EXISTENCE RESULTS FOR SECOND-ORDER 
MULTI-POINT IMPULSIVE TIME SCALE BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

İsmail Yaslan and Esma Tozak

Abstract. In this paper, we consider nonlinear second order \( m \)-point impulsive time scale boundary value problems on infinite intervals. We establish criteria for the existence of positive solutions to the nonlinear impulsive time scale boundary value problems on infinite intervals by using a result from the theory of fixed point index.

1. Introduction

Impulsive problems describe processes which experience a sudden change in their states at certain moments. Impulsive differential equations have been developed in modeling impulsive problems in physics, chemical technology, population dynamics, ecology, biological systems, biotechnology, industrial robotics, optimal control, economics, and so forth. For the introduction of the theory of impulsive differential equations, we refer to the books [4, 14, 17]. Especially, the study of impulsive dynamic equations on time scales has also attracted much attention since it provides an unifying structure for differential equations in the continuous cases and finite difference equations in the discrete cases, see [1, 2, 3, 22, 21, 8, 10, 11, 12, 13, 16, 18, 19, 20, 24, 25] and references therein. Some basic definitions and theorems on time scales can be found in the books [5, 6]. In recent years, there are a few authors studied the existence of positive solutions for time scale boundary value problems on infinite intervals.

2010 Mathematics Subject Classification. Primary 34B18; Secondary 34B37; 34N05.

Key words and phrases. Boundary value problems, cone, fixed point theorems, impulsive dynamic equations, positive solutions, time scales.

Communicated by İlkyaslı KARACA, İzmir, Turkey.
Zhao, Ge [27] studied the existence of at least three positive solutions for the nonlinear time scale boundary value problems

\[
\begin{align*}
\begin{cases}
(\varphi_p(u^{\Delta}(t)))^{\nabla} + q(t) f(u(t), u^{\Delta}(t)) = 0, & t \in [0, \infty)_{\mathbb{T}} \\
u(0) = \beta u^{\Delta}(\eta), & \lim_{t \to \infty} u^{\Delta}(t) = 0
\end{cases}
\end{align*}
\]

by using Leggett-Williams fixed point theorem, where \( \varphi_p(s) = |s|^{p-2}s, p > 1 \).

Zhao, Ge [28] considered the following \( m \)-point boundary value problem on time scale

\[
\begin{align*}
\begin{cases}
(\varphi_p(u^{\Delta}(t)))^{\nabla} + h(t) f(t, u(t), u^{\Delta}(t)) = 0, & t \in [0, \infty)_{\mathbb{T}} \\
u(0) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), & u^{\Delta}(\infty) = \sum_{i=1}^{m-2} \beta_i u^{\Delta}(\eta_i),
\end{cases}
\end{align*}
\]

where \( u^{\Delta}(\infty) = \lim_{t \to \infty} u^{\Delta}(t) \), \( \varphi_p(s) = |s|^{p-2}s, p > 1 \), \( \eta_1, \eta_2, \ldots, \eta_{m-2} \in \mathbb{T}, \sigma(0) < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty \), \( \alpha_i \geq 0, \beta_i \geq 0 \) for \( i = 1, 2, \ldots, m - 2 \). They obtained the criteria for the existence of positive solutions by using Avery-Peterson theorem.

Karaca, Tokmak [9] discussed the nonlinear multi-point impulsive time scale boundary value problems

\[
\begin{align*}
\begin{cases}
(\varphi(x^{\Delta}(t)))^{\nabla} + \varphi(t) f(t, x(t), x^{\Delta}(t)) = 0, & t \in (0, \infty)_{\mathbb{T}} \\
x(0) = \sum_{i=1}^{m-2} \alpha_i x^{\Delta}(\eta_i), & \lim_{t \to \infty} x^{\Delta}(t) = 0,
\end{cases}
\end{align*}
\]

where \( f \in C([0, \infty)_{\mathbb{T}} \times [0, \infty)_{\mathbb{T}} \times [0, \infty), [0, \infty)) \). \( \alpha_i \geq 0 \ (1 \leq i \leq m - 2) \ 0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty \), \( \varphi : \mathbb{R} \to \mathbb{R} \) is an increasing homeomorphism and positive homomorphism with \( \varphi(0) = 0 \). They established the sufficient conditions for the existence of three positive solutions for \( m \)-point time scale boundary value problems on infinite intervals by using the Leggett-Williams fixed point theorem and five functionals fixed point theorem.

