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COMMON FIXED POINTS FOR TWO WEAK SUBSEQUENTIAL CONTINUOUS MAPPINGS

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ABSTRACT. In this paper we are concerned with the existence and uniqueness of common fixed points for a pair of mappings satisfying an implicit relation under new concepts. Our results present an interesting contribution in the fixed point theory's area.

1. Introduction

In 1986, Jungck [**5**] introduced the notion of compatible mappings. Inspired by the above work, many authors developed much weaker conditions. One of the most interesting generalization is subcompatibility introduced in [**2**] ([**3**]). Again, Pathak et al. [**10**] introduced the notions of R-weak commutativity of type $(\mathcal{A}_f)$ and $(\mathcal{A}_g)$ for obtaining common fixed point theorems. Motivated by the above concepts, Kumar ([**6**], [**7**]) gave the notion of R-weak commutativity of type $(\mathcal{P})$. On the other hand, Pant [**8**] initiated the study of fixed points for discontinuous mappings by using the concept of reciprocal continuity. Recently, in [**2**] ([**3**]), we suggested the notion of subsequential continuity which represents a legitimate generalization of the concept of reciprocal continuity. More recently, Pant et al. [**9**] introduced the notion of weak reciprocal continuity and obtained fixed point theorems by employing the new notion. Quite recently, Gopal et al. [**4**] presented their new notions of sequential continuity of type $(\mathcal{A}_f)$ and $(\mathcal{A}_g)$. Appeared in 2016 in [**1**], the new concept of weak subsequential continuity represents a genuine reasonable generalization of weak reciprocal continuity (resp. sequential continuity of type

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$(\mathcal{A}_f)$ or $(\mathcal{A}_g)$). This definition makes up an addition to develop the literature of fixed point theory.

2. Preliminaries

Let us start by stating some needed definitions.

Definition 2.1. ([**5**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called compatible if and only if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Definition 2.2. ([**2, 3**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called subcompatible if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, $t \in \mathcal{X}$ and which satisfy

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0.$$

Definition 2.3. ([**8**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$.

Definition 2.4. ([**2, 3**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be subsequentially continuous if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$ and satisfy $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$.

Definition 2.5. ([**9**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ will be called weakly reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$.

Definition 2.6. ([**4**]) A pair (f, g) of self-mappings defined on a metric space $(\mathcal{X}, d)$ is said to be sequentially continuous of type $(\mathcal{A}_f)$ if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} ggx_n = gt$.

Definition 2.7. ([**4**]) A pair (f, g) of self-mappings defined on a metric space $(\mathcal{X}, d)$ is said to be sequentially continuous of type $(\mathcal{A}_g)$ if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} gfx_n = gt$ and $\lim_{n\to\infty} ffx_n = ft$.

Definition 2.8. ([**10**]) Let $(\mathcal{X}, d)$ be a metric space and let f, g be self-mappings of $\mathcal{X}$. The mappings f and g are said to be R-weakly commuting of type $(\mathcal{A}_f)$ if there exists a positive real number R such that

$$d(fgx, ggx) \leqslant Rd(fx, gx) \tag{2.1}$$

for all $x \in \mathcal{X}$. f and g are said to be R-weakly commuting of type $(\mathcal{A}_f)$ if (2.1) holds for some real number $R > 0$.

Definition 2.9. ([**10**]) Let $(\mathcal{X}, d)$ be a metric space and let f, g be self-mappings of $\mathcal{X}$. The mappings f and g are said to be R-weakly commuting of type $(\mathcal{A}_g)$ if there exists a positive real number R such that

$$d(gfx, ffx) \leqslant Rd(fx, gx) \tag{2.2}$$

for all $x \in \mathcal{X}$. f and g are said to be R-weakly commuting of type $(\mathcal{A}_g)$ if (2.2) holds for some real number $R > 0$.

Definition 2.10. ([**6, 7**]) A pair of self-mappings (f, g) of a metric space $(\mathcal{X}, d)$ is said to be R-weakly commuting of type $(\mathcal{P})$ if there exists some $R > 0$ such that

$$d(ffx, ggx) \leqslant Rd(fx, gx)$$

for all $x \in \mathcal{X}$.

Definition 2.11. ([**1**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called **weakly subsequentially continuous** if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$ and which satisfy $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$.

