially, the study of impulsive dynamic equations on time scales has also attracted much attention since it provides an unifying structure for differential equations in the continuous cases and finite difference equations in the discrete cases, see [**1, 2, 3, 22, 21, 8, 10, 11, 12, 13, 16, 18, 19, 20, 24, 25**] and references therein. Some basic definitions and theorems on time scales can be found in the books [**5, 6**]. In recent years, there are a few authors studied the existence of positive solutions for time scale boundary value problems on infinite intervals.

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Zhao, Ge [**27**] studied the existence of at least three positive solutions for the nonlinear time scale boundary value problems

$$\begin{cases} (\varphi_p(u^{\Delta}(t)))^{\nabla} + q(t)f\left(u(t), u^{\Delta}(t)\right) = 0, \ \ t \in [0,\infty)_{\mathbb{T}} \\ u(0) = \beta u^{\Delta}(\eta), \quad \lim\limits_{t\to\infty} u^{\Delta}(t) = 0 \end{cases}$$

by using Leggett-Williams fixed point theorem, where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$.

Zhao, Ge [**28**] considered the following m-point boundary value problem on time scale

$$\begin{cases} (\varphi_p(u^{\Delta}(t)))^{\nabla} + h(t)f\left(t, u(t), u^{\Delta}(t)\right) = 0, \ \ t \in [0,\infty)_{\mathbb{T}} \\ u(0) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u^{\Delta}(\infty) = \sum\limits_{i=1}^{m-2} \beta_i u^{\Delta}(\eta_i), \end{cases}$$

where $u^{\Delta}(\infty) = \lim\limits_{t\to\infty} u^{\Delta}(t)$,$\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta_1, \eta_2, \dots, \eta_{m-2} \in \mathbb{T}$, $\sigma(0) < \eta_1 < \eta_2 < \dots < \eta_{m-2} < \infty$, $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ for $i = 1, 2, \dots, m-2$. They obtained the criteria for the existence of positive solutions by using Avery-Peterson theorem.

Karaca, Tokmak [**9**] discussed the nonlinear multi-point impulsive time scale boundary value problems

$$\begin{cases} (\varphi(x^{\Delta}(t)))^{\nabla} + \phi(t)f\left(t, x(t), x^{\Delta}(t)\right) = 0, \ \ t \in (0,\infty)_{\mathbb{T}} \\ x(0) = \sum\limits_{i=1}^{m-2} \alpha_i x^{\Delta}(\eta_i), \quad \lim\limits_{t\to\infty} x^{\Delta}(t) = 0, \end{cases}$$

where $f \in C\left([0,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty)\right)$. $\alpha_i \geqslant 0$ $(1 \leqslant i \leqslant m-2)$ $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < \infty$, $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$. They established the sufficient conditions for the existence of three positive solutions for m-point time scale boundary value problems on infinite intervals by using the Leggett-Williams fixed point theorem and five functionals fixed point theorem.

Yaslan, Haznedar [**23**] obtained the criteria for the existence of at least one, two and three positive solutions to the nonlinear impulsive multi-point time scale boundary value problems

$$\begin{cases} (\varphi(y^{\Delta}(t))^{\nabla} + h(t)f\left(t, y(t), y^{\Delta}(t)\right) = 0, \ \ t \in [a,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1, 2, ..., n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \ \ k = 1, 2, ..., n \\ y(a) - \beta y^{\Delta}(a) = \sum\limits_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), \quad \lim\limits_{t\to\infty} y^{\Delta}(t) = 0, \quad m \geqslant 3 \end{cases}$$

by using Leray-Schauder fixed point theorem, Avery-Henderson fixed point theorem and the five functionals fixed point theorem, respectively, where $\alpha_i \geqslant 0$ $(1 \leqslant i \leqslant m-2)$, $\beta \geqslant 0$, $\eta_1, \eta_2, \dots, \eta_{m-2} \in \mathbb{T}$, $0 \leqslant a < \eta_1 < \eta_2 < \dots < \eta_{m-2} < \infty$, $f \in C\left([a,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty)\right)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$.

