

ARITHMETICS OF SOME SEQUENCES VIA 2-DETERMINANTS

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ABSTRACT. We extend our investigation of 2-determinants, which we defined in a previous paper. For a linear homogenous recurrence of the second order, we consider relations between different sequences satisfying the same linear homogeneous recurrence of the second order. After we prove a generalized identity of d'Ocagne, we derive, from a single identity, a number of classical identities (and their generalizations) such as d'Ocagne's, Cassini's, Catalan's, and Vajda's. Along the way, the corresponding combinatorial interpretations in terms of restricted words over a finite alphabet are stated for the sequences we investigate.

1. Introduction and preliminaries

We continue our investigation of 2-determinants, defined in [7]. For a linear homogenous recurrence of the second order, we consider relations between different sequences satisfying the same recurrence. Linear homogenous recurrences of the second order are much studied objects and there is an overwhelming amount of literature containing various formulas involving sequences defined recurrently (as an introduction to the topic, we recommend [14], [9], [1], and [4]).

Our method produces a number of identities involving Fibonacci numbers and polynomials, bisection of Fibonacci numbers, positive integers, Pell numbers, Jacobsthal numbers, Mersenne numbers, and Chebyshev polynomials of the second kind. For these and some other objects, we derive variations of well-known identities, such as d'Ocagne's, Cassini's, Vajda's, and Catalan's identities.

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Let x and y be integer-valued variables. We consider the following recurrence of the second order:

$$(1.1) \quad a_{n+1}(x, y) = x \cdot a_n(x, y) + y \cdot a_{n-1}(x, y), \quad n > 0,$$

where

$$(1.2) \quad a_0(x, y) = 0, \quad a_1(x, y) = 1.$$

We investigate mutual connections between three sequences (a) , (b) , and (c) which satisfy (1.1), and where (a) also satisfies (1.2). Our main result is the following general identity.

THEOREM 1.1 (A generalized identity of d'Ocagne). *Let (a) be a sequence satisfying Equation (1.1) and Equation (1.2), and let $(b) = (b_0, b_1, \dots)$ and $(c) = (c_0, c_1, \dots)$ both satisfy Equation (1.1). Then*

$$(1.3) \quad \begin{vmatrix} b_k & b_{k+m} \\ c_k & c_{k+m} \end{vmatrix} = (-y)^k \cdot a_m \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}.$$

Once we establish the identity (1.3), from this single identity we will derive a number of identities involving some classical sequences of numbers. We note that this formula is, in a sense, asymmetric, since factors $(-y)^k$ and a_m appear only on the right-hand side of the equation. Moreover, on the right-hand side only two initial terms of sequences (b) and (c) appear.

We also note that the left-hand side of (1.3) contains two arbitrary parameters k and m . Furthermore, in Theorem 2.3, a formula containing four different parameters will be proved. Extra parameters allow us to derive a number of identities concerning classical sequences as special cases of this particular identity.

We see that the fundamental role in our investigation is played by sequences that satisfy (1.1) and (1.2). In this section we closely investigate such sequences. In particular, we give explicit formulas for them and combinatorial interpretations of such sequences in terms of restricted words over a finite alphabet.

In the cases under consideration, x and y will always have fixed values, so that we can write a_n instead of $a_n(x, y)$, omitting x and y to simplify notation.

The following result is proved in Proposition 11, Proposition 12, and Proposition 16 in [8].

PROPOSITION 1.1. *Let (a) be a sequence satisfying (1.1) and (1.2).*

(1) *The following explicit formula holds*

$$(1.4) \quad a_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} \cdot x^{n-2k-1} \cdot y^k.$$

(2) *If $x > 0$ and $y > 0$, then a_{n+1} equals the number of words of length n over the alphabet $\{0, 1, \dots, x-1, x, \dots, x+y-1\}$ in which letters $0, 1, \dots, x-1$ avoid runs of odd lengths.*

(3) *If $x > 0$, $y < 0$, and $-y < x$, then a_n is the number of words of length $n-1$ over $\{0, 1, \dots, x-1\}$ with no subword of the form $0i$, where $i \in \{1, 2, \dots, -y\}$.*

Some well-known integer sequences are given by a linear homogeneous recurrence of the second order, for instance, Fibonacci numbers, Fibonacci polynomials, Jacobsthal numbers, and Pell numbers. They are obtained when $x > 0, y > 0$. The first example concerning the case $x > 0, y < 0$ shows that positive integers also satisfy (1.1). Chebyshev polynomials of the second kind, bisection of Fibonacci numbers, and Mersenne numbers also belong to this class.