According to definitions 2.5, 2.6, 2.7, 2.11, it can easily seen that weakly reciprocally continuous mappings are weakly subsequentially continuous mappings. Also, if in our definition we have $\lim_{n\to\infty} fgx_n = ft$, then evidently sequentially continuous of type $(\mathcal{A}_f)$ mappings imply our definition (alternately, if we have $\lim_{n\to\infty} gfx_n = gt$, then sequentially continuous of type $(\mathcal{A}_g)$ mappings imply our definition). To see that the converse implications are not true in general, let us give the next example which fulfills our desire.

Example 2.1. Let $\mathcal{X} = [0, 6]$ and let d be the usual metric on $\mathcal{X}$. We define f, $g : \mathcal{X} \to \mathcal{X}$ as follows:

$$fx = \begin{cases} 3 - x \text{ if } x \in [0, 3] \\ \dfrac{9}{x} \text{ if } x \in (3, 6], \end{cases} \qquad gx = \begin{cases} 3 + x \text{ if } x \in [0, 3) \\ \dfrac{27}{x^2} \text{ if } x \in [3, 6]. \end{cases}$$

First, we readily see that f and g are not continuous at $x = 3$. It can also be noted that f and g are weakly subsequentially continuous. To see this, let $\{x_n\}$ be the sequence in $\mathcal{X}$ given by $x_n = 3 + \dfrac{1}{n}$ for $n = 1, 2, \ldots$. Then

$$fx_n = \frac{9}{x_n} \to 3 = t \text{ as } n \to \infty,$$

$$gx_n = \frac{27}{x_n^2} \to 3 = t \text{ as } n \to \infty$$

and

$$fgx_n = f(\frac{27}{x_n^2}) = 3 - \frac{27}{x_n^2} \to 0 = f(3) \text{ as } n \to \infty,$$

but

$$gfx_n = g(\frac{9}{x_n}) = 3 + \frac{9}{x_n} \to 6 \neq 3 = g(3) \text{ as } n \to \infty.$$

Again, it is obvious that f and g are not sequentially continuous of type $(\mathcal{A}_f)$ because

$$ggx_n = g(\frac{27}{x_n^2}) = 3 + \frac{27}{x_n^2} \to 6 \neq 3 = g(3) \text{ as } n \to \infty.$$

Finally, we can check that f and g are not weakly reciprocally continuous by giving the sequence $x_n = \frac{1}{n}$ for $n = 1, 2, \ldots$. Then

$$fx_n = 3 - x_n \to 3 = t \text{ as } n \to \infty,$$

$$gx_n = 3 + x_n \to 3 = t \text{ as } n \to \infty$$

but

$$fgx_n = f(3 + x_n) = \frac{9}{3 + x_n} \to 3 \neq 0 = f(3) \text{ as } n \to \infty$$

and

$$gfx_n = g(3 - x_n) = 6 - x_n \to 6 \neq 3 = g(3) \text{ as } n \to \infty.$$

3. Implicit Relations

Motivated by [**11**], let us consider $\mathcal{F}$ the set of all continuous functions $F : \mathbb{R}^6 \to \mathbb{R}$ such that

(1) (F_1): F is increasing in variable t_6,
(2) (F_2): there exists $h \in [0, 1)$ such that for all $u, v \geqslant 0$, $F(u, v, v, u, 0, u + v) \leqslant 0$ implies $u \leqslant hv$,
(3) (F_3): $F(t, t, 0, 0, t, t) > 0$ for all $t > 0$,
(4) (F_4): $F(t, 0, t, 0, t, 0) > 0$ for all $t > 0$.

Example 3.1. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{\frac{t_2 + t_3 + t_4}{3}, \frac{t_5 + t_6}{2}\}$, where $k \in [0, 1)$.

(1) (F_1): Obvious.
(2) (F_2): Let $u, v \geqslant 0$, $F(u, v, v, u, 0, u + v) = u - k \max\{\frac{u + 2v}{3}, \frac{u + v}{2}\} \leqslant 0$. If $u > v$, then $v < u \leqslant \frac{k}{2 - k} v < v$, a contradiction. Hence $u \leqslant v$ which implies $u \leqslant hv$, where $0 \leqslant h = \frac{2k}{3 - k} < 1$.
(3) (F_3): $F(t, t, 0, 0, t, t) = t - k \max\{\frac{t}{3}, t\} = t(1 - k) > 0$ for all $t > 0$.
(4) (F_4): $F(t, 0, t, 0, t, 0) = t - k \max\{\frac{t}{3}, \frac{t}{2}\} = t(1 - \frac{k}{2}) > 0$ for all $t > 0$.

