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POLYNOMIALS ASSOCIATED WITH CLOSED NEIGHBORHOOD CORONA AND NEIGHBORHOOD COMPLEMENT CORONA OF GRAPHS

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Abstract. The closed neighborhood of a vertex $v \in V(G)$ is, $N_G[v] = N_G(v) \cup \{v\}$, where $N_G(v)$ is the set of all vertices which are adjacent to v. Motivated by the concept of closed neighborhood, we define a new corona called as, closed neighborhood corona. Further, we study polynomials associated with adjacency matrix, Laplacian matrix and signless Laplacian matrix of the same structure. Also, we study polynomials associated with adjacency matrix, Laplacian matrix and signless Laplacian matrix of the structure of neighborhood complement corona.

1. Introduction

In a graph G, two vertices are neighbors, whenever there is an edge connecting them. The set of all neighbors of a vertex is called its neighborhood. By taking into account of the concept of neighborhood, in 2011, Indulal [**6**] defined a new corona product, called it as neighborhood corona, hence studied the adjacency, Laplacian and signless Laplacian spectrum when both the graphs forming the corona product

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are regular. Later, in 2014, Liu and Zhou [**7**] studied the same structure for two arbitrary graphs. Adiga et al. [**1, 2**] defined some new variants of neighborhood coronae and studied adjacency, Laplacian and signless Laplacian polynomial for arbitrary graphs. Inline with the concept of neighborhood complement, Rakshit and Subba Krishna [**9**] defined neighborhood complement corona and hence studied adjacency, Laplacian and signless Laplacian spectrum of regular graphs.

Motivated by the closed neighborhood concept, in the present work we define a new variant of neighborhood corona, called closed neighborhood corona, hence study adjacency, Laplacian and signless Laplacian polynomial for arbitrary graphs and the corresponding spectrum of regular graphs. Further, we study neighborhood complement corona for arbitrary graphs and obtain their adjacency, Laplacian and signless Laplacian polynomials and prove the results for regular graphs. The results due to Rakshit and Subba Krishna [**9**] becomes particular cases of our results. The signless Laplacian spectrum for neighborhood complement corona of two regular graphs given in [**9**] is prone to some errors and we have given here the corrected version of the same.

2. Preliminaries

Throughout the paper, we consider simple, finite and undirected graphs. A graph is a pair $G = \big(V(G), E(G)\big)$ of sets such that, the elements of $E(G)$ are 2-element subsets of $V(G)$. The elements of $V(G)$ are vertices and that of $E(G)$ are edges of the graph G. Two vertices are adjacent, or neighbors, whenever there is an edge between them. The number of neighbors of a vertex is called its degree. If all the vertices in the graph have same degree then graph is called a regular graph. The set of all neighbors of a vertex v is its neighborhood denoted by $N_G(v)$. Closed neighborhood of a vertex v is, $N_G[v] = N_G(v) \cup \{v\}$. For a graph G with n vertices $v_1, v_2, \ldots, v_n$, the adjacency matrix is defined as, $A(G) = [a_{ij}]_{n\times n}$ in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent, and 0 otherwise. Laplacian and signless Laplacian matrices are defined as: $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively, where $D(G)$ is the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$, where $d_i = d_G(v_i)$ is the degree of a vertex v_i. For any matrix $M_{n\times n}$, the polynomial associated with it is given by, $\phi\,(M; x) = det\,(xI_n - M)$. Thus, $\phi\,(A(G); x)\,, \phi\,(L(G); x)$ and $\phi\,(Q(G); x)$ denotes adjacency polynomial, Laplacian polynomial and signless Laplacian polynomial of G, respectively. Their roots are adjacency eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of G, respectively. Denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, by

$$\begin{array}{ccccccc}
\lambda_1(G) & \geqslant & \lambda_2(G) & \geqslant & \ldots & \geqslant & \lambda_n(G), \\
\mu_1(G) & \leqslant & \mu_2(G) & \leqslant & \ldots & \leqslant & \mu_n(G), \\
\gamma_1(G) & \geqslant & \gamma_2(G) & \geqslant & \ldots & \geqslant & \gamma_n(G).
\end{array}$$

It is noted that, $\mu_i = r - \lambda_i$ and $\gamma_i = r + \lambda_i$, if the degree of all vertices of G is r. The collection of distinct eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ together with their corresponding multiplicities form the A-spectrum, L-spectrum and Q-spectrum of G, respectively. The complement $\overline{G}$ of a graph G has the same vertices as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The Kronecker product $C \otimes D$ of two matrices $C = [c_{ij}]_{m\times n}$ and $D = [d_{ij}]_{p\times q}$ is the $mp \times nq$ matrix obtained from C by replacing each entry c_{ij} by $c_{ij}D$ [**5**]. For matrices C, D, E and F such that products CE and DF exist, $(C\otimes D)\,(E\otimes F) = CE\otimes DF$, $(C \otimes D)^{-1} = C^{-1} \otimes D^{-1}$ and $(C \otimes D)^T = C^T \otimes D^T$. Let $\mathbf{1}_n$ denotes the column vector of dimension n and J denotes all $1'$s matrix. For undefined graph theoretical terminologies and notations, we follow the book [**4**].