In the next example, Equation (1.4) is used to give an explicit expression for each of the classical sequences involved. These formulas are well known and most of them have been widely described in numerous literature ([14], [10], [1], [2], [3], [5], [6]). Here, ten examples are listed in order to emphasize the idea that all of them hold a common origin in Equation (1.4). Along the way, the corresponding combinatorial interpretations are stated for each of the sequences.

We start with the most important two: Fibonacci polynomials and Chebyshev polynomials of the second kind. Some of the examples that follow are just particular cases of these two.

EXAMPLE 1.1. (1) If $x > 0$ and $y = 1$, then $a_{n+1} = F_n(x), n \geq 1$, where $F_n(x)$ is the n th Fibonacci polynomial. Also, Equation (1.4) gives the explicit expression for $F_n(x)$ (see [2]):

$$F_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^{n-2k-1}.$$

In terms of restricted words, if $x > 0$ is an integer, then $F_n(x)$ equals the number of words of length n over $\{0, 1, \dots, x-1\}$ in which 0 avoids runs of odd lengths.

(2) If $x = 2z$ and $y = -1$, then $a_n = U_n(z)$, where $U_n(z)$ is the Chebyshev polynomial of the second kind. From Equation (1.4), we get the following well-known formula (see [3]):

$$U_n(z) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cdot \binom{n-k-1}{k} (2z)^{n-2k-1}.$$

In terms of restricted words, if $z > 0$ is an integer, then $U_{n+1}(z)$ equals the number of words of length n over the alphabet $\{0, 1, \dots, 2z-1\}$ avoiding the subword 01.

(3) Particular case of (1) is when $x = 1$ and $y = 1$. In this case we have $a_{n+1} = F_n, n \geq 1$. Also, Equation (1.4) is the standard expression for the Fibonacci numbers in terms of the binomial coefficients (see Identity (54) of [14]):

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

Combinatorially, the Fibonacci number F_n equals the number of binary words of length n avoiding a run of zeros of odd length.

- (4) For $x = 2$ and $y = 1$, we have $a_{n+1} = P_n$, where P_n , $n \geq 0$, is the n th Pell number. From Equation (1.4), we get (see [5])

$$P_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2k-1} \cdot \binom{n-k-1}{k}.$$

Also, P_n equals the number of ternary words of length n in which 0 avoids runs of odd lengths. The Pell numbers are sometimes called “silver Fibonacci numbers”.

- (5) For $x = 1$ and $y = 2$, we have $a_{n+1} = J_n$, where J_n , ($n = 0, 1, 2, \dots$), are the Jacobsthal numbers. From Equation (1.4), we obtain (see [6])

$$J_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^k \binom{n-k-1}{k-1}.$$

Also, the number J_n equals the number of ternary words of length n in which 0 and 1 avoid runs of odd lengths.

- (6) If $x = 2$ and $y = 2$, then a_{n+1} is the number of ways to tile a board of length n using tiles of two colors of length 1 and 2. Also,

$$a_{n+1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-k-1} \cdot \binom{n-k-1}{k}, \quad n > 0.$$

- (7) If $x = 2$ and $y = -1$, then $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 2a_n - a_{n-1}$, which is the recurrence for non-negative integers. Thus, we have

$$n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cdot 2^{n-2k-1} \binom{n-k-1}{k}, \quad n > 0.$$

This formula for n may seem rather complex, but its combinatorial meaning is very simple. Namely, n equals the number of binary words of length $n - 1$ avoiding 01, which is obvious.