EXAMPLE 3.2. $F(t_1, t_2, t_3, t_4, t_5, t_6) = kt_1 - t_2 - t_3 - t_4 - t_5 - t_6$, where $k > 5$.

(1) (F_1): Obvious.
(2) (F_2): Let $u, v \geqslant 0$ and $F(u, v, v, u, 0, u + v) = u(k - 2) - 3v \leqslant 0$ which implies $u \leqslant hv$, where $h = \dfrac{3}{k-2} \in [0, 1)$.
(3) (F_3): $F(t, t, 0, 0, t, t) = t(k - 3) > 0$ for all $t > 0$.
(4) (F_4): $F(t, 0, t, 0, t, 0) = t(k - 2) > 0$ for all $t > 0$.

EXAMPLE 3.3. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - k\dfrac{t_3t_4 + t_5t_6}{1 + t_2}$, where $k \in [0, 1)$.

(1) (F_1): Obvious.
(2) (F_2): Let u, $v \geqslant 0$ be and $F(u, v, v, u, 0, u + v) = u^2 - k\dfrac{uv}{1 + v} \leqslant 0$. If $u > 0$, then $u \leqslant k\dfrac{v}{1 + v}$, which implies $u \leqslant hv$, where $0 \leqslant h = k < 1$. If $u = 0$, then $u \leqslant hv$.
(3) (F_3): $F(t, t, 0, 0, t, t) = t^2(1 - \dfrac{k}{1 + t}) > 0$ for all $t > 0$.
(4) (F_4): $F(t, 0, t, 0, t, 0) = t^2 > 0$ for all $t > 0$.

Now, let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. Let us define the set

$$S = \{\{x_n\} \subseteq \mathcal{X} : \text{ if there holds } \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

then there holds

$$\lim_{n\to\infty} fgx_n = ft \text{ or } \lim_{n\to\infty} gfx_n = gt\}.$$

Suppose that $f\mathcal{X} \subseteq g\mathcal{X}$, there exists a sequence $\{x_i\}_{i=0}^{\infty}$, such that x_{n+1} is the pre-image under g of fx_n, that is

$$(a)\ fx_0 = gx_1, fx_1 = gx_2, \ldots, fx_n = gx_{n+1}, \ldots.$$

Let us define the set U to be the set of all sequences $\{x_n\}$ defined by (a). Let us define the sequence $\{y_n\} \subseteq \mathcal{X}$ by $y_n = fx_n = gx_{n+1}, n = 0, 1, 2, \ldots$.

4. Main Results

THEOREM 4.1. *Let f and g be weakly subsequentially continuous self-mappings of a complete metric space $(\mathcal{X}, d)$ such that $f\mathcal{X} \subseteq g\mathcal{X}$, $U \cap S \neq \emptyset$ and*

$$(4.1)\quad F(d(fx, fy), d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(gx, fy)) \leqslant 0$$

for all x, y in $\mathcal{X}$ and $F \in \mathcal{F}$. If f and g are either R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ or R-weakly commuting of type $(\mathcal{P})$ then f and g have a unique common fixed point.

PROOF. We choose an arbitrary x_0 such that the corresponding sequence $\{x_n\}$ defined in (a) belongs to $U \cap S$. Then, as in [**9**], by a routine calculation it follows that $\{y_n\}$ defined above is a Cauchy sequence. Since $\mathcal{X}$ is complete, $\{y_n\}$ converges to a point t in $\mathcal{X}$. Moreover, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = t$.

Now, suppose that f and g are R-weakly commuting of type $(\mathcal{A}_g)$. Weak subsequential continuity of f and g implies that $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$. Let us first assume that $\lim_{n\to\infty} gfx_n = gt$. Then R-weak commutativity of type $(\mathcal{A}_g)$ yields $d(ffx_n, gfx_n) \leqslant Rd(fx_n, gx_n)$. Taking the limit as $n \to \infty$, we obtain $\lim_{n\to\infty} ffx_n = gt$. Using (4.1) we get

$$F(d(ft, ffx_n), d(gt, gfx_n), d(ft, gt), d(ffx_n, gfx_n),\\ d(ft, gfx_n), d(gt, ffx_n)) \leqslant 0.$$

Making $n \to \infty$ we get

$$F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leqslant 0,$$

that is $ft = gt$. Again, by virtue of R-weak commutativity of type $(\mathcal{A}_g)$, $d(fft, gft) \leqslant Rd(ft, gt)$. This yields $fft = gft$ or $fgt = fft = gft = ggt$. By (4.1) we have

$$F(d(ft, fft), d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(gt, fft))\\ = F(d(ft, fft), d(ft, fft), 0, 0, d(ft, fft), d(ft, fft)) \leqslant 0,$$

i.e., $ft = fft = gft$.