Definition 2.1. ([**6**]) Given a graph G_1 on n_1 vertices and m_1 edges with the vertex set $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, and G_2 be another graph on n_2 vertices, then the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and making all the vertices in the i^{th} copy of G_2 adjacent with the neighbors of v_i, for $i = 1, 2, \ldots, n_1$, is called as the neighborhood corona of two graphs, denoted by $G_1 \star G_2$.

Definition 2.2. ([**8**]) Given a graph G on n vertices with the graph matrix M, where M is viewed as a matrix over the field of rational functions $\mathbb{C}(x)$ with $det(xI_n - M)$ non zero. The M-coronal $\Gamma_M(x) \in \mathbb{C}(x)$ of G is,

$$\Gamma_M(x) = \mathbf{1}_n^T\Big(xI_n - M\Big)^{-1}\mathbf{1}_n.$$

If M has a constant row sum r, then $\Gamma_M(x) = \dfrac{n}{x-r}$.

Proposition 2.1 (Schur Complement [**3**]). *Suppose that the order of all four matrices D_{11}, D_{12}, D_{21} and D_{22} satisfy the rules of operations on matrices. Then we have*

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \begin{cases} |D_{22}|\;|D_{11} - D_{12}D_{22}^{-1}D_{21}|, & \text{if } D_{22} \text{ is a non-singular matrix,} \\ |D_{11}|\;|D_{22} - D_{21}D_{11}^{-1}D_{12}|, & \text{if } D_{11} \text{ is a non-singular matrix.} \end{cases}$$

Proposition 2.2 ([**4**]). *If G is an r-regular graph on n vertices with the adjacency eigenvalues: $r, \lambda_2(G), \lambda_3(G), \ldots, \lambda_n(G)$. Then the adjacency eigenvalues of $\overline{G}$ are: $n - r - 1, -1 - \lambda_2(G), -1 - \lambda_3(G), \ldots, -1 - \lambda_n(G)$.*

3. Closed neighborhood corona of two graphs

Given a graph G_1, the neighborhood corona focus only on neighbors of a vertex, in forming the corona product with the graph G_2. Closed neighborhood of a vertex include the vertex itself along with its neighbors. Motivated by this, we define a new variation of corona of two graphs, called as closed neighborhood corona.

Definition 3.1. Given a graph G_1 on n_1 vertices and m_1 edges with the vertex set, $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and let G_2 be another graph on n_2 vertices and m_2 edges. Then, the graph obtained by taking a copy of G_1, n_1 copies of G_2 and then making all vertices in the i^{th} copy of G_2 adjacent to the neighbors of v_i including v_i itself, for $i = 1, 2, \ldots, n_1$, is called as closed neighborhood corona of G_1 and G_2, denoted by $G_1\boxed{\star}G_2$.

It is noted that $G_1\boxed{\star}G_2$ has $n_1 + n_1n_2$ vertices and $m_1 + n_1m_2 + n_1n_2 + 2m_1n_2$ edges.

EXAMPLE 3.1. Let C_n denotes the cycle on n vertices and K_n denotes complete graph on n vertices. Fig. 1 depicts $C_6 \boxed{\star} K_2$.

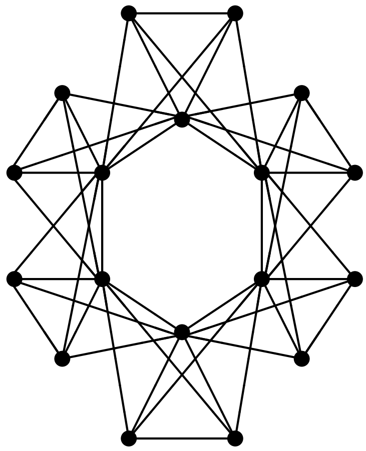

Figure 1: $C_6 \boxed{\star} K_2$.

3.1. Adjacency polynomial of closed neighborhood corona.

THEOREM 3.1. *If G_1 and G_2 are any two graphs on n_1 and n_2 vertices respectively, then the spectral polynomial of $A\big(G_1 \boxed{\star} G_2\big)$ is*

$$\phi\left(A(G_1 \boxed{\star} G_2);\, x\right) = \phi(A(G_2);\, x)^{n_1} \prod_{i=1}^{n_1} \Big(x - \lambda_i(G_1) - \Gamma_{A(G_2)}(x)\ \ (1 + \lambda_i(G_1))^2\Big).$$

PROOF. The adjacency matrix of $G_1 \boxed{\star} G_2$ is,

$$A\left(G_1 \boxed{\star} G_2\right) = \begin{pmatrix} A(G_1) & (I_{n_1} + A(G_1)) \otimes \mathbf{1}_{n_2}^T \\ (I_{n_1} + A(G_1)) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes A(G_2) \end{pmatrix}.$$

The spectral polynomial is,

$$\begin{aligned} \phi\left(A\left(G_1 \boxed{\star} G_2\right);\, x\right) &= det\left(x\, I_{n_1+n_1n_2} - A\left(G_1 \boxed{\star} G_2\right)\right) \\ &= det\left(\begin{array}{c|c} xI_{n_1} - A(G_1) & -\left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2}^T \\ \hline -\left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes \left(xI_{n_2} - A(G_2)\right) \end{array}\right). \end{aligned}$$