- (8) If $x = 3$ and $y = -1$, then we have that a_{n+1} is the bisection of Fibonacci numbers, that is, $a_{n+1} = F_{2n}$. From Equation (1.4), we obtain

$$F_{2n} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cdot 3^{n-2k-1} \cdot \binom{n-k-1}{k}.$$

Also, F_{2n} equals the number of ternary words of length $n - 1$ avoiding 01.

- (9) If $x = 3$ and $y = -2$, then $a_n = 2^n - 1$. These numbers are usually called Mersenne numbers. We have

$$2^n - 1 = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-2)^k \cdot 3^{n-2k-1} \cdot \binom{n-k-1}{k}.$$

Also, $2^{n+1} - 1$ equals the number of ternary words of length n avoiding 01 and 02.

(10) If $x = 4$ and $y = -3$, then $a_n = \frac{3^n - 1}{2}$. Next we have

$$\frac{3^n - 1}{2} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-3)^k \cdot 4^{n-2k-1} \cdot \binom{n-k-1}{k}.$$

Also, $\frac{3^n - 1}{2}$ equals the number of quaternary words of length n avoiding 01, 02 and 03.

2. Identities

Unless stated otherwise, throughout this section, we assume that (a) is a sequence satisfying Equation (1.1) and Equation (1.2), and that $(b) = (b_0, b_1, \dots)$ and $(c) = (c_0, c_1, \dots)$ both satisfy Equation (1.1). Note that we do not have any assumptions about the initial conditions for the sequences (b) and (c) .

In our previous paper [7], we defined the notion of an n -determinant, and used it to derive numerous identities related to some sequences given by linear homogeneous recurrences of the second order. Here, we use the results obtained via 2-determinants to derive a number of identities related to some classical sequences. For the definition of an n -determinant and related results, we refer the reader to [7]. We remark here that the matrix methods are widely used when it comes to proving some of the classical identities. For example, see [9] for a demonstration of how powerful these methods can be in simplifying proofs of some of the identities.

The starting point of our investigation is the following general theorem that relates sequences satisfying the same linear homogeneous recurrence of the second order.

THEOREM 2.1 ([8], Proposition 8). *Let $(u) = (u_0, u_1, \dots)$ and $(v) = (v_0, v_1, \dots)$ be any two sequences, and let $(b) = (b_0, b_1, \dots)$ and $(c) = (c_0, c_1, \dots)$ be two sequences both satisfying the same recurrence:*

$$\begin{aligned} b_{n+1} &= u_n b_n + v_{n-1} \cdot b_{n-1}, \quad n > 0, \\ c_{n+1} &= u_n c_n + v_{n-1} \cdot c_{n-1}, \quad n > 0. \end{aligned}$$

Then, for $k \leq n + 2$, we have

$$(2.1) \quad \begin{vmatrix} b_k & b_{n+2} \\ c_k & c_{n+2} \end{vmatrix} = (-1)^k \cdot v_1 \cdot v_2 \cdots v_k \cdot a_{n-k+2} \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix},$$

where $a_0 = 0, a_1 = 1, a_2 = u_{k+1}, a_i = v_{k+i-2} a_{i-2} + u_{k+i-1} a_{i-1}, i > 2$.

The basic result that we use to investigate sequences of numbers is the following direct corollary of the previous theorem.

THEOREM 2.2 (A generalized identity of d’Ocagne). *Let (a) be a sequence satisfying Equation (1.1) and Equation (1.2), and let $(b) = (b_0, b_1, \dots)$ and $(c) = (c_0, c_1, \dots)$ both satisfy Equation (1.1). Then*

$$(2.2) \quad \begin{vmatrix} b_k & b_{k+m} \\ c_k & c_{k+m} \end{vmatrix} = (-y)^k \cdot a_m \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}.$$

PROOF. The statement follows from the previous theorem by setting, for all $n, u_n = x, v_n = y$, and $n + 2 = k + m$. □

The importance of Equation (2.2) lies in the fact that there is an extra term a_m on the right-hand side of the formula. This gives us a lot of freedom in choosing concrete values of the sequence a_m , yielding many identities as special cases of Equation (2.2). We will derive some of these identities now. Note that when it comes to the indices that appear on the left-hand side of Equation (2.2), they are not as general as indices appearing in some well-known identities, such as Identity (70) of [9], but the extra term a_m on the right-hand side makes up for the lack of full generality of the indices. Moreover, there are four different parameters appearing as indices in the identity given in Theorem 2.3. Extra parameters allow us to derive a number of identities concerning classical sequences as special cases of this particular identity.