Yaslan, Haznedar [23] obtained the criteria for the existence of at least one, two and three positive solutions to the nonlinear impulsive multi-point time scale boundary value problems

\[
\begin{align*}
\begin{cases}
(\varphi(y^{\Delta}(t)))^{\nabla} + h(t) f(t, y(t), y^{\Delta}(t)) = 0, & t \in [a, \infty)_{\mathbb{T}}, \ t \neq t_k, \ k = 1, 2, \ldots, n \\
y(t_k^-) - y(t_k^+) = I_k(y(t_k)), & k = 1, 2, \ldots, n \\
y(a) - \beta y^{\Delta}(a) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), & \lim_{t \to \infty} y^{\Delta}(t) = 0, \ m \geq 3
\end{cases}
\end{align*}
\]

by using Leray-Schauder fixed point theorem, Avery-Henderson fixed point theorem and the five functionals fixed point theorem, respectively, where \( \alpha_i \geq 0 \ (1 \leq i \leq m - 2) \), \( \beta \geq 0 \), \( \eta_1, \eta_2, \ldots, \eta_{m-2} \in \mathbb{T}, 0 \leq a < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty \), \( f \in C([a, \infty)_{\mathbb{T}} \times [0, \infty)_{\mathbb{T}} \times [0, \infty), [0, \infty)) \) and \( \varphi : \mathbb{R} \to \mathbb{R} \) is an increasing homeomorphism and positive homomorphism with \( \varphi(0) = 0 \).
Karaca, Sinanoglu [26] investigated the criteria for the existence of at least one positive solution to the m-point time scale boundary value problems

\[
\begin{align*}
(\varphi_p(u^\Delta(t)) + h(t)f(t, u(t), u^\Delta(t))) &= 0, \quad t \in (0, \infty)_T, \quad t \neq t_k, \quad k = 1, 2, \ldots, n \\
u(0) &= \sum_{i=1}^{m-2} \alpha_i u^\Delta(\eta_i), \quad u^\Delta(\infty) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \\
u(t_k^+) - u(t_k^-) &= I_k(u(t_k)), \quad \varphi_p(u^\Delta(t_k^+)) - \varphi_p(u^\Delta(t_k^-)) = -f_k(u(t_k)), \quad k \in \mathbb{N}
\end{align*}
\]

by using the four functionals fixed point theorem, where \(\eta_1, \ldots, \eta_{m-2} \in \mathbb{T}, \sigma(0) < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < \infty, u^\Delta(\infty) = \lim_{t \to \infty} u^\Delta(t), \varphi_p(s) = |s|^{p-2}s, \quad p > 1, \quad I_k \in C([0, \infty), [0, \infty)) \) and \(T_k \in C([0, \infty), [0, \infty))\).

We consider the following boundary value problem (BVP)

\[
\begin{align*}
\begin{cases}
\nu^\nabla(t) + h(t)f(t, y(t), \nu^\Delta(t)) &= 0, \quad t \in [a, \infty)_T, \quad t \neq t_k, \quad k = 1, 2, \ldots, n \\
y(t_k^+) - y(t_k^-) &= I_k(y(t_k)), \quad k = 1, 2, \ldots, n \\
y(a) - \gamma \nu^\Delta(a) &= \sum_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i), \quad \lim_{t \to \infty} \nu^\Delta(t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i), \quad m \geq 3
\end{cases}
\end{align*}
\]

where \(T\) is a time scale, \(\alpha_i \geq 0, \beta_i \geq 0 (1 \leq i \leq m - 2), \gamma \geq 0, 0 \leq a < \eta_1 < \ldots < \eta_{m-2} < \infty\) and \(f \in C([a, \infty)_T \times [0, \infty) \times [0, \infty), [0, \infty))\).