Applying Proposition 2.1, we have

$$\phi\left(A\left(G_1 \boxed{\star} G_2\right);\, x\right) = \phi\left(A\left(G_2\right);\, x\right)^{n_1} det\left(xI_{n_1} - A(G_1) - S\right),$$

where

$$\begin{aligned} S &= \left[-\left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2}^T\right] \left[I_{n_1} \otimes \left(xI_{n_2} - A(G_2)\right)\right]^{-1} \left[-\left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2}\right] \\ &= \Big(I_{n_1} + A(G_1)\Big)^2 \otimes \mathbf{1}_{n_2}^T \Big(xI_{n_2} - A(G_2)\Big)^{-1} \mathbf{1}_{n_2} \\ &= \Gamma_{A(G_2)}(x)\ \Big(I_{n_1} + A(G_1)\Big)^2. \end{aligned}$$

Therefore,

$$\phi\left(A\left(G_1 \boxed{\star} G_2\right);\, x\right) = \phi(A\left(G_2\right);\, x)^{n_1} det\left(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)\left(I_{n_1} + A(G_1)\right)^2\right).$$

Expanding the determinant of RHS in terms of products of $\lambda_i(G_1)$, the result follows. □

Corollary 3.1. *If G_1 is r_1-regular graph and G_2 is r_2-regular graph, then the eigenvalues of $A\left(G_1 \boxed{\star} G_2\right)$ are $\lambda_i(G_2)$, $i = 2, 3, \ldots, n_2$, each occuring n_1 times and*

$$\frac{r_2 + \lambda_i(G_1) \pm \sqrt{\left(r_2 - \lambda_i(G_1)\right)^2 + 4n_2\left(\lambda_i(G_1) + 1\right)^2}}{2}, \quad i = 1, 2, \ldots, n_1.$$

Proof. As G_2 is r_2-regular, substituting $\Gamma_{A(G_2)}(x) = \dfrac{n_2}{x - r_2}$ in Theorem 3.1, result follows. □

3.2. Laplacian polynomial of closed neighborhood corona.

Theorem 3.2. *If G_1 is an r_1-regular graph on n_1 vertices and G_2 is any graph on n_2 vertices, then the spectral polynomial of $L\left(G_1 \boxed{\star} G_2\right)$ is,*

$$\begin{aligned}
&\phi\left(L\left(G_1 \boxed{\star} G_2\right);\, x\right) \\
&= \left(\phi\left(L\left(G_2\right);\, x - r_1 - 1\right)\right)^{n_1} \\
&\prod_{i=1}^{n_1}\left(x - n_2 - r_1 n_2 - \mu_i(G_1) - \Gamma_{L(G_2)}(x - r_1 - 1)\ \left(r_1 - \mu_i(G_1) + 1\right)^2\right).
\end{aligned}$$

Proof. The Laplacian matrix of $G_1 \boxed{\star} G_2$ is,

$$L\left(G_1 \boxed{\star} G_2\right) = \begin{pmatrix} n_2\left(D(G_1) + I_{n_1}\right) + L(G_1) & -\left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2}^T \\ -\left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes L(G_2) + \left(D(G_1) + I_{n_1}\right) \otimes I_{n_2}. \end{pmatrix}$$

The spectral polynomial is

$$\begin{aligned}
&\phi\left(L\left(G_1 \boxed{\star}\, G_2\right);\, x\right) \\
= \;& det\left(x\, I_{n_1 + n_1 n_2} - L\left(G_1 \boxed{\star} G_2\right)\right) \\
= \;& det\begin{pmatrix} xI_{n_1} - n_2\left(D(G_1) + I_{n_1}\right) - L(G_1) & \left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2}^T \\ \left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2} & xI_{n_1 n_2} - I_{n_1} \otimes L(G_2) - \left(D(G_1) + I_{n_1}\right) \otimes I_{n_2} \end{pmatrix}.
\end{aligned}$$

As G_1 is r_1-regular graph, we have

$$\begin{aligned}
&\phi\left(L\left(G_1 \boxed{\star} G_2\right);\, x\right) \\
= \;& det\begin{pmatrix} xI_{n_1} - n_2\left(r_1 + 1\right) I_{n_1} - L(G_1) & \left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2}^T \\ \left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2} & xI_{n_1 n_2} - I_{n_1} \otimes L(G_2) - \left(r_1 + 1\right) I_{n_1} \otimes I_{n_2} \end{pmatrix} \\
= \;& det\left(\begin{array}{c|c} xI_{n_1} - n_2\left(r_1 + 1\right) I_{n_1} - L(G_1) & \left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2}^T \\ \hline \left(I_{n_1} + A(G_1)\right) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes \left((x - r_1 - 1)I_{n_2} - L(G_2)\right) \end{array}\right).
\end{aligned}$$