In Section 4 of [7], we stated a number of d'Ocagne's identities for Fibonacci numbers and polynomials, Lucas and Chebyshev polynomials. We illustrate Equation (2.2) by deriving identities for some sequences described in the preceding section.

To clarify the name of the identity given by Equation (2.2), we prove that d'Ocagne's identity for Fibonacci numbers is a particular case of this identity.

COROLLARY 2.1 (d'Ocagne's identity). *The following formula holds*

$$\begin{vmatrix} F_{k+1} & F_{k+m+1} \\ F_k & F_{k+m} \end{vmatrix} = (-1)^k \cdot F_m,$$

PROOF. When $x = y = 1$, Equation (2.2) becomes the recurrence for Fibonacci numbers. Hence, $a_m = F_m$, $m \geq 0$. For sequences (b) and (c), we again choose the Fibonacci numbers with the initial conditions such that the determinant on the right-hand side of Equation (2.2) is equal to 1. For instance, we can choose $c_0 = 0, c_1 = 1$, and $b_0 = 1, b_1 = 1$, that is, $b_k = F_{k+1}$ and $c_k = F_k$. We thus obtain d'Ocagne's identity. \square

We note one more consequence of Equation (2.2). Namely, if we know sequences (b) and (c) from the previous theorem, we can determine the sequence (a).

COROLLARY 2.2. *If (a), (b), and (c) are sequences satisfying (2.2), then the members of the sequence (a) are rational functions of numbers $b_i, c_i, (i = 0, 1, \dots)$, with denominator equal to $(-y)^k \cdot (b_0 c_1 - b_1 c_0)$.*

REMARK 2.1. Besides d'Ocagne's identity, three of the most important identities are: Cassini's, Catalan's, and Vajda's. All these identities can be derived from the generalized identity of d'Ocagne (2.2). What we show here is that these important identities hold for each sequence satisfying (1.1) and (1.2), i.e. they are, in a sense, consequences of the homogenous linear recurrence of the second order, not of the particular coefficients.

For $m = 1$, we have $a_1 = 1$, so that from Equation (2.2) we get the following identity (see Identity (70) of [9]).

PROPOSITION 2.1 (A Cassini-like identity). *The following formula holds*

$$\begin{vmatrix} b_k & b_{k+1} \\ c_k & c_{k+1} \end{vmatrix} = (-y)^k \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}.$$

It is easy to see that for $x = y = 1, b_0 = 1, b_1 = 1, c_0 = 0,$ and $c_1 = 1,$ we obtain the standard Cassini's identity for Fibonacci numbers.

We now consider the Lucas numbers $L_0 = 2, L_1 = 1, L_2 = 3, \dots$ We know that these numbers satisfy the same recurrence as the Fibonacci numbers do. We thus take $x = y = 1$ and obtain the following identity (see Identity (16b) of [14]).

IDENTITY 2.1 (A Cassini-like identity for Fibonacci and Lucas numbers).

$$L_k \cdot F_{k+1} - L_{k+1} \cdot F_k = 2 \cdot (-1)^k.$$

In Example 1.1, we have seen that the Jacobsthal numbers $J_n, (n = 0, 1, 2, \dots),$ are obtained from Equation (1.4) for $x = 1, y = 2.$ If we take $b_0 = J_1, b_1 = J_2, c_0 = J_0, c_1 = J_1,$ we obtain the following identity (see Identity (2.5) of [6]).