Next, assume that $\lim_{n\to\infty} fgx_n = ft$. $f\mathcal{X} \subseteq g\mathcal{X}$ implies that there is a point $u \in \mathcal{X}$ such that $ft = gu$. Then $\lim_{n\to\infty} fgx_n = gu$. By virtue of (a) this also yields $\lim_{n\to\infty} ffx_n = gu$. Hence R-weak commutativity of type $(\mathcal{A}_g)$ implies that $d(ffx_n, gfx_n) \leqslant Rd(fx_n, gx_n)$. Taking the limit as $n \to \infty$ we get $\lim_{n\to\infty} gfx_n = ft = gu$. On using (4.1), we find

$$F(d(fu, ffx_n), d(gu, gfx_n), d(fu, gu), d(ffx_n, gfx_n),\\ d(fu, gfx_n), d(gu, ffx_n)) \leqslant 0.$$

On letting $n \to \infty$ we obtain

$$F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leqslant 0,$$

so $fu = gu$. Again, R-weak commutativity of type $(\mathcal{A}_g)$ implies that $d(ffu, gfu) \leqslant Rd(fu, gu)$, which yields $ffu = gfu$ and $fgu = ffu = gfu = ggu$. Finally, on using (4.1), we get

$$F(d(fu, ffu), d(gu, gfu), d(fu, gu), d(ffu, gfu), d(fu, gfu), d(gu, ffu))\\ = F(d(fu, ffu), d(fu, ffu), 0, 0, d(fu, ffu), d(fu, ffu)) \leqslant 0,$$

which is a contradiction. Thus $fu = ffu = gfu$.

Suppose that f and g are R-weakly commuting of type $(\mathcal{A}_f)$. Weak subsequential continuity of f and g implies that $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$. Let $\lim_{n\to\infty} gfx_n = gt$. R-weak commutativity of type $(\mathcal{A}_f)$ yields

$d(ggx_n, fgx_n) \leqslant Rd(fx_n, gx_n)$. Taking the limit as $n \to \infty$ and by virtue of (a), we obtain $\lim_{n\to\infty} fgx_n = gt$. On using (4.1), we get

$$F(d(ft, fgx_n), d(gt, ggx_n), d(ft, gt), d(fgx_n, ggx_n), d(ft, ggx_n), d(gt, fgx_n)) \leqslant 0.$$

Making $n \to \infty$, we get

$$F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leqslant 0,$$

a contradiction. Hence $ft = gt$. Again, by virtue of R-weak commutativity of type $(\mathcal{A}_f)$, $d(ggt, fgt) \leqslant Rd(ft, gt)$. This yields $ggt = fgt$
and $gft = ggt = fgt = fft$. By (4.1), we have

$$\begin{aligned} &F(d(ft, fft), d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(gt, fft)) \\ &= F(d(ft, fft), d(ft, fft), 0, 0, d(ft, fft), d(ft, fft)) \leqslant 0, \end{aligned}$$

that is $ft = fft = gft$.

Next, assume that $\lim_{n\to\infty} fgx_n = ft$. Then $f\mathcal{X} \subseteq g\mathcal{X}$ implies that there is an element $u \in \mathcal{X}$ such that $ft = gu$. Hence R-weak commutativity of type $(\mathcal{A}_f)$ implies that $d(ggx_n, fgx_n) \leqslant Rd(fx_n, gx_n)$. Letting $n \to \infty$ we get
$\lim_{n\to\infty} ggx_n = ft = gu$. On using (4.1), we get

$$F(d(fu, fgx_n), d(gu, ggx_n), d(fu, gu), d(fgx_n, ggx_n), d(fu, ggx_n), d(gu, fgx_n)) \leqslant 0.$$