Applying Proposition 2.1, we have

$$\phi\left(L\left(G_1\boxed{\star}G_2\right);\,x\right)=$$
$$\phi\left(L\left(G_2\right);\,x-r_1-1\right)^{n_1}\,det\left((x-n_2-r_1n_2)I_{n_1}-L(G_1)-S\right),$$

where

$$\begin{aligned}S&=\left[(I_{n_1}+A(G_1))\otimes\mathbf{1}_{n_2}^T\right]\left[I_{n_1}\otimes((x-r_1-1)I_{n_2}-L(G_2))\right]^{-1}\left[(I_{n_1}+A(G_1))\otimes\mathbf{1}_{n_2}\right]\\&=(I_{n_1}+A(G_1))^2\otimes\mathbf{1}_{n_2}^T\left((x-r_1-1)I_{n_2}-L(G_2)\right)^{-1}\mathbf{1}_{n_2}\\&=\Gamma_{L(G_2)}(x-r_1-1)\ (I_{n_1}+A(G_1))^2.\end{aligned}$$

Therefore,

$$\phi\left(L\left(G_1\boxed{\star}G_2\right);\,x\right)=$$
$$\phi\left(L\left(G_2\right);\,x-r_1-1\right)^{n_1}det\Big((x-n_2-r_1n_2)I_{n_1}-L(G_1)-\Gamma_{L(G_2)}(x-r_1-1)\left(I_{n_1}+A(G_1)\right)^2\Big).$$

Expanding the determinant of RHS in terms of products of $\mu_i(G_1)$ by using the relation $\lambda_i(G_1)=r_1-\mu_i(G_1)$, the result follows. □

Corollary 3.2. *If G_1 is r_1-regular graph and G_2 is r_2-regular graph, then the eigenvalues of $L\left(G_1\boxed{\star}G_2\right)$ are $\mu_i(G_2)+r_1+1$, $i=2,3,\ldots,n_2$, each occuring n_1 times and*

$$\frac{(n_2+1)(r_1+1)+\mu_i(G_1)\pm\sqrt{\left((n_2+1)(r_1+1)+\mu_i(G_1)\right)^2+4k_i}}{2},$$

where $k_i=\mu_i(G_1)\left(n_2\mu_i(G_1)-(2n_2+1)(r_1+1)\right)$, $i=1,2,\ldots,n_1$.

Proof. As G_2 is r_2-regular, substituting $\Gamma_{L(G_2)}(x-r_1-1)=\dfrac{n_2}{(x-r_1-1)}$ in Theorem 3.2, result follows. □

3.3. Signless Laplacian polynomial of closed neighborhood corona.

Theorem 3.3. *If G_1 is an r_1-regular graph on n_1 vertices and G_2 is any graph on n_2 vertices, then the spectral polynomial of $Q\big(G_1\boxed{\star}G_2\big)$ is*

$$\phi\left(Q\big(G_1\boxed{\star}G_2\big);\,x\right)=$$
$$\left(\phi\left(Q\left(G_2\right);\,x-r_1-1\right)\right)^{n_1}$$
$$\prod_{i=1}^{n_1}\Big(x-n_2-r_1n_2-\gamma_i(G_1)-\Gamma_{Q(G_2)}(x-r_1-1)\left(\gamma_i(G_1)-r_1+1\right)^2\Big).$$

Proof. The signless Laplacian matrix of $G_1\boxed{\star}G_2$ is,

$$Q\left(G_1\boxed{\star}G_2\right)=\begin{pmatrix}n_2\left(D(G_1)+I_{n_1}\right)+Q(G_1) & (I_{n_1}+A(G_1))\otimes\mathbf{1}_{n_2}^T\\(I_{n_1}+A(G_1))\otimes\mathbf{1}_{n_2} & I_{n_1}\otimes Q(G_2)+\left(D(G_1)+I_{n_1}\right)\otimes I_{n_2}\end{pmatrix}.$$

The spectral polynomial is,

$$\begin{aligned}
&\phi\left(Q\left(G_1\boxed{\star}G_2\right);\, x\right)\\
&= det\left(x\, I_{n_1+n_1n_2} - Q\left(G_1\boxed{\star}G_2\right)\right)\\
&= det\begin{pmatrix} xI_{n_1} - n_2\left(D(G_1)+I_{n_1}\right) - Q(G_1) & -\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2}^T \\ -\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2} & xI_{n_1n_2} - I_{n_1}\otimes Q(G_2) - \left(D(G_1)+I_{n_1}\right)\otimes I_{n_2}\end{pmatrix}.
\end{aligned}$$

As G_1 is r_1-regular graph, we have

$$\begin{aligned}
&\phi\left(Q\left(G_1\boxed{\star}G_2\right);\, x\right)\\
&= det\begin{pmatrix} xI_{n_1} - n_2(r_1+1)I_{n_1} - Q(G_1) & -\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2}^T \\ -\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2} & xI_{n_1n_2} - I_{n_1}\otimes Q(G_2) - (r_1+1)I_{n_1}\otimes I_{n_2}\end{pmatrix}\\
&= det\begin{pmatrix} xI_{n_1} - n_2(r_1+1)I_{n_1} - Q(G_1) & -\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2}^T \\ -\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2} & I_{n_1}\otimes\left((x-r_1-1)I_{n_2} - Q(G_2)\right)\end{pmatrix}.
\end{aligned}$$