We will assume that the following conditions are satisfied:

1. \(H_1\) \(h \in C([a, \infty)_T, [0, \infty)), \quad h(s) \nabla s < \infty\);
2. \(H_2\) \(f(t, (1+t)u, v) \leq \omega(\max(\{|u|, |v|\}))\) with \(\omega \in C([0, \infty), [0, \infty))\) nondecreasing;
3. \(H_3\) \(\sum_{\alpha_i < t_k < \infty} I_k(y(t_k)) < \infty, \quad I_k \in C(\mathbb{R}, \mathbb{R}^+), \quad t_k \in [a, \infty)_T \) and \(y(t_k^+) = \lim_{h \to 0} y(t_k + h), \quad y(t_k^-) = \lim_{h \to 0} y(t_k - h)\) represent the right and left limits of \(y(t)\) at \(t = t_k, \quad k = 1, \ldots, n\).

2. Preliminaries

To state and prove the main results of this paper, we will need the following lemmas.

**Lemma 2.1.** Assume \(H_3\) holds. If \(x \in C([a, \infty)_T, [0, \infty))\) and \(\int_a^\infty x(t) \nabla t < \infty\), then the boundary value problem

\[
\begin{align*}
\begin{cases}
\nu^\nabla(t) + x(t) &= 0, \quad t \in [a, \infty)_T, \quad t \neq t_k, \quad k = 1, 2, \ldots, n \\
y(t_k^+) - y(t_k^-) &= I_k(y(t_k)), \quad k = 1, 2, \ldots, n \\
y(a) - \gamma \nu^\Delta(a) &= \sum_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i), \quad \lim_{t \to \infty} \nu^\Delta(t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i), \quad m \geq 3
\end{cases}
\end{align*}
\]
has a unique solution

\[ y(t) = (\gamma - a) \int_{a}^{t} x(s) \nabla s + t \int_{a}^{t} x(s) \nabla s + \int_{a}^{t} sx(s) \nabla s + (\gamma + t - a) \sum_{i=1}^{m-2} \beta_i y(\eta_i) \]

(2.1) \[ + \sum_{i=1}^{m-2} \alpha_i \left[ \sum_{j=1}^{m-2} \beta_j y(\eta_j) + \int_{\eta_i}^{t} x(s) \nabla s \right] + \sum_{a < t_k < t} I_k(y(t_k)). \]

**Proof.** Since we have \( y^\Delta (t) = -x(t) \) for \( t \in [a, \infty) \), we get

(2.2) \[ y^\Delta (t) = \lim_{t \to \infty} y^\Delta (t) + \int_{a}^{t} x(\xi) \nabla \xi \]

By using the second boundary condition we obtain

\[ y^\Delta (t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_{a}^{t} x(\xi) \nabla \xi. \]

Integrating the above equality from \( a \) to \( t \), we have

\[ y(t) - y(a) - \sum_{a < t_k < t} I_k(y(t_k)) = (t-a) \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_{a}^{t} x(s) \nabla s \Delta \xi. \]

By using the first boundary condition we obtain

\[ y(t) = \gamma y^\Delta (a) + \sum_{i=1}^{m-2} \alpha_i y^\Delta (\eta_i) + (t-a) \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_{a}^{t} (s-a)x(s) \nabla s \]

\[ + (t-a) \int_{a}^{t} x(s) \nabla s + \sum_{a < t_k < t} I_k(y(t_k)). \]

Thus, from (2.2) we have (2.1). \( \square \)

By Lemma 2.1, the solutions of the BVP (1.1) are the fixed points of the operator \( A \) defined by

\[ Ag(t) = (\gamma - a) \int_{a}^{\infty} h(s)f(s,y(s),y^\Delta (s)) \nabla s + t \int_{a}^{\infty} h(s)f(s,y(s),y^\Delta (s)) \nabla s \]

\[ + \int_{a}^{t} sh(s)f(s,y(s),y^\Delta (s)) \nabla s + (\gamma + t - a) \sum_{i=1}^{m-2} \beta_i y(\eta_i) \]

\[ + \sum_{i=1}^{m-2} \alpha_i \left[ \sum_{j=1}^{m-2} \beta_j y(\eta_j) + \int_{\eta_i}^{\infty} h(s)f(s,y(s),y^\Delta (s)) \nabla s \right] + \sum_{a < t_k < t} I_k(y(t_k)). \]
Let $B$ be the Banach space defined by
\[
B = \left\{ y \in C^\Delta([a, \infty)) : \sup_{t \in [a, \infty)} \frac{y(t)}{1+t} < \infty, \lim_{t \to \infty} y^\Delta(t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i) \right\}
\]
with the norm $\|y\| = \max \left\{ \|y\|_1, \|y^\Delta\|_\infty \right\}$, where
\[
\|y\|_1 = \sup_{t \in [a, \infty)} \frac{|y(t)|}{1+t}, \quad \|y^\Delta\|_\infty = \sup_{t \in [a, \infty)} |y^\Delta(t)|
\]
and define the cone $P \subset B$ by
\[
P = \left\{ y \in B : y(a) - \gamma y^\Delta(a) = \sum_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i), \ y \text{ is concave, non-decreasing and nonnegative on } [a, \infty) \right\}.
\]