Taking the limit as $n \to \infty$, we obtain

$$F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leqslant 0,$$

a contradiction so that $fu = gu$. Again, using R-weak commutativity of type $(\mathcal{A}_f)$ we have $d(ggu, fgu) \leqslant Rd(fu, gu)$. This yields $ggu = fgu$
and $gfu = ggu = fgu = ffu$. By (4.1), we have

$$\begin{aligned} &F(d(fu, ffu), d(gu, gfu), d(fu, gu), d(ffu, gfu), d(fu, gfu), d(gu, ffu)) \\ &= F(d(fu, ffu), d(fu, ffu), 0, 0, d(fu, ffu), d(fu, ffu)) \leqslant 0, \end{aligned}$$

i.e., $fu = ffu = gfu$.

Finally, suppose that f and g are R-weakly commuting of type $(\mathcal{P})$. Now, weak subsequential continuity of f and g implies that $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$. Let us first assume that $\lim_{n\to\infty} gfx_n = gt$. By virtue of (a) and R-weak commutativity of type $(\mathcal{P})$, we have $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = gt$. Using (4.1), we get

$$F(d(ft, ffx_n), d(gt, gfx_n), d(ft, gt), d(ffx_n, gfx_n), d(ft, gfx_n), d(gt, ffx_n)) \leqslant 0.$$

On letting $n \to \infty$, we obtain

$$F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leqslant 0,$$

that is $ft = gt$. Again, by virtue of R-weak commutativity of type $(\mathcal{P})$,
$d(fft = ggt) \leqslant Rd(ft, gt)$. This implies that $fft = ggt$

and $fgt = fft = ggt = gft$. Also, using (4.1), we find

$$\begin{aligned}&F(d(ft,fft),d(gt,gft),d(ft,gt),d(fft,gft),d(ft,gft),d(gt,fft))\\&=F(d(ft,fft),d(ft,fft),0,0,d(ft,fft),d(ft,fft))\leqslant 0,\end{aligned}$$

a contradiction. Hence $ft = fft = gft$.

Now, assume that $\lim_{n\to\infty} fgx_n = ft$. $f\mathcal{X} \subseteq g\mathcal{X}$ implies that there exists a point $u \in \mathcal{X}$ which verifies $ft = gu$. By virtue of (a) and R-weak commutativity of type $(\mathcal{P})$, we get $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = ft = gu$. We assert that $fu = gu$. Let on contrary that $fu \neq gu$. Using (4.1), we obtain

$$\begin{aligned}&F(d(fu,fgx_n),d(gu,ggx_n),d(fu,gu),d(fgx_n,ggx_n),\\&d(fu,ggx_n),d(gu,fgx_n))\leqslant 0.\end{aligned}$$

At infinity, we get

$$F(d(fu,gu),0,d(fu,gu),0,d(fu,gu),0)\leqslant 0,$$

i.e., $fu = gu$. Again, by virtue of R-weak commutativity of type $(\mathcal{P})$,
$d(ffu, ggu) \leqslant Rd(fu, gu)$. This yields $ffu = ggu$
and $fgu = ffu = ggu = gfu$. On using (4.1), we get

$$\begin{aligned}&F(d(fu,ffu),d(gu,gfu),d(fu,gu),d(ffu,gfu),d(fu,gfu),d(gu,ffu))\\&=F(d(fu,ffu),d(fu,ffu),0,0,d(fu,ffu),d(fu,ffu))\leqslant 0,\end{aligned}$$

that is, $fu = ffu = gfu$.

Uniqueness of the common fixed point follows immediately by (F_3) and (4.1). □

To illustrate our Theorem, we give the following example.

EXAMPLE 4.1. Let $\mathcal{X} = [1,3]$ and let d be the usual metric on $\mathcal{X}$. Define $f, g : \mathcal{X} \to \mathcal{X}$ as follows:

$$fx=\begin{cases}1 \text{ if } x=1\\ \dfrac{3}{2} \text{ if } x\in(1,2]\\ 1 \text{ if } x\in(2,3],\end{cases}\qquad gx=\begin{cases}1 \text{ if } x=1\\ 3 \text{ if } x\in(1,2]\\ \dfrac{x}{2} \text{ if } x\in(2,3].\end{cases}$$