Applying Proposition 2.1, we have

$\phi\left(Q\left(G_1\boxed{\star}G_2\right);\, x\right) =$

$\phi\left(Q\left(G_2\right);\, x-r_1-1\right)^{n_1} det\left((x-n_2-r_1n_2)I_{n_1} - Q(G_1) - S\right),$

where

$S =$

$\left[-\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2}^T\right]\left[I_{n_1}\otimes\left((x-r_1-1)I_{n_2}-Q(G_2)\right)\right]^{-1}\left[-\left(I_{n_1}+A(G_1)\right)\otimes \mathbf{1}_{n_2}\right]$

$= \left(I_{n_1}+A(G_1)\right)^2\otimes \mathbf{1}_{n_2}^T\left((x-r_1-1)I_{n_2}-Q(G_2)\right)^{-1}\mathbf{1}_{n_2}$

$= \Gamma_{Q(G_2)}(x-r_1-1)\left(I_{n_1}+A(G_1)\right)^2.$

Therefore,

$$\begin{aligned}
\phi\left(Q\left(G_1\boxed{\star}G_2\right);\, x\right) &= \phi\left(Q(G_2);\, x-r_1-1\right)^{n_1}\\
&\quad det\left((x-n_2-r_1n_2)I_{n_1} - Q(G_1) - \Gamma_{Q(G_2)}(x-r_1-1)\left(I_{n_1}+A(G_1)\right)^2\right).
\end{aligned}$$

Expanding the determinant of RHS in terms of products of $\gamma_i(G_1)$ by using the relation $\lambda_i(G_1) = \gamma_i(G_1) - r_1$, the result follows. □

Corollary 3.3. *If G_1 is r_1-regular graph and G_2 is r_2-regular graph, then the eigenvalues of $Q\left(G_1\boxed{\star}G_2\right)$ are $\gamma_i(G_2)+r_1+1$, $i = 2, 3, \ldots, n_2$, each occuring n_1 times and*

$$\frac{(n_2+1)(r_1+1)+2r_2+\gamma_i(G_1) \pm \sqrt{\left((n_2+1)(r_1+1)+2r_2+\gamma_i(G_1)\right)^2 - 4k_i}}{2},$$

where $k_i = 2n_2r_2(r_1+1) + (2n_2r_1+2r_2-2n_2+r_1+1)\gamma_i(G_1) + (4r_1-\gamma_i(G_1)^2)n_2$, $i = 1, 2, \ldots, n_1$.

Proof. As G_2 is r_2-regular, substituting

$$\Gamma_{Q(G_2)}(x-r_1-1) = \frac{n_2}{(x-r_1-1-2r_2)}$$

in Theorem 3.3, result follows. □

4. Neighborhood complement corona of two graphs

In 2015, Rakshit et al. [**9**] defined the neighborhood complement corona $G_1 \overline{\star} G_2$ and hence studied adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum of those structures when both the graphs taken into account are regular. In this section we study neighborhood complement corona $G_1 \overline{\star} G_2$ for their adjacency polynomial (both G_1 and G_2 are arbitrary); Laplacian polynomial and signless Laplacian polynomial (G_2 is arbitrary). The results due to Rakshit et al. [**9**] are particular cases when both the graphs are regular.

Definition 4.1. ([**9**]) Let G_1 be the graph with vertices $v_1, v_2, \ldots, v_{n_1}$ and G_2 be another graph. Then, the neighborhood complement corona $G_1 \overline{\star} G_2$ of two graphs G_1 and G_2 is obtained by taking n_1 copies of G_2 and joining each vertex v_j in G_1 to every vertex of i-th copy of G_2, provided the vertices v_i and v_j are non-adjacent in G_1 or $i = j$ (see Fig. 2).

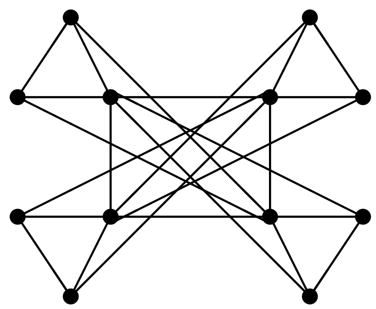

Figure 2: $C_4 \overline{\star} K_2$.

4.1. Adjacency polynomial of neighborhood complement corona.

Theorem 4.1. *If G_1 and G_2 are any two graphs on n_1 and n_2 vertices respectively, then the spectral polynomial of $A\big(G_1 \overline{\star} G_2\big)$ is,*

$$\phi\left(A\big(G_1 \overline{\star} G_2\big);\, x\right) = \phi\left(A\left(G_2\right);\, x\right)^{n_1} \, det\left(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)\ \left(I_{n_1} + A(\overline{G_1})\right)^2\right).$$

Proof. The adjacency matrix of $G_1 \overline{\star} G_2$ is,

$$A\left(G_1 \overline{\star} G_2\right) = \begin{pmatrix} A(G_1) & (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2}^T \\ (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes A(G_2) \end{pmatrix}.$$