**Lemma 2.2.** If $y \in P$, then we have $\|y\|_1 \leq M\|y^\Delta\|_\infty$, where
\[
M = \max \left\{ \gamma - a + \sum_{i=1}^{m-2} \alpha_i, 1 \right\}.
\]

**Proof.** For $y \in P$ and $t \in [a, \infty)$, we have
\[
\frac{y(t)}{1+t} = \frac{1}{1+t} \left( \int_a^t y^\Delta(s) \Delta s + \gamma y^\Delta(a) + \sum_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i) \right) \leq \frac{t - a + \gamma + \sum_{i=1}^{m-2} \alpha_i}{1+t} \|y^\Delta\|_\infty
\]
\[
\leq M\|y^\Delta\|_\infty.
\]
Hence, the proof is complete. \qed

**Lemma 2.3.** If (H1)-(H3) hold, then the operator $A : P \to P$ is completely continuous.

**Proof.** First, we will show that $A : P \to P$. For $y \in P$, we have
\[
(Ay)(a) - \gamma (Ay)^\Delta(a) = \sum_{i=1}^{m-2} \alpha_i (Ay)^\Delta(\eta_i),
\]
\[
(Ay)^\Delta(t) = -h(t)f \left( t, y(t), y^\Delta(t) \right) \leq 0,
\]
\[
(Ay)^\Delta(t) = \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_a^t h(s)f \left( s, y(s), y^\Delta(s) \right) \nabla s \geq 0,
\]
\[
(Ay)(a) = \gamma \left( \int_a^\infty x(s) \nabla s + \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \sum_{i=1}^{m-2} \alpha_i \left[ \sum_{j=1}^{m-2} \beta_j y(\eta_j) + \int_{\eta_i}^\infty x(s) \nabla s \right] \right) + \sum_{0 < t_k < t} I_k(y(t_k)) \geq 0.
\]
Hence, $A : P \to P$. 

Now, we will show that $A : P \to P$ is continuous. If $y_n \to y$ as $n \to \infty$ in $P$, then there exists $\tau$ such that $\sup_{n \in \mathbb{N}} \|y_n\| < \tau$. From (H2), for all $t \in [a, \infty)_T$ we have
\[
f(t, y_n(t), y_n^\Delta(t)) \leq \omega \left( \max \left\{ \frac{|y_n(t)|}{1 + t}, |y_n^\Delta(t)| \right\} \right) \leq \omega(\|y_n\|) < \omega(\tau) \quad \text{and} \quad f(t, y(t), y^\Delta(t)) \leq \omega(\|y\|) < \omega(\tau)
\]
by the continuity of norm function. Since
\[
\int_t^\infty h(s)f(s, y_n(s), y_n^\Delta(s)) - f(s, y(s), y^\Delta(s)) \, ds \leq 2\omega(\tau) \int_a^\infty h(s) \, ds < \infty
\]
by using (H1), we get
\[
\| (Ay_n)^\Delta - (Ay)^\Delta \| \to 0,
\]
as $n \to \infty$. Since
\[
\|Ay_n - Ay\| \leq M \| (Ay_n)^\Delta - (Ay)^\Delta \| \to 0,
\]
$A : P \to P$ is continuous.

Now we will show that the image of any bounded subset of $P$ under $A$ is relatively compact in $P$. If $\Omega$ is any bounded subset of $P$, then there exists $K > 0$ such that $\|y\| \leq K$ for $\forall y \in \Omega$. By (H1) and (H2), for $\forall y \in \Omega$, we have
\[
\| (Ay)^\Delta \|_\infty \leq \sum_{i=1}^{m-2} \beta_i |y(\eta_i) - y(\eta_i)| + \int_a^\infty h(s)f(s, y(s), y^\Delta(s)) \, ds \leq K \sum_{i=1}^{m-2} \beta_i (1 + \eta_i) + \omega(K) \int_a^\infty h(s) \, ds < \infty
\]
Since $\|A\Omega\| \leq M \| (A\Omega)^\Delta \|_\infty < \infty$, $A\Omega$ is uniformly bounded.