IDENTITY 2.2 (A Cassini-like identity for Jacobsthal numbers).

$$J_{k+1}^2 - J_k \cdot J_{k+2} = (-2)^k.$$

Since Chebyshev polynomials $T_n(z)$ of the first kind satisfy the same recurrence as $U_n(x)$ do, by taking

$$T_0(0) = 1, T_1(1) = z, x = 2z, y = -1,$$

we obtain the following identity.

IDENTITY 2.3 (A Cassini-like identity for Chebyshev polynomials of the first kind).

$$T_{k+1}^2(z) - T_k(z) \cdot T_{k+2}(z) = 1 - z^2.$$

Next, we assume that $k \geq p.$ If we replace k by $k - p$ in Equation (2.2), then we get

$$\begin{vmatrix} b_{k-p} & b_{k+m-p} \\ c_{k-p} & c_{k+m-p} \end{vmatrix} = (-y)^{k-p} \cdot a_m \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}.$$

If we apply Equation (2.2) to the right-hand side of the previous equality, we obtain the following universal property. This is a well known index reduction formula (cf. Identity (70) of [9]).

PROPOSITION 2.2 (Index reduction formula). *If $k \geq p,$ then*

$$\begin{vmatrix} b_k & b_{k+m} \\ c_k & c_{k+m} \end{vmatrix} = (-y)^p \cdot \begin{vmatrix} b_{k-p} & b_{k-p+m} \\ c_{k-p} & c_{k-p+m} \end{vmatrix}.$$

In particular, if $k = p,$ then by using the index reduction formula, we can write d'Ocagne's identity in the form:

$$\begin{vmatrix} b_k & b_{k+m} \\ c_k & c_{k+m} \end{vmatrix} = (-y)^k \cdot \begin{vmatrix} b_0 & b_m \\ c_0 & c_m \end{vmatrix}.$$

By comparing the last equality with Equation (2.2), we get the following identity.

PROPOSITION 2.3 (A reduced identity of d'Ocagne).

$$a_m \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix} = \begin{vmatrix} b_0 & b_m \\ c_0 & c_m \end{vmatrix}.$$

We illustrate this formula with two identities. The first identity concerns Fibonacci numbers.

IDENTITY 2.4. *For arbitrary non-negative integers m , p , and q , where $m > p$, the following holds*

$$F_m \cdot \begin{vmatrix} F_p & F_{p+1} \\ F_q & F_{q+1} \end{vmatrix} = \begin{vmatrix} F_p & F_{m+p} \\ F_q & F_{m+q} \end{vmatrix}.$$

The next identity concerns Chebyshev polynomials $U_n(x)$ of the second kind.

IDENTITY 2.5. *For arbitrary non-negative integers m , p , and q , where $m > p$, the following holds*

$$U_m(x) \cdot \begin{vmatrix} U_p(x) & U_{p+1}(x) \\ U_q(x) & U_{q+1}(x) \end{vmatrix} = \begin{vmatrix} U_p(x) & U_{m+p}(x) \\ U_q(x) & U_{m+q}(x) \end{vmatrix}.$$

In the next result, we introduce two more parameters in the formula (2.2). In this theorem we take $(c) = (a)$.

THEOREM 2.3 (Four parameter theorem). *For $m \geq k, p \geq q$, the following formula holds*

$$\begin{vmatrix} b_{k+p} & b_{m+p} \\ a_{k+q} & a_{m+q} \end{vmatrix} = (-y)^{k+q} \cdot a_{m-k} \cdot b_{p-q}.$$

PROOF. We only need to use the index reduction formula twice. We have

$$\begin{vmatrix} b_{k+p} & b_{m+p} \\ a_{k+q} & a_{m+q} \end{vmatrix} = (-y)^k \cdot \begin{vmatrix} b_p & b_{m+p-k} \\ a_q & a_{m+q-k} \end{vmatrix}.$$

Applying the index reduction formula on the right-hand side of the last equation, we obtain

$$\begin{vmatrix} b_{k+p} & b_{m+p} \\ a_{k+q} & a_{m+q} \end{vmatrix} = (-y)^{k+q} \cdot \begin{vmatrix} b_{p-q} & b_{m+p-k-q} \\ a_0 & a_{m-k} \end{vmatrix},$$

and since $a_0 = 0$, the assertion follows. \square

By replacing m by $m+k$, p by $p+q$, and finally $k+q$ by k , we obtain the following corollary of the previous theorem.