Then f and g satisfy all conditions of the above theorem and have the common fixed point $x = 1$. It can be verified that f and g satisfy condition (4.1) with $F = t_1 - k\max\{t_2, t_3, t_4, t_5, t_6\}$ where $k \in [\frac{1}{4}, \frac{1}{2})$. Furthermore, f and g are R-weakly commuting of type $(\mathcal{A}_g)$. It can be noted that f and g are weakly subsequentially continuous. At this end, let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that $x_n = 2 + \frac{1}{n}$ for $n = 1, 2, \ldots$. Then, $fx_n = 1 \to 1 = t$, $gx_n = \frac{x_n}{2} \to 1 = t$ and $gfx_n = g(1) = 1$, but $fgx_n = f(\frac{x_n}{2}) = \frac{3}{2} \neq 1 = f(1)$. On the other hand, we have $f\mathcal{X} = \{1, \frac{3}{2}\} \subseteq g\mathcal{X} = [1, \frac{3}{2}] \cup \{3\}$.

Now, we give some results.

Corollary 4.1. *Let f and g be weakly subsequentially continuous mappings from a complete metric space $(\mathcal{X}, d)$ into itself such that $f\mathcal{X} \subseteq g\mathcal{X}$, $U \cap S \neq \emptyset$ and*

$$d(fx, fy) \leqslant k \max\{\frac{d(gx, gy) + d(fx, gx) + d(fy, gy)}{3}, \frac{d(fx, gy) + d(gx, fy)}{2}\}$$

for all x, y in $\mathcal{X}$, where $k \in [0, 1)$. If f and g are either R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ or R-weakly commuting of type $(\mathcal{P})$ then f and g have a unique common fixed point.

Proof. Use Theorem 4.1 and Example 3.1. □

Corollary 4.2. *Let f and g be weakly subsequentially continuous mappings from a complete metric space $(\mathcal{X}, d)$ into itself such that $f\mathcal{X} \subseteq g\mathcal{X}$, $U \cap S \neq \emptyset$ and*

$$d(fx, fy) \leqslant \frac{1}{k}[d(gx, gy) + d(fx, gx) + d(fy, gy) + d(fx, gy) + d(gx, fy)]$$

for all x, y in $\mathcal{X}$, where $k > 5$. If f and g are either R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ or R-weakly commuting of type $(\mathcal{P})$ then f and g have a unique common fixed point.

Proof. Use Theorem 4.1 and Example 3.2. □

Corollary 4.3. *Let f and g be weakly subsequentially continuous mappings from a complete metric space $(\mathcal{X}, d)$ into itself such that $f\mathcal{X} \subseteq g\mathcal{X}$, $U \cap S \neq \emptyset$ and*

$$d^2(fx, fy) \leqslant k[\frac{d(fx, gx)d(fy, gy) + d(fx, gy)d(gx, fy)}{1 + d(gx, gy)}]$$

for all x, y in $\mathcal{X}$, where $k \in [0, 1)$. If f and g are either R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ or R-weakly commuting of type $(\mathcal{P})$ then f and g have a unique common fixed point.

Proof. Use Theorem 4.1 and Example 3.3. □

In the following, we will prove a common fixed point theorem for a subcompatible pair of self-mappings.

Theorem 4.2. *Let f and g be weakly reciprocally continuous subcompatible self-mappings of a metric space $(\mathcal{X}, d)$ satisfying $f\mathcal{X} \subseteq g\mathcal{X}$ and*

$$F(d(fx, fy), d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(gx, fy)) \leqslant 0 \tag{4.2}$$

for all x, y in $\mathcal{X}$, where F is continuous and satisfies only (F_3) and (F_4). If f and g are R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ then f and g have a unique common fixed point.

Proof. Since f and g are weakly reciprocally continuous there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$ and which satisfy $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$. Let $\lim_{n\to\infty} gfx_n = gt$. Then R-weak commutativity of type $(\mathcal{A}_g)$ yields $d(ffx_n, gfx_n) \leqslant Rd(fx_n, gx_n)$. Taking the limit as

$n \to \infty$, we get $\lim\limits_{n\to\infty} ffx_n = gt$. By (4.2) we have

$$F(d(ft, ffx_n), d(gt, gfx_n), d(ft, gt), d(ffx_n, gfx_n),$$
$$d(ft, gfx_n), d(gt, ffx_n)) \leqslant 0.$$

At infinity we get

$$F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leqslant 0,$$

a contradiction so that $ft = gt$. Again, by virtue of R-weak commutativity of type $(\mathcal{A}_g)$, $d(fft, gft) \leqslant Rd(ft, gt)$, which yields $fft = gft$
and $fgt = fft = gft = ggt$. Using (4.2), we obtain

$$F(d(ft, fft), d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(gt, fft))$$
$$= F(d(ft, fft), d(ft, fft), 0, 0, d(ft, fft), d(ft, fft)) \leqslant 0,$$

a contradiction. Hence $ft = fft = gft$.