Karaca, Sinanoglu [**26**] investigated the criteria for the existence of at least one positive solution to the m-point time scale boundary value problems

$$\begin{cases} (\varphi_p(u^\Delta(t))^\nabla + h(t)f\left(t, u(t), u^\Delta(t)\right) = 0, \ \ t \in (0,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1,2,...,n \\ u(0) = \sum\limits_{i=1}^{m-2} \alpha_i u^\Delta(\eta_i), \quad u^\Delta(\infty) = \sum\limits_{i=1}^{m-2} \beta_i u(\eta_i), \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \ \ \varphi_p(u^\Delta(t_k^+)) - \varphi_p(u^\Delta(t_k^-)) = -\overline{I}_k(u(t_k)), \ \ k \in \mathbb{N} \end{cases}$$

by using the four functionals fixed point theorem, where $\eta_1, \ldots, \eta_{m-2} \in \mathbb{T}$, $\sigma(0) < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty$, $u^\Delta(\infty) = \lim\limits_{t\to\infty} u^\Delta(t)$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $I_k \in C([0,\infty),[0,\infty))$ and $\overline{I}_k \in C([0,\infty),[0,\infty))$.

We consider the following boundary value problem (BVP)

$$\begin{cases} y^{\Delta\nabla}(t) + h(t)f\left(t, y(t), y^\Delta(t)\right) = 0, \ \ t \in [a,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1,2,...,n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \ \ k = 1,2,...,n \\ y(a) - \gamma y^\Delta(a) = \sum\limits_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i), \quad \lim\limits_{t\to\infty} y^\Delta(t) = \sum\limits_{i=1}^{m-2} \beta_i y(\eta_i), \quad m \geqslant 3 \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ $(1 \leqslant i \leqslant m-2)$, $\gamma \geqslant 0$, $0 \leqslant a < \eta_1 < \ldots < \eta_{m-2} < \infty$ and $f \in C\left([a,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty)\right)$.

We will assume that the following conditions are satisfied:

(H1) $h \in C\left([a,\infty)_{\mathbb{T}}, [0,\infty)\right)$, $\int\limits_a^\infty h(s)\nabla s < \infty$;

(H2) $f(t,(1+t)u,v) \leqslant \omega\left(\max\{|u|,|v|\}\right)$ with $\omega \in C\left([0,\infty),[0,\infty)\right)$ nondecreasing;

(H3) $\sum\limits_{a<t_k<\infty} I_k(y(t_k)) < \infty$, $I_k \in C\left(\mathbb{R},\mathbb{R}^+\right)$, $t_k \in [a,\infty)_{\mathbb{T}}$ and $y(t_k^+) = \lim\limits_{h\to 0} y(t_k + h)$, $y(t_k^-) = \lim\limits_{h\to 0} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1,...,n$.

2. Preliminaries

To state and prove the main results of this paper, we will need the following lemmas.

Lemma 2.1. *Assume (H3) holds. If* $x \in C\left([a,\infty)_{\mathbb{T}}, [0,\infty)\right)$ *and* $\int\limits_a^\infty x(t)\nabla t < \infty$, *then the boundary value problem*

$$\begin{cases} y^{\Delta\nabla}(t) + x(t) = 0, \ \ t \in [a,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1,2,...,n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \ \ k = 1,2,...,n \\ y(a) - \gamma y^\Delta(a) = \sum\limits_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i), \quad \lim\limits_{t\to\infty} y^\Delta(t) = \sum\limits_{i=1}^{m-2} \beta_i y(\eta_i), \quad m \geqslant 3 \end{cases}$$

has a unique solution

$$
\begin{aligned}
y(t) &= (\gamma-a)\int_a^{\infty} x(s)\nabla s + t\int_t^{\infty} x(s)\nabla s + \int_a^{t} s x(s)\nabla s + (\gamma+t-a)\sum_{i=1}^{m-2}\beta_i y(\eta_i) \\
&+ \sum_{i=1}^{m-2}\alpha_i\left[\sum_{j=1}^{m-2}\beta_j y(\eta_j) + \int_{\eta_i}^{\infty} x(s)\nabla s\right] + \sum_{a<t_k<t} I_k(y(t_k)).
\end{aligned}
\tag{2.1}
$$

PROOF. Since we have $y^{\Delta\nabla}(t) = -x(t)$ for $t \in [a,\infty)_{\mathbb{T}}$, we get

$$
y^{\Delta}(t) = \lim_{t\to\infty} y^{\Delta}(t) + \int_t^{\infty} x(\xi)\nabla\xi \tag{2.2}
$$