The spectral polynomial is,

$$\begin{aligned} \phi\left(A\left(G_1 \overline{\star} G_2\right);\, x\right) &= det\left(x\, I_{n_1+n_1 n_2} - A\left(G_1 \overline{\star} G_2\right)\right) \\ &= det\left(\begin{array}{c|c} xI_{n_1} - A(G_1) & -\left(J_{n_1} - A(G_1)\right) \otimes \mathbf{1}_{n_2}^T \\ \hline -\left(J_{n_1} - A(G_1)\right) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes \left(xI_{n_2} - A(G_2)\right) \end{array}\right). \end{aligned}$$

Applying Proposition 2.1, we have

$$\phi\left(A\left(G_1 \overline{\star} G_2\right);\, x\right) \quad = \quad \phi\left(A\left(G_2\right);\, x\right)^{n_1} det\left(xI_{n_1} - A(G_1) - S\right),$$

where,

$S =$
$\left[-\left(J_{n_1} - A(G_1)\right) \otimes \mathbf{1}_{n_2}^T\right] \left[I_{n_1} \otimes \left(xI_{n_2} - A(G_2)\right)\right]^{-1} \left[-\left(J_{n_1} - A(G_1)\right) \otimes \mathbf{1}_{n_2}\right]$
$= \left(J_{n_1} - A(G_1)\right)^2 \otimes \mathbf{1}_{n_2}^T \left(xI_{n_2} - A(G_2)\right)^{-1} \mathbf{1}_{n_2}$
$= \Gamma_{A(G_2)}(x)\ \left(J_{n_1} - A(G_1)\right)^2$
$= \Gamma_{A(G_2)}(x)\ \left(I_{n_1} + A(\overline{G_1})\right)^2.$

Therefore,

$$\begin{aligned}\phi\left(A\left(G_1 \overline{\star} G_2\right);\, x\right) \quad &= \quad \phi\left(A\left(G_2\right);\, x\right)^{n_1} \\ &\quad det\left(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)\ \left(I_{n_1} + A(\overline{G_1})\right)^2\right).\end{aligned}$$

Hence, the result follows. □

Corollary 4.1. *If G_1 and G_2 are r_1 and r_2-regular graphs, respectively, then the eigenvalues of $A\left(G_1 \overline{\star} G_2\right)$ are $\lambda_i(G_2)$, $i = 2, 3, \ldots, n_2$, each occuring n_1 times,*

$$\frac{r_2 + r_1 \pm \sqrt{(r_2 - r_1)^2 + 4n_2(n_1 - r_1)^2}}{2}$$

and

$$\frac{r_2 + \lambda_i(G_1) \pm \sqrt{\left(r_2 - \lambda_i(G_1)\right)^2 + 4n_2\lambda_i(G_1)^2}}{2}, \quad i = 2, 3, \ldots, n_1.$$

Proof. Both G_1 and G_2 are regular graphs. Therefore substituting $\Gamma_{A(G_2)}(x) = \dfrac{n_2}{x - r_2}$ in Theorem 4.1, using the relation between $\lambda_i(G)$ and $\lambda_i(\overline{G})$ from Proposition 2.2 and equating the RHS to zero, we will arrive at the spectrum of $A(G_1 \overline{\star} G_2)$. □

Theorem 3.1 of [**9**] becomes the particular case of Theorem 4.1 and is observed from Corollary 4.1.

4.2. Laplacian polynomial of neighborhood complement corona.

Theorem 4.2. *If G_1 is an r_1-regular graph on n_1 vertices and G_2 is any graph on n_2 vertices, then the spectral polynomial of $L\left(G_1 \overline{\star} G_2\right)$ is,*

$$\begin{aligned}\phi\left(L(G_1 \overline{\star} G_2);\, x\right) \quad &= \quad \left(\phi\left(L\left(G_2\right);\, x - n_1 + r_1\right)\right)^{n_1} \\ &\quad det\left((x - n_1 n_2 + r_1 n_2)I_{n_1} - L(G_1) - \Gamma_{L(G_2)}(x - n_1 + r_1)\left(J_{n_1} - A(G_1)\right)^2\right).\end{aligned}$$

Proof. The Laplacian matrix of $G_1 \overline{\star} G_2$ is

$$L\left(G_1 \overline{\star} G_2\right) = \begin{pmatrix} n_2\left(n_1 I_{n_1} - D(G_1)\right) + L(G_1) & -\left(J_{n_1} - A(G_1)\right) \otimes \mathbf{1}_{n_2}^T \\ -\left(J_{n_1} - A(G_1)\right) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes L(G_2) + \left(n_1 I_{n_1} - D(G_1)\right) \otimes I_{n_2} \end{pmatrix}.$$

The spectral polynomial is

$$\begin{aligned}
&\phi\left(L\left(G_1 \bar{\star} G_2\right);\, x\right)\\
= \;& det\left(x\, I_{n_1+n_1n_2} - L\left(G_1 \bar{\star} G_2\right)\right)\\
= \;& det\begin{pmatrix} xI_{n_1} - n_2\left(n_1 I_{n_1} - D(G_1)\right) - L(G_1) & (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2}^T \\ (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2} & xI_{n_1n_2} - I_{n_1} \otimes L(G_2) - \left(n_1 I_{n_1} - D(G_1)\right) \otimes I_{n_2} \end{pmatrix}.
\end{aligned}$$