Now, we show that $A\Omega$ is equicontinuous on $[a, \infty)_T$. For any $R > 0$, $t, p \in [a, R]_T$, and for all $y \in \Omega$, without loss of generality we may assume that $t < p$. By (H2), we have
\[
\| (Ay)^\Delta(t) - (Ay)^\Delta(p) \| = \left| \int_t^p h(s)f(s, y(s), y^\Delta(s)) \, ds \right| \leq \omega(K) \int_t^p h(s) \, ds \to 0,
\]
uniformly as $t \to p$. Since $\| (Ay)^\Delta(t) - (Ay)^\Delta(p) \| \to 0$, uniformly as $t \to p$, we obtain $\| (Ay)(t) - (Ay)(p) \| \to 0$, uniformly as $t \to p$, by Lemma 2.2. Thus, $A\Omega$ is equicontinuous on any compact interval of $[a, \infty)_T$. 
Now, we show that $A\Omega$ is equiconvergent on $[a, \infty)_T$. For any $y \in \Omega$, we have

$$\left| (Ay)^\Delta(t) - (Ay)^\Delta(\infty) \right| = \left| \int_t^\infty h(s)f(s, y(s), y^\Delta(s)) \nabla s \right| \to 0$$

as $t \to \infty$. Then, we obtain $\| (Ay)(t) - (Ay)(\infty) \| \to 0$, as $t \to \infty$, by Lemma 2.2. Therefore $A\Omega$ is equiconvergent on $[a, \infty)_T$.

Hence, the operator $A : P \to P$ is completely continuous. \hfill \Box

### 3. Positive solutions

**Definition 3.1.** Remember that a subset $K \neq \emptyset$ of $X$ is called a retract of $X$ if there is a continuous map $R : X \to K$, a retraction, such that $Rx = x$ on $K$. Let $X$ be a Banach space, $K \subset X$ retract, $\Omega \subset K$ open and $f : \overline{\Omega} \to K$ compact and such that $Fix(f) \cap \partial \Omega = \emptyset$. Then we can define an integer $i_K(f, \Omega)$ which has the following properties.

(a) $i_K(f, \Omega) = 1$ for $f(\Omega) \in \Omega$.
(b) Let $f : \Omega \to K$ be a continuous function and assume that $Fix(f)$ is a compact subset of $\Omega$. Let $\Omega_1$ and $\Omega_2$ be disjoint open subsets of $\Omega$ such that $Fix(f) \subset \Omega_1 \cup \Omega_2$. Then we obtain $i_K(f, \Omega) = i_K(f, \Omega_1) + i_K(f, \Omega_2)$.
(c) Let $G$ be an open subset of $K \times [0, 1]$ and $F : G \to K$ be a compact map. Assume that $Fix(F)$ is a compact subset of $G$. If $G_i = \{x : (x, t) \in G\}$ and $F_i = F(\cdot, t)$, then we have $i_K(F_0, G_0) = i_K(F_1, G_1)$.
(d) If $K_0 \subset K$ is a retract of $K$ and $F(\overline{\Omega}) \subset K_0$, then $i_K(F, \Omega) = i_{K_0}(F, \Omega \cap K_0)$.

We will apply the following well-known result of the fixed point theorems to prove the existence of positive solutions to the (1.1).

**Lemma 3.1.** [7, 15] Let $P$ be a cone in a Banach space $\mathcal{B}$, and let $D$ be an open, bounded subset of $\mathcal{B}$ with $D_P := D \cap P \neq \emptyset$ and $\overline{D_P} \neq P$. Assume that $A : \overline{D_P} \to P$ is a compact map such that $y \neq Ay$ for $y \in \partial D_P$. The following results hold.