By using the second boundary condition we obtain

$$
y^{\Delta}(t) = \sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_t^{\infty} x(\xi)\nabla\xi.
$$

Integrating the above equality from a to t, we have

$$
y(t) - y(a) - \sum_{a<t_k<t} I_k\left(y(t_k)\right) = (t-a)\sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_a^{t}\int_{\xi}^{\infty} x(s)\nabla s\Delta\xi.
$$

By using the first boundary condition we obtain

$$
\begin{aligned}
y(t) &= \gamma y^{\Delta}(a) + \sum_{i=1}^{m-2}\alpha_i y^{\Delta}(\eta_i) + (t-a)\sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_a^{t}(s-a)x(s)\nabla s \\
&+ (t-a)\int_t^{\infty} x(s)\nabla s + \sum_{a<t_k<t} I_k\left(y(t_k)\right).
\end{aligned}
$$

Thus, from (2.2) we have (2.1). $\square$

By Lemma 2.1, the solutions of the BVP (1.1) are the fixed points of the operator A defined by

$$
\begin{aligned}
Ay(t) &= (\gamma-a)\int_a^{\infty} h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s + t\int_t^{\infty} h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s \\
&+ \int_a^{t} s h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s + (\gamma+t-a)\sum_{i=1}^{m-2}\beta_i y(\eta_i) \\
&+ \sum_{i=1}^{m-2}\alpha_i\left[\sum_{j=1}^{m-2}\beta_j y(\eta_j) + \int_{\eta_i}^{\infty} h(s)f\left(s,y(s),y^{\Delta}(s)\right)\nabla s\right] + \sum_{a<t_k<t} I_k(y(t_k)).
\end{aligned}
$$

Let $\mathcal{B}$ be the Banach space defined by

$$\mathcal{B} = \left\{ y \in C^{\Delta}\left([a,\infty)\right) : \sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{y(t)}{1+t} < \infty, \quad \lim_{t\to\infty} y^{\Delta}(t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i) \right\}$$

with the norm $\|y\| = \max\left\{ \|y\|_1, \|y^{\Delta}\|_{\infty} \right\}$, where

$$\|y\|_1 = \sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{|y(t)|}{1+t}, \qquad \|y^{\Delta}\|_{\infty} = \sup_{t\in[a,\infty)_{\mathbb{T}}} |y^{\Delta}(t)|$$

and define the cone $P \subset \mathcal{B}$ by

$$\begin{aligned} P \;=\; & \left\{ y \in \mathcal{B} : y(a) - \gamma y^{\Delta}(a) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), \;\; y \text{ is concave, non-decreasing and} \right. \\ & \left. \text{nonnegative on } [a,\infty)_{\mathbb{T}} \right\}. \end{aligned} \tag{2.3}$$

LEMMA 2.2. *If $y \in P$, then we have $\|y\|_1 \leqslant M \|y^{\Delta}\|_{\infty}$, where*

$$M = \max\left\{ \gamma - a + \sum_{i=1}^{m-2} \alpha_i, 1 \right\}. \tag{2.4}$$

PROOF. For $y \in P$ and $t \in [a,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} \frac{y(t)}{1+t} &= \frac{1}{1+t}\left(\int_a^t y^{\Delta}(s)\Delta s + \gamma y^{\Delta}(a) + \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i) \right) \leqslant \frac{t - a + \gamma + \sum\limits_{i=1}^{m-2} \alpha_i}{1+t} \|y^{\Delta}\|_{\infty} \\ &\leqslant M \|y^{\Delta}\|_{\infty}. \end{aligned}$$

Hence, the proof is complete. □

LEMMA 2.3. *If (H1)-(H3) hold, then the operator $A : P \to P$ is completely continuous.*

PROOF. First, we will show that $A : P \to P$. For $y \in P$, we have

$(Ay)(a) - \gamma (Ay)^{\Delta}(a) = \sum\limits_{i=1}^{m-2} \alpha_i (Ay)^{\Delta}(\eta_i)$,

$(Ay)^{\Delta\nabla}(t) = -h(t) f\left(t, y(t), y^{\Delta}(t)\right) \leqslant 0$,

$(Ay)^{\Delta}(t) = \sum\limits_{i=1}^{m-2} \beta_i y(\eta_i) + \int\limits_t^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s \geqslant 0$,

$$\begin{aligned} (Ay)(a) &= \gamma\Bigg(\int_a^{\infty} x(s)\nabla s + \sum_{i=1}^{m-2} \beta_i y(\eta_i) \Bigg) + \sum_{i=1}^{m-2} \alpha_i \Bigg[\sum_{j=1}^{m-2} \beta_j y(\eta_j) + \int_{\eta_i}^{\infty} x(s)\nabla s \Bigg] \\ &+ \sum_{a<t_k<t} I_k(y(t_k)) \geqslant 0. \end{aligned}$$

Hence, $A : P \to P$.