IDENTITY 2.6 (A Vajda-like identity). *For sequences (a) and (b) we have*

$$(2.3) \quad \begin{vmatrix} b_{k+p} & b_{k+m+p} \\ a_k & a_{k+m} \end{vmatrix} = (-y)^k \cdot a_m \cdot b_p.$$

It is clear that, if $x = 1, y = 1$ and $a_i = F_i, b_i = F_i (i = 0, 1, \dots)$, we obtain the well known Vajda's identity for Fibonacci numbers.

Also, it is clear that, if $x = 1, y = 1$ and $a_i = F_i, b_i = L_i (i = 0, 1, \dots)$, where L_i is the i th Lucas number, we obtain the following identity (see Identity (19b) in [14]).

IDENTITY 2.7 (A Vajda-like identity for Fibonacci and Lucas numbers).

$$\begin{vmatrix} L_{k+p} & L_{k+m+p} \\ F_k & F_{k+m} \end{vmatrix} = (-1)^k \cdot F_m \cdot L_p.$$

The following three identities are special cases of a Vajda-like identity for non-Fibonacci numbers.

IDENTITY 2.8 (A Vajda-like identity for Mersenne numbers). *If $a_i = 2^{i-1}$, ($i \geq 1$), then*

$$\begin{vmatrix} 2^{k+p} - 1 & 2^{k+m+p} - 1 \\ 2^k - 1 & 2^{k+m} - 1 \end{vmatrix} = 2^k \cdot (2^m - 1) \cdot (2^p - 1).$$

IDENTITY 2.9 (A Vajda-like identity for positive integers). *If $a_i = i$, $i \geq 0$, then*

$$\begin{vmatrix} k + p & k + m + p \\ k & k + m \end{vmatrix} = m \cdot p.$$

If we take $(a) = (b)$, $m = p = r$, $k = n - r$ in (2.3), we obtain the following identity.

IDENTITY 2.10 (A Catalan-like identity).

$$\begin{vmatrix} a_n & a_{n+r} \\ a_{n-r} & a_n \end{vmatrix} = (-y)^{n-r} \cdot a_r^2.$$

It is clear that this identity generalizes the standard Catalan's identity for Fibonacci numbers, which is obtained for $(a) = (F)$ and $y = 1$. We illustrate this case with several identities. The first identity is for Jacobsthal numbers (cf. [6]).

IDENTITY 2.11 (A Catalan-like identity for Jacobsthal numbers). *If $a_n = J_n$, then*

$$J_n^2 - J_{n-i} \cdot J_{n+i} = (-2)^{n-i} \cdot J_i^2.$$

IDENTITY 2.12 (A Catalan-like identity for Pell numbrs). *If $a_n = P_n$, then*

$$P_n^2 - P_{n-i} \cdot P_{n+i} = (-1)^{n-i} \cdot P_i^2.$$

See [5] for this and some other identities involving Pell numbers.

IDENTITY 2.13 (A Catalan-like identity for $U_n(z)$). *Assume that $a_n = U_n(z)$. Then*

$$U_n^2(z) - U_{n-i}(z) \cdot U_{n+i}(z) = (-1)^{n-i} \cdot U_i^2(z).$$

See [13] for the direct proof of the previous identity.

Finally, we illustrate the identity from Theorem 2.3 by two well known identities for Fibonacci numbers. For the first identity, we set $(a) = (b) = (F)$, $p = m$, $q = k$ and $y = 1$, then we have the following result ([11], page 77).

IDENTITY 2.14.

$$F_{k+m}^2 - F_{m-k}^2 = F_{2k} \cdot F_{2m}.$$

Also, if $p = m + 1$, $q = k + 1$, and $y = 1$, then we have the identity from [12], page 63.

IDENTITY 2.15.

$$F_{k+m+1}^2 + F_{m-k}^2 = F_{2k+1} \cdot F_{2m+1}.$$

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