Next, assume that $\lim\limits_{n\to\infty} fgx_n = ft$. Since $f\mathcal{X} \subseteq g\mathcal{X}$ then, there is an element $u \in \mathcal{X}$ such that $ft = gu$. By virtue of subcompatibility and R-weak commutativity of type $(\mathcal{A}_g)$, $\lim\limits_{n\to\infty} ffx_n = \lim\limits_{n\to\infty} gfx_n = ft = gu$. By (4.2), we have

$$F(d(fu, ffx_n), d(gu, gfx_n), d(fu, gu), d(ffx_n, gfx_n),$$
$$d(fu, gfx_n), d(gu, ffx_n)) \leqslant 0.$$

Making $n \to \infty$, we obtain

$$F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leqslant 0,$$

a contradiction. Hence $fu = gu$. Again, by virtue of R-weak commutativity of type $(\mathcal{A}_g)$ we get $ffu = gfu$ and $fgu = ffu = gfu = ggu$. On using (4.2), we obtain

$$F(d(fu, ffu), d(gu, gfu), d(fu, gu), d(ffu, gfu), d(fu, gfu), d(gu, ffu))$$
$$= F(d(fu, ffu), d(fu, ffu), 0, 0, d(fu, ffu), d(fu, ffu)) \leqslant 0,$$

that is, $fu = ffu = gfu$.

Finally, suppose that f and g are R-weakly commuting of type $(\mathcal{A}_f)$, then, we have $d(fgx_n, ggx_n) \leqslant Rd(fx_n, gx_n)$. Now, weak reciprocal continuity implies that $\lim\limits_{n\to\infty} fgx_n = ft$ or $\lim\limits_{n\to\infty} gfx_n = gt$. Let $\lim\limits_{n\to\infty} gfx_n = gt$. By virtue of subcompatibility, we have $\lim\limits_{n\to\infty} fgx_n = gt$ and consequently $\lim\limits_{n\to\infty} ggx_n = gt$. Using (4.2), we get

$$F(d(ft, fgx_n), d(gt, ggx_n), d(ft, gt), d(fgx_n, ggx_n),$$
$$d(ft, ggx_n), d(gt, fgx_n)) \leqslant 0.$$

At infinity we get

$$F(d(ft, gt), 0, d(ft, gt), 0, d(ft, gt), 0) \leqslant 0,$$

i.e., $ft = gt$. Again, by virtue of R-weak commutativity of type $(\mathcal{A}_f)$, $ggt = fgt$ and $gft = ggt = fgt = fft$. On using (4.2), we obtain

$$\begin{aligned}&F(d(ft, fft), d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(gt, fft))\\&= F(d(ft, fft), d(ft, fft), 0, 0, d(ft, fft), d(ft, fft)) \leqslant 0,\end{aligned}$$

that is, $ft = fft = gft$.

Next, suppose that $\lim_{n\to\infty} fgx_n = ft$. $f\mathcal{X} \subseteq g\mathcal{X}$ implies that, there exists some $u \in \mathcal{X}$ such that $ft = gu$. By virtue of R-weak commutativity of type $(\mathcal{A}_f)$, we have $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = ft = gu$. Using (4.2), we get

$$\begin{aligned}&F(d(fu, fgx_n), d(gu, ggx_n), d(fu, gu), d(fgx_n, ggx_n),\\&d(fu, ggx_n), d(gu, fgx_n)) \leqslant 0.\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$F(d(fu, gu), 0, d(fu, gu), 0, d(fu, gu), 0) \leqslant 0,$$

i.e., $fu = gu$. Again, by virtue of R-weak commutativity of type $(\mathcal{A}_f)$ we get $ggu = fgu$ and $gfu = ggu = fgu = ffu$. We assert that $fu = ffu = gfu$. Let on contrary that $fu \neq ffu$. On using (4.2), we obtain

$$\begin{aligned}&F(d(fu, ffu), d(gu, gfu), d(fu, gu), d(ffu, gfu), d(fu, gfu), d(gu, ffu))\\&= F(d(fu, ffu), d(fu, ffu), 0, 0, d(fu, ffu), d(fu, ffu)) \leqslant 0,\end{aligned}$$

a contradiction. Hence $fu = ffu = gfu$.