Now, we will show that $A : P \to P$ is continuous. If $y_n \to y$ as $n \to \infty$ in P, then there exists τ such that $\sup_{n\in\mathbb{N}} \|y_n\| < \tau$. From (H2), for all $t \in [a,\infty)_{\mathbb{T}}$ we have $f\left(t, y_n(t), y_n^{\Delta}(t)\right) \leqslant \omega\left(\max\left\{\frac{|y_n(t)|}{1+t}, |y_n^{\Delta}(t)|\right\}\right) \leqslant \omega\left(\|y_n\|\right) < \omega(\tau)$ and $f\left(t, y(t), y^{\Delta}(t)\right) \leqslant \omega\left(\|y\|\right) < \omega(\tau)$ by the continuity of norm function. Since

$$\int_t^{\infty} h(s)|f\left(s, y_n(s), y_n^{\Delta}(s)\right) - f\left(s, y(s), y^{\Delta}(s)\right)|\nabla s \leqslant 2\omega(\tau)\int_a^{\infty} h(s)\nabla s < \infty$$

by using (H1), we get

$$\begin{aligned}
\left|(Ay_n)^{\Delta}(t) - (Ay)^{\Delta}(t)\right| &\leqslant \sum_{i=1}^{m-2}\left|\beta_i\big(y_n(\eta_i) - y(\eta_i)\big)\right| \\
&\quad + \int_t^{\infty} h(s)\left|f\left(s, y_n(s), y_n^{\Delta}(s)\right) - f\left(s, y(s), y^{\Delta}(s)\right)\right|\nabla s \\
&\to 0,\ \ n \to \infty
\end{aligned}$$

by using the Lebesgue dominated convergence theorem. Hence, we obtain

$$\|(Ay_n)^{\Delta} - (Ay)^{\Delta}\|_{\infty} \to 0,$$

as $n \to \infty$. Since

$$\|Ay_n - Ay\| \leqslant M\|(Ay_n)^{\Delta} - (Ay)^{\Delta}\|_{\infty} \to 0,$$

$A : P \to P$ is continuous.

Now we will show that the image of any bounded subset of P under A is relatively compact in P. If Ω is any bounded subset of P, then there exists $K > 0$ such that $\|y\| \leqslant K$ for $\forall y \in \Omega$. By (H1) and (H2), for $\forall y \in \Omega$, we have

$$\begin{aligned}
\|(Ay)^{\Delta}\|_{\infty} &= \sum_{i=1}^{m-2}\beta_i y(\eta_i) + \int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s \\
&\leqslant K\sum_{i=1}^{m-2}\beta_i(1+\eta_i) + \omega(K)\int_a^{\infty} h(s)\nabla s < \infty
\end{aligned}$$

Since $\|A\Omega\| \leqslant M\|(A\Omega)^{\Delta}\|_{\infty} < \infty$, $A\Omega$ is uniformly bounded.

Now, we show that $A\Omega$ is equicontinuous on $[a,\infty)_{\mathbb{T}}$. For any $R > 0$, $t, p \in [a, R]_{\mathbb{T}}$, and for all $y \in \Omega$, without loss of generality we may assume that $t < p$. By (H2), we have

$$\left|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(p)\right| = \left|\int_t^{p} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right| \leqslant \omega(K)\int_t^{p} h(s)\nabla s \to 0,$$

uniformly as $t \to p$. Since $\|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(p)\|_{\infty} \to 0$, uniformly as $t \to p$, we obtain $\|(Ay)(t) - (Ay)(p)\| \to 0$, uniformly as $t \to p$, by Lemma 2.2. Thus, $A\Omega$ is equicontinuous on any compact interval of $[a,\infty)_{\mathbb{T}}$.