As G_1 is r_1-regular graph, we have

$$\begin{aligned}
&\phi\left(L\left(G_1 \bar{\star} G_2\right);\, x\right)\\
= \;& det\begin{pmatrix} xI_{n_1} - n_2(n_1 - r_1)I_{n_1} - L(G_1) & (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2}^T \\ (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2} & xI_{n_1n_2} - I_{n_1} \otimes L(G_2) - (n_1 - r_1)I_{n_1} \otimes I_{n_2} \end{pmatrix}\\
= \;& det\begin{pmatrix} xI_{n_1} - n_2(n_1 - r_1)I_{n_1} - L(G_1) & (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2}^T \\ (J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2} & I_{n_1} \otimes ((x - n_1 + r_1)I_{n_2} - L(G_2)) \end{pmatrix}.
\end{aligned}$$

Applying Proposition 2.1, we have

$$\begin{aligned}
&\phi\left(L\left(G_1 \bar{\star} G_2\right);\, x\right) =\\
&\phi\left(L\left(G_2\right);\, x - n_1 + r_1\right)^{n_1} det\left((x - n_1n_2 + r_1n_2)I_{n_1} - L(G_1) - S\right),
\end{aligned}$$

where,

$S =$

$\left[(J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2}^T\right]\left[I_{n_1} \otimes ((x - n_1 + r_1)I_{n_2} - L(G_2))\right]^{-1}\left[(J_{n_1} - A(G_1)) \otimes \mathbf{1}_{n_2}\right]$

$= (J_{n_1} - A(G_1))^2 \otimes \mathbf{1}_{n_2}^T\left((x - n_1 + r_1)I_{n_2} - L(G_2)\right)^{-1}\mathbf{1}_{n_2}$

$= \Gamma_{L(G_2)}(x - n_1 + r_1)\;\left(J_{n_1} - A(G_1)\right)^2.$

Therefore,

$$\begin{aligned}
\phi\left(L(G_1 \bar{\star} G_2);\, x\right) \;=\; & \left(\phi\left(L\left(G_2\right);\, x - n_1 + r_1\right)\right)^{n_1}\\
& det\left((x - n_1n_2 + r_1n_2)I_{n_1} - L(G_1) - \Gamma_{L(G_2)}(x - n_1 + r_1)\left(J_{n_1} - A(G_1)\right)^2\right).
\end{aligned}$$

Hence, the result follows. □

Corollary 4.2. *If G_1 is r_1-regular graph and G_2 is r_2-regular graph, then the eigenvalues of $L\left(G_1 \bar{\star} G_2\right)$ are $\mu_i(G_2) + n_1 - r_1$, $i = 2, 3, \ldots, n_2$, each occuring n_1 times, 0 and $(n_1 - r_1)(n_2 + 1)$, each occuring one time and*

$$\frac{(n_1 - r_1)(n_2 + 1) + \mu_i(G_1) \pm \sqrt{\left((n_1 - r_1)(n_2 + 1) + \mu_i(G_1)\right)^2 + 4k_i}}{2},$$

where, $k_i = n_2\left(n_1 - \mu_i(G_1)\right)\left(2r_1 - \mu_i(G_1) - n_1\right) - \mu_i(G_1)\;(n_1 - r_1)$, for $i = 2, 3, \ldots, n_1$.

Proof. Both G_1 and G_2 are regular graphs. Therefore substituting $\Gamma_{L(G_2)}(x - n_1 + r_1) = \dfrac{n_2}{x - n_1 + r_1}$ and $J_{n_1} - A(G_1) = I_{n_1} + A(\overline{G_1})$ in Theorem 4.2 and then, using the relation between $\lambda_i(G)$ and $\lambda_i(\overline{G})$ from Proposition 2.2 and $\mu_i(G_1) = r_1 - \lambda_i(G_1)$ and equating the RHS to zero, we will arrive at the spectrum of $L(G_1 \bar{\star} G_2)$. □

Theorem 3.3 of [**9**] becomes the particular case of Theorem 4.2.

4.3. Signless Laplacian polynomial of neighborhood complement corona.

THEOREM 4.3. *If G_1 is an r_1-regular graph on n_1 vertices and G_2 is any graph on n_2 vertices, then the spectral polynomial of $Q\big(G_1\overline{\star}G_2\big)$ is,*

$$\phi\left(Q(G_1\overline{\star}G_2);\, x\right) = \left(\phi(Q(G_2);\, x-n_1+r_1)\right)^{n_1} det\Big((x-n_1n_2+r_1n_2)I_{n_1} - Q(G_1) - \Gamma_{Q(G_2)}(x-n_1+r_1)\ \ (J_{n_1}-A(G_1))^2\Big).$$

PROOF. The signless Laplacian matrix of $G_1\overline{\star}G_2$ is,

$$Q\left(G_1\overline{\star}G_2\right) = \begin{pmatrix} n_2\left(n_1I_{n_1} - D(G_1)\right) + Q(G_1) & (J_{n_1}-A(G_1))\otimes \mathbf{1}_{n_2}^T \\ (J_{n_1}-A(G_1))\otimes \mathbf{1}_{n_2} & I_{n_1}\otimes Q(G_2) + (n_1I_{n_1}-D(G_1))\otimes I_{n_2} \end{pmatrix}.$$