Uniqueness of the common fixed point follows easily by (F_3) and (4.2). □

The next example illustrates our result.

Example 4.2. Endow $\mathcal{X} = [0, 10]$ with the absolute value metric and define $f, g : \mathcal{X} \to \mathcal{X}$ by

$$fx = \begin{cases} 1 \text{ if } x \in [0,1] \\ \frac{4}{3} \text{ if } x \in (1,5] \\ 1 \text{ if } x \in (5,10], \end{cases} \quad gx = \begin{cases} 1 \text{ if } x \in [0,1] \\ 7 \text{ if } x \in (1,5] \\ \frac{x+1}{6} \text{ if } x \in (5,10]. \end{cases}$$

Then f and g are certainly R-weakly commuting of type $(\mathcal{A}_g)$ since

$$d(ffx, gfx) \leqslant Rd(fx, gx) \text{ for all } x \in \mathcal{X}.$$

Moreover, f and g are subcompatible. To this end, consider the sequence $x_n = 1 - \frac{1}{n}$ for $n = 1, 2, \ldots$. Then $fx_n = 1 = gx_n$ and $fgx_n = gfx_n = 1$. Thus $|fgx_n - gfx_n| = 0$. To see that f and g are weakly reciprocally continuous, consider $x_n = 5 + \frac{1}{n}$ for $n = 1, 2, \ldots$. Then $fx_n = 1$, $gx_n = \frac{x_n + 1}{6} \to 1$, and $gfx_n = 1 = g(1)$; whereas

$$fgx_n = f(\frac{x_n + 1}{6}) = \frac{4}{3} \neq 1 = f(1).$$

On the other hand, observe that

$f\mathcal{X} = \{1, \frac{4}{3}\} \subseteq g\mathcal{X} = [1, \frac{11}{6}] \cup \{7\}$. Finally, we can check that condition (4.2) is verified for all x, $y \in \mathcal{X}$ with $k \in [\frac{3}{35}, 1)$. Consequently, all conditions of theorem 4.2 are satisfied and $x = 1$ is the unique common fixed point.

Finally, we end our paper by giving some results.

COROLLARY 4.4. *Let f and g be weakly reciprocally continuous subcompatible self-mappings of a metric space $(\mathcal{X}, d)$ satisfying $f\mathcal{X} \subseteq g\mathcal{X}$ and the inequality*

$$d(fx, fy) \leqslant k \max\{\frac{d(gx, gy) + d(fx, gx) + d(fy, gy)}{3}, \frac{d(fx, gy) + d(gx, fy)}{2}\}$$

for all x, y in $\mathcal{X}$, where $k \in [0, 1)$. If f and g are R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ then f and g have a unique common fixed point.

PROOF. Use Theorem 4.2 and Example 3.1. □

COROLLARY 4.5. *Let f and g be weakly reciprocally continuous subcompatible self-mappings of a metric space $(\mathcal{X}, d)$ satisfying $f\mathcal{X} \subseteq g\mathcal{X}$ and the inequality*

$$d(fx, fy) \leqslant \frac{1}{k}[d(gx, gy) + d(fx, gx) + d(fy, gy) + d(fx, gy) + d(gx, fy)]$$

for all x, y in $\mathcal{X}$, where $k > 3$. If f and g are R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ then f and g have a unique common fixed point.

PROOF. Use Theorem 4.2 and Example 3.2. □

COROLLARY 4.6. *Let f and g be weakly reciprocally continuous subcompatible self-mappings of a metric space $(\mathcal{X}, d)$ satisfying $f\mathcal{X} \subseteq g\mathcal{X}$ and the inequality*

$$d^2(fx, fy) \leqslant k[\frac{d(fx, gx)d(fy, gy) + d(fx, gy)d(gx, fy)}{1 + d(gx, gy)}]$$

for all x, y in $\mathcal{X}$, where $k \in [0, 1)$. If f and g are R-weakly commuting of type $(\mathcal{A}_g)$ or R-weakly commuting of type $(\mathcal{A}_f)$ then f and g have a unique common fixed point.

PROOF. Use Theorem 4.2 and Example 3.3. □

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