Now, we show that $A\Omega$ is equiconvergent on $[a,\infty)_{\mathbb{T}}$. For any $y \in \Omega$, we have

$$|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(\infty)| = \left| \int_t^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s \right| \to 0$$

as $t \to \infty$. Then, we obtain $\|(Ay)(t) - (Ay)(\infty)\| \to 0$, as $t \to \infty$, by Lemma 2.2. Therefore $A\Omega$ is equiconvergent on $[a,\infty)_{\mathbb{T}}$.

Hence, the operator $A : P \to P$ is completely continuous. $\square$

3. Positive solutions

Definition 3.1. Remember that a subset $K \neq \emptyset$ of X is called a retract of X if there is a continuous map $R : X \to K$, a retraction, such that $Rx = x$ on K. Let X be a Banach space, $K \subset X$ retract, $\Omega \subset K$ open and $f : \overline{\Omega} \to K$ compact and such that $Fix(f) \cap \partial\Omega = \emptyset$. Then we can define an integer $i_K(f,\Omega)$ which has the following properties.

(a) $i_X(f,\Omega) = 1$ for $f(\overline{\Omega}) \in \Omega$.

(b) Let $f : \Omega \to K$ be a continuous function and assume that $Fix(f)$ is a compact subset of Ω. Let Ω_1 and Ω_2 be disjoint open subsets of Ω such that $Fix(f) \subset \Omega_1 \cup \Omega_2$. Then we obtain $i_K(f,\Omega) = i_K(f,\Omega_1) + i_K(f,\Omega_2)$.

(c) Let G be an open subset of $K \times [0,1]$ and $F : G \to K$ be a continuous map. Assume that $Fix(F)$ is a compact subset of G. If $G_t = \{x : (x,t) \in G\}$ and $F_t = F(.,t)$, then we have $i_K(F_0, G_0) = i_K(F_1, G_1)$.

(d) If $K_0 \subset K$ is a retract of K and $F(\overline{\Omega}) \subset K_0$, then $i_K(F,\Omega) = i_{K_0}(F, \Omega \cap K_0)$.

We will apply the following well-known result of the fixed point theorems to prove the existence of positive solutions to the (1.1).

Lemma 3.1. [**7, 15**] *Let P be a cone in a Banach space $\mathcal{B}$, and let D be an open, bounded subset of $\mathcal{B}$ with $D_P := D \cap P \neq \emptyset$ and $\overline{D}_P \neq P$. Assume that $A : \overline{D}_P \to P$ is a compact map such that $y \neq Ay$ for $y \in \partial D_P$. The following results hold.*

(i) *If $\|Ay\| \leqslant \|y\|$ for $y \in \partial D_P$, then $i_P(A, D_P) = 1$.*

(ii) *If there exists an $b \in P \setminus \{0\}$ such that $y \neq Ay + \lambda b$ for all $y \in \partial D_P$ and all $\lambda > 0$, then $i_P(A, D_P) = 0$.*

(iii) *Assume U be open in P such that $\overline{U}_P \subset D_P$. If $i_P(A, D_P) = 1$ and $i_P(A, U_P) = 0$, then A has a fixed point in $D_P \setminus \overline{U}_P$. The same result holds if $i_P(A, D_P) = 0$ and $i_P(A, U_P) = 1$.*

For the cone P given in (2.3) and any positive real number r, define the convex set

$$P_r := \{y \in P : \|y\| < r\}$$

and the set

$$\Omega_r := \{y \in P : \min_{t \in [\eta_1, \infty)} \frac{y(t)}{1+t} < Mr, \quad \min_{t \in [\eta_1, \infty)} y^{\Delta}(t) < r\}$$

where M is defined in (2.4). The following results are proved in [**15**].

LEMMA 3.2. *The set Ω_r has the following properties.*

(i) Ω_r is open relative to P.

(ii) $P_{Mr} \subset \Omega_r \subset P_r$

(iii) $y \in \partial\Omega_r$ if and only if $\min\limits_{t\in[\eta_1,\infty)} \dfrac{y(t)}{1+t} = Mr$.

(iv) If $y \in \partial\Omega_r$, then $Mr(1+t) \leqslant y(t) \leqslant r$ for $t \in [\eta_1, \infty)$.