(i) If $\| Ay \| \leq \| y \|$ for $y \in \partial D_P$, then $i_P(A, D_P) = 1$.
(ii) If there exists an $b \in P \setminus \{0\}$ such that $y \neq Ay + \lambda b$ for all $y \in \partial D_P$ and all $\lambda > 0$, then $i_P(A, D_P) = 0$.
(iii) Assume $U$ be open in $P$ such that $\overline{U} \subset D_P$. If $i_P(A, D_P) = 1$ and $i_P(A, U_P) = 0$, then $A$ has a fixed point in $D_P \setminus \overline{U}$. The same result holds if $i_P(A, D_P) = 0$ and $i_P(A, U_P) = 1$.

For the cone $P$ given in (2.3) and any positive real number $r$, define the convex set

$$P_r := \{ y \in P : \| y \| < r \}$$

and the set

$$\Omega_r := \{ y \in P : \min_{t \in [a, \infty)} \frac{y(t)}{1 + t} \leq M, \min_{t \in [a, \infty)} y^\Delta(t) < r \}$$
where \( M \) is defined in (2.4). The following results are proved in [15].

**Lemma 3.2.** The set \( \Omega_r \) has the following properties.

(i) \( \Omega_r \) is open relative to \( P \).

(ii) \( P_{Mr} \subset \Omega_r \subset P_r \).

(iii) \( y \in \partial \Omega_r \) if and only if \( \min_{t \in [\eta, \infty)} \frac{y(t)}{1 + t} = Mr \).

(iv) If \( y \in \partial \Omega_r \), then \( Mr(1 + t) \leq y(t) \leq r \) for \( t \in [\eta, \infty) \).

For convenience, we introduce the following notations. Let

\[
N = \int_{a}^{\infty} h(s) \nabla s
\]

and

\[
L = \int_{\eta}^{\infty} h(s) \nabla s.
\]

**Lemma 3.3.** If the conditions

\[
f_0^r \leq \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN} \quad \text{and} \quad y \neq Ay \text{ for } y \in \partial P_r,
\]

hold, then \( i_P(A, P_r) = 1 \).

**Proof.** If \( y \in \partial P_r \), then we have \( 0 \leq \frac{y(t)}{1 + t} \leq r \) and \( 0 \leq y^\Delta(t) \leq r \).

\[
\|Ay\| \leq M \sup_{t \in [a, \infty) \cap \eta_r} |Ay^\Delta(t)|
\]

\[
= M \left( \sum_{i=1}^{m-2} \beta_i y(\eta_i) + \int_{a}^{\infty} h(s) f \left( s, y(s), y^\Delta(s) \right) \nabla s \right)
\]

\[
\leq M \left( r \sum_{i=1}^{m-2} \beta_i(1 + \eta_i) + \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN} r \int_{a}^{\infty} h(s) \nabla s \right)
\]

\[
= r = \|y\|.
\]
It follows that \(|Ay| \leq |y|\) for \(y \in \partial P_r\). By Lemma 3.1(i), we get \(i_P(A, P_r) = 1\). □

**Lemma 3.4.** If the conditions

\[
f^l_{M_r} \geq M\frac{1 + \eta_1}{\alpha_1 L} \text{ and } y \neq Ay \text{ for } y \in \partial \Omega_r,
\]

hold, then \(i_P(A, \Omega_r) = 0\).

**Proof.** Let \(h(t) \equiv 1\) for \(t \in [a, \infty)\), then \(h \in \partial \Omega_1\). Assume there exist \(y_0 \in \partial \Omega_r\) and \(\lambda_0 > 0\) such that \(y_0 = Ay_0 + \lambda_0 b\). Then for \(t \in [a, \infty)\) we have

\[
y_0(t) = Ay_0(t) + \lambda_0 b(t) \\
\geq \alpha_1 \int_{n_1}^{\infty} h(s)f(s, y(s), y^\lambda(s)) \nabla s + \lambda_0 \\
\geq \alpha_1 \int_{n_1}^{\infty} h(s) \nabla s + \lambda_0 \\
\geq \alpha_1 M \frac{1 + \eta_1}{\alpha_1 L} \int_{n_1}^{\infty} h(s) \nabla s + \lambda_0 \\
= Mr(1 + \eta_1) + \lambda_0.
\]

But this implies that \(Mr(1 + \eta_1) \geq Mr(1 + \eta_1) + \lambda_0\), a contradiction. Hence, \(y_0 \neq Ay_0 + \lambda_0 b\) for \(y_0 \in \partial \Omega_r\) and \(\lambda_0 > 0\), so by Lemma 3.1(ii), we get \(i_P(A, \Omega_r) = 0\). □

**Theorem 3.1.** Assume (H1)-(H3) hold. Let \(M, N\) and \(L\) be as in (2.4), (3.1), and (3.2), respectively. Suppose that one of the following conditions holds.