The spectral polynomial is

$$\begin{aligned} &\phi\left(Q\left(G_1\overline{\star}G_2\right);\, x\right) \\ =\ & det\left(x\, I_{n_1+n_1n_2} - Q\left(G_1\overline{\star}G_2\right)\right) \\ =\ & det\begin{pmatrix} xI_{n_1} - n_2\left(n_1I_{n_1}-D(G_1)\right) - Q(G_1) & -\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2}^T \\ -\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2} & xI_{n_1n_2} - I_{n_1}\otimes Q(G_2) - \left(n_1I_{n_1}-D(G_1)\right)\otimes I_{n_2} \end{pmatrix}. \end{aligned}$$

As G_1 is r_1-regular graph, we have

$$\begin{aligned} &\phi\left(Q\left(G_1\overline{\star}G_2\right);\, x\right) \\ =\ & det\begin{pmatrix} xI_{n_1} - n_2(n_1-r_1)\ I_{n_1} - Q(G_1) & -\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2}^T \\ -\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2} & xI_{n_1n_2} - I_{n_1}\otimes Q(G_2) - (n_1-r_1)I_{n_1}\otimes I_{n_2} \end{pmatrix} \\ =\ & det\left(\begin{array}{c|c} xI_{n_1} - n_2(n_1-r_1)\ I_{n_1} - Q(G_1) & -\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2}^T \\ \hline -\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2} & I_{n_1}\otimes\left((x-n_1+r_1)I_{n_2} - Q(G_2)\right) \end{array}\right). \end{aligned}$$

Applying Proposition 2.1, we have

$$\phi\left(Q\left(G_1\overline{\star}G_2\right);\, x\right) = \phi\left(Q\left(G_2\right);\, x-n_1+r_1\right)^{n_1} det\left((x-n_1n_2+r_1n_2)I_{n_1} - Q(G_1) - S\right)$$

where

$$\begin{aligned} S =\ & \\ & \left[-\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2}^T\right]\ \left[I_{n_1}\otimes\left((x-n_1+r_1)I_{n_2}-Q(G_2)\right)\right]^{-1}\ \left[-\left(J_{n_1}-A(G_1)\right)\otimes \mathbf{1}_{n_2}\right] \\ =\ & \left(J_{n_1}-A(G_1)\right)^2 \otimes \mathbf{1}_{n_2}^T\left((x-n_1+r_1)I_{n_2}-Q(G_2)\right)^{-1}\mathbf{1}_{n_2} \\ =\ & \Gamma_{Q(G_2)}(x-n_1+r_1)\ \ \left(J_{n_1}-A(G_1)\right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\phi\left(Q\big(G_1\overline{\star}G_2\big);\, x\right) = \\ &\left(\phi\left(Q\left(G_2\right);\, x-n_1+r_1\right)\right)^{n_1} \\ &det\Big((x-n_1n_2+r_1n_2)I_{n_1} - Q(G_1) - \Gamma_{Q(G_2)}(x-n_1+r_1)\ \ \left(J_{n_1}-A(G_1)\right)^2\Big). \end{aligned}$$

Hence, the result follows. □

COROLLARY 4.3. *If G_1 and G_2 are r_1 and r_2-regular graphs, respectively, then the eigenvalues of $Q\left(G_1\overline{\star}G_2\right)$ are $\gamma_i(G_2)+n_1-r_1$, $i=2,3,\ldots,n_2$, each occuring n_1 times and*

$$\frac{n_2(n_1-r_1)+n_1+r_1+2r_2 \pm \sqrt{\left(n_2(n_1-r_1)+n_1+r_1+2r_2\right)^2+8k_i}}{2},$$

where $k_i = [n_2 r_2(r_1 - n_1) + r_1^2 - r_1(2r_2 + n_1)]$ *and*

$$\frac{(n_1 - r_1)(n_2 + 1) + 2r_2 + \gamma_i(G_1) \pm \sqrt{((n_1 - r_1)(n_2 + 1) + 2r_2 + \gamma_i(G_1))^2 + 4s_i}}{2},$$

where $s_i = n_2(2n_1r_1 - 2n_1r_2 + 2r_1r_2 - n_1^2 + \gamma_i(G_1)^2) - \gamma_i(G_1)(2r_1n_2 + 2r_2 + n_1 - r_1)$, $i = 2, 3, \ldots, n_1$.

Proof. Both G_1 and G_2 are regular graphs. Substituting $\Gamma_{Q(G_2)}(x - n_1 + r_1) = \dfrac{n_2}{x - n_1 + r_1 - 2r_2}$, $J_{n_1} - A(G_1) = I_{n_1} + A(\overline{G_1})$, in Theorem 4.3 and then, using the relation between $\lambda_i(G)$ and $\lambda_i(\overline{G}$ from Proposition 2.2, $\gamma_i(G_1) = r_1 + \lambda_i(G_1)$ and equating the RHS to zero, we arrive at the spectrum of $Q(G_1 \overline{\star} G_2)$. □

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