For convenience, we introduce the following notations. Let

$$f_{Mr}^r := \min\left\{ \min_{t\in[\eta_1,\infty)} \frac{f(t,u,v)}{r} : u \in [0, Mr(1+\eta_1)],\ \ v \in [0,r] \right\},$$

$$f_0^r := \max\left\{ \max_{t\in[a,\infty)} \frac{f(t,u,v)}{r} : u \in [0,\infty),\ \ v \in [0,r] \right\}.$$

In the next two lemmas, we give conditions on f guaranteeing that $i_P(A, P_r) = 1$ or $i_P(A, \Omega_r) = 0$. Define

$$N = \int_a^\infty h(s)\nabla s \tag{3.1}$$

and

$$L = \int_{\eta_1}^\infty h(s)\nabla s. \tag{3.2}$$

LEMMA 3.3. *If the conditions*

$$f_0^r \leqslant \frac{1 - M\sum\limits_{i=1}^{m-2} \beta_i(1+\eta_i)}{MN} \quad \text{and } y \neq Ay \text{ for } y \in \partial P_r,$$

hold, then $i_P(A, P_r) = 1$.

PROOF. If $y \in \partial P_r$, then we have $0 \leqslant \frac{y(t)}{1+t} \leqslant r$ and $0 \leqslant y^\Delta(t) \leqslant r$.

$$\begin{aligned}
\|Ay\| &\leqslant M \sup_{t\in[a,\infty)_{\mathbb{T}}} |Ay^\Delta(t)| \\
&= M\left(\sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_a^\infty h(s) f\left(s, y(s), y^\Delta(s)\right)\nabla s \right) \\
&\leqslant M\left(r\sum_{i=1}^{m-2} \beta_i(1+\eta_i) + \frac{1 - M\sum\limits_{i=1}^{m-2}\beta_i(1+\eta_i)}{MN} r \int_a^\infty h(s)\nabla s \right) \\
&= r = \|y\|.
\end{aligned}$$

It follows that $\|Ay\| \leqslant \|y\|$ for $y \in \partial P_r$. By Lemma 3.1(i), we get $i_P(A, P_r) = 1$. $\square$

Lemma 3.4. *If the conditions*

$$f^r_{Mr} \geqslant M\frac{1+\eta_1}{\alpha_1 L} \text{ and } y \neq Ay \text{ for } y \in \partial\Omega_r,$$

hold, then $i_P(A, \Omega_r) = 0$.

Proof. Let $b(t) \equiv 1$ for $t \in [a, \infty)_{\mathbb{T}}$, then $b \in \partial P_1$. Assume there exist $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$ such that $y_0 = Ay_0 + \lambda_0 b$. Then for $t \in [a, \infty)_{\mathbb{T}}$ we have

$$\begin{aligned}
y_0(t) &= Ay_0(t) + \lambda_0 b(t) \\
&\geqslant \alpha_1 \int_{\eta_1}^{\infty)} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s + \lambda_0 \\
&\geqslant \alpha_1 r f^r_{Mr} \int_{\eta_1}^{\infty)} h(s) \nabla s + \lambda_0 \\
&\geqslant \alpha_1 r M \frac{1+\eta_1}{\alpha_1 L} \int_{\eta_1}^{\infty)} h(s) \nabla s + \lambda_0 \\
&= Mr(1+\eta_1) + \lambda_0.
\end{aligned}$$

But this implies that $Mr(1+\eta_1) \geqslant Mr(1+\eta_1) + \lambda_0$, a contradiction. Hence, $y_0 \neq Ay_0 + \lambda_0 b$ for $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$, so by Lemma 3.1(ii), we get $i_P(A, \Omega_r) = 0$. $\square$

Theorem 3.1. *Assume (H1)-(H3) hold. Let M, N and L be as in* (2.4), (3.1), *and* (3.2), *respectively. Suppose that one of the following conditions holds.*

$(C1)$ *There exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < c_2 < ec_3$ such that*

$$f^{c_1}_{Mc_1}, f^{c_3}_{Mc_3} \geqslant M\frac{1+\eta_1}{\alpha_1 L}, f^{c_2}_0 \leqslant \frac{1 - M\sum\limits_{i=1}^{m-2}\beta_i(1+\eta_i)}{MN}, \text{ and } y \neq Ay \text{ for } y \in \partial P_{c_2}.$$