(C1) There exist constants \(c_1, c_2, c_3 \in \mathbb{R}\) with \(0 < c_1 < c_2 < cc_3\) such that

\[
f^l_{M_{c_1}} \geq M\frac{1 + \eta_1}{\alpha_1 L}, f^ {c_2}_0 \leq \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN}, \text{ and } y \neq Ay \text{ for } y \in \partial P_{c_2}.
\]

(C2) There exist constants \(c_1, c_2, c_3 \in \mathbb{R}\) with \(0 < c_1 < cc_2 \text{ and } c_2 < c_3\) such that

\[
f^l_{M_{c_1}} \geq M\frac{1 + \eta_1}{\alpha_1 L}, f^ {c_2}_0 \leq \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN}, f^ {c_3}_{M_{c_2}} \geq M\frac{1 + \eta_1}{\alpha_1 L}, \text{ and } y \neq Ay \text{ for } y \in \partial \Omega_{c_2}.
\]

Then (1.1) has two positive solutions. Additionally, if in (C2) the condition

\[
f^ {c_1}_0 \leq \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN}
\]

was also satisfied, then (1.1) has three positive solutions.
is replaced by

\[ f_{c_1}^c < \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN}, \]

then (1.1) has a third positive solution in \( P_{c_1} \).

**Proof.** Assume that \( (C1) \) holds. We show that either \( A \) has a fixed point in \( \partial \Omega_{c_1} \) or in \( P_{c_2} \setminus \overline{\Omega}_{c_1} \). If \( y \neq Ay \) for \( y \in \partial \Omega_{c_1} \), then by Lemma 3.4, we have \( i_P(A, \Omega_{c_1}) = 0 \). Since

\[ f_{c_2}^c \leq \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN} \]

and \( y \neq Ay \) for \( y \in \partial P_{c_2} \), from Lemma 3.3 we get \( i_P(A, \Omega_{c_2}) = 1 \). By Lemma 3.2(ii) and \( c_1 < c_2 \), we have \( \overline{\Omega}_{c_2} \subset \overline{P}_{c_1} \subset P_{c_2} \). From Lemma 3.1(iii), \( A \) has a fixed point in \( P_{c_2} \setminus \overline{\Omega}_{c_1} \). If \( y \neq Ay \) for \( y \in \partial \Omega_{c_2} \), then from Lemma 3.4 \( i_P(A, \Omega_{c_2}) = 0 \). By Lemma 3.2(ii) and \( c_2 < M_{c_3} \), we get \( \overline{P}_{c_2} \subset P_{M_{c_3}} \subset \Omega_{c_3} \). From Lemma 3.1(iii), \( A \) has a fixed point in \( \Omega_{c_3} \setminus \overline{P}_{c_2} \). The proof is similar when \( (C2) \) holds and we omit it here.

As a special case of Theorem 3.1, we have the following result.

**Theorem 3.2.** Suppose (H1)-(H3) hold. Let \( M, N \) and \( L \) be as in (2.4), (3.1), and (3.2), respectively. Assume that one of the following conditions holds.

(C3) There exist constants \( c_1, c_2 \in \mathbb{R} \) with \( 0 < c_1 < c_2 \) such that

\[ f_{M_{c_1}}^c \geq M \frac{1 + \eta_1}{\alpha_1 L} \quad \text{and} \quad f_{c_0}^c \leq \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN}. \]

(C4) There exist constants \( c_1, c_2 \in \mathbb{R} \) with \( 0 < c_1 < M_{c_2} \) such that

\[ f_{c_0}^c \leq \frac{1 - M \sum_{i=1}^{m-2} \beta_i(1 + \eta_i)}{MN} \quad \text{and} \quad f_{M_{c_2}}^c \geq M \frac{1 + \eta_1}{\alpha_1 L}. \]

Then (1.1) has a positive solution.

**References**


Received by editors 06.03.2021; Revised version 15.05.2021; Available online 24.05.2021.

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