$(C2)$ *There exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < ec_2$ and $c_2 < c_3$ such that*

$$f^{c_1}_0, f^{c_3}_0 \leqslant \frac{1 - M\sum\limits_{i=1}^{m-2}\beta_i(1+\eta_i)}{MN}, f^{c_2}_{Mc_2} \geqslant M\frac{1+\eta_1}{\alpha_1 L}, \text{ and } y \neq Ay \text{ for } y \in \partial\Omega_{c_2}.$$

Then (1.1) *has two positive solutions. Additionally, if in* $(C2)$ *the condition*

$$f^{c_1}_0 \leqslant \frac{1 - M\sum\limits_{i=1}^{m-2}\beta_i(1+\eta_i)}{MN}$$

is replaced by

$$f_0^{c_1} < \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1+\eta_i)}{MN},$$

then (1.1) *has a third positive solution in* P_{c_1}.

Proof. Assume that $(C1)$ holds. We show that either A has a fixed point in $\partial\Omega_{c_1}$ or in $P_{c_2} \setminus \overline{\Omega}_{c_1}$. If $y \neq Ay$ for $y \in \partial\Omega_{c_1}$, then by Lemma 3.4, we have $i_P(A, \Omega_{c_1}) = 0$. Since

$$f_0^{c_2} \leqslant \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1+\eta_i)}{MN}$$

and $y \neq Ay$ for $y \in \partial P_{c_2}$, from Lemma 3.3 we get $i_P(A, P_{c_2}) = 1$. By Lemma 3.2(ii) and $c_1 < c_2$, we have $\overline{\Omega}_{c_1} \subset \overline{P}_{c_1} \subset P_{c_2}$. From Lemma 3.1$(iii)$, A has a fixed point in $P_{c_2} \setminus \overline{\Omega}_{c_1}$. If $y \neq Ay$ for $y \in \partial\Omega_{c_3}$, then from Lemma 3.4 $i_P(A, \Omega_{c_3}) = 0$. By Lemma 3.2(ii) and $c_2 < Mc_3$, we get $\overline{P}_{c_2} \subset P_{Mc_3} \subset \Omega_{c_3}$. From Lemma 3.1$(iii)$, A has a fixed point in $\Omega_{c_3} \setminus \overline{P}_{c_2}$. The proof is similar when $(C2)$ holds and we omit it here. □

As a special case of Theorem 3.1, we have the following result.

Theorem 3.2. *Suppose (H1)-(H3) hold. Let* M, N *and* L *be as in* (2.4), (3.1), *and* (3.2), *respectively. Assume that one of the following conditions holds.*

$(C3)$ *There exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < c_2$ *such that*

$$f_{Mc_1}^{c_1} \geqslant M\frac{1+\eta_1}{\alpha_1 L} \text{ and } f_0^{c_2} \leqslant \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1+\eta_i)}{MN}.$$

$(C4)$ *There exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < Mc_2$ *such that*

$$f_0^{c_1} \leqslant \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1+\eta_i)}{MN} \text{ and } f_{Mc_2}^{c_2} \geqslant M\frac{1+\eta_1}{\alpha_1 L}.$$

Then (1.1) *has a positive solution.*

References

[1] R. P. Agarwal, M. Benchohra, D. O'Regan and A. Ouahab. Second order impulsive dynamic equations on time scales. *Funct. Differ. Equ.*, **11**(3-4)(2004), 223-234.

[2] M. Benchohra, S. K. Ntouyas and A. Ouahab. Existence results for second order boundary value problem of impulsive dynamic equations on time scales. *J. Math. Anal. Appl.*, **296**(1)(2004), 65–73.

[3] M. Benchohra, S. K. Ntouyas and A. Ouahab. Extremal solutions of second order impulsive dynamic equations on time scales. *J. Math. Anal. Appl.*, **324**(1)(2006), 425–434.

[4] M. Benchohra, J. Henderson and S. Ntouyas *Impulsive Differential Equations and Inclusions.* Hindawi Publishing Corporation, New York, 2006.

[5] M. Bohner and A. Peterson. *Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[6] M. Bohner and A. Peterson (eds.). *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[7] D. Guo and V. Lakshmikantham. *Nonlinear Problems in Abstract Cones*, Academic press, San Diego, 1988.

[8] J. Henderson. Double solutions of impulsive dynamic boundary value problems on a time scale. *J. Difference Equ. Appl.*, **8**(4)(2002), 345-356.

[9] I. Y. Karaca and F. Tokmak. Existence of three positive solutions for m-point time scale boundary value problems on infinite intervals. *Dyn. Sys. Appl.*, **20**(1-4)(2011), 355–368.

[10] I. Y. Karaca, O. B. Ozen and F. Tokmak Fen. Multiple positive solutions of boundary value problems for p-Laplacian impulsive dynamic equations on time scales. *Fixed Point Theory*, **15**(2)(2014), 475–486.

[11] I. Y. Karaca and F. Tokmak Fen. Multiple positive solutions for nonlinear second-order m-point impulsive boundary value problems on time scales. *Filomat*, **29**(4)(2015), 817–827.

[12] I. Y. Karaca and F. Tokmak Fen. On positive solutions of nonlinear third-order impulsive boundary value problems on time scales. *Mediterr. J. Math.*, **13**(6)(2016), 4447–4461.

[13] I. Y. Karaca and F. Tokmak Fen. Existence of positive solutions for nonlinear third-order m-point impulsive boundary value problems on time scales. *Ukr. Math. J.*, **68**(3)(2016), 458–474.

[14] V. Lakshmikantham, D. Bainov and P. Simeonov. *Theory of Impulsive Differential Equations*. World Scientific, Singapore, 1989.

[15] K. Q. Lan. Multiple positive solutions of semilinear differential equations with singularities. *J. London Math. Soc.*, **63**(3)(2001), 690–704.

[16] O. B. Ozen, I. Y. Karaca and F. Tokmak. Fen. Existence results for p-Laplacian boundary value problems of impulsive dynamic equations on time scales. *Adv. Differ. Equ.*, **2013**, 334(2013). https://doi.org/10.1186/1687-1847-2013-334.

[17] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*. World Scientific, Singapore, 1995.

[18] F. Tokmak and I. Y. Karaca. Positive solutions of an impulsive second-order boundary value problem on time scales. *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.* , **20**(6)(2013), 695–708.

[19] F. Tokmak and I. Y. Karaca. Existence of positive solutions for p-Laplacian impulsive boundary value problems on time scales. *J. Inequal. Appl.*, **2014**, 196(2014). https://doi.org/10.1186/1029-242X-2014-196

[20] F. Tokmak Fen and I. Y. Karaca. On positive solutions of m-point boundary value problems for p- Laplacian impulsive dynamic equations on time scales. *Indian J. Pure Appl. Math.*, **46**(5)(2015), 723–738.

[21] F. Tokmak Fen and I. Y. Karaca. Positive solutions for a fourth-order multipoint impulsive BVP with p-Laplacian on time scales. *The seventh international conference on dynamic systems and applications and The fifth international confernce on neural, paralel and scientific computations, Atlanta, 27 - 30 May 2015*, (Vol. 7, pp. 83-89), 2016.

[22] F. Tokmak Fen and I. Y. Karaca. Existence of positive solutions for nonlinear second-order impulsive boundary value problems on time scales. *Mediterr. J. Math.* , **13**(1)(2016), 191–204.

[23] İ. Yaslan and Z. Haznedar. Existence of positive solutions for second-order impulsive time scale boundary value problems on infinite intervals. *Filomat*, **28**(10)(2014), 2163-2173.

[24] İ. Yaslan. Existence of positive solutions for second-order impulsive boundary value problems on time scales. *Mediterr. J. Math.*, **13**(4)(2016), 1613-1624.

[25] İ. Yaslan. Positive solutions for multi point impulsive boundary value problems on time scales. *Journal of Nonlinear Functional Analysis*, **2019**(2019), ID: 5 https://doi.org/10.23952/jnfa.2019.5

[26] I. Yaslan Karaca and A. Sinanoglu. Positive solutions of impulsive time-scale boundary value problems with p-Laplacian on the half-line. *Filomat*, **33**(2)(2019), 415-433.

[27] X. Zhao and W. Ge. Multiple positive solutions for time scale boundary value problems on infinite intervals. *Acta Appl. Math.*, **106**, 265(2009), https://doi.org/10.1007/s10440-008-9294-3.

[28] X. Zhao and W. Ge. Unbounded positive solutions for m-point time scale boundary value problems on infinite intervals. *J. Appl. Math. Comput.*, **33**(1-2)(2010), 103–123.

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