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GRAPHS WITH SMALL AND LARGE HOP DOMINATION NUMBERS

D. Anusha, J. John and S. Joseph Robin

Abstract. A set $S \subseteq V(G)$ of a graph G is a *hop dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called the *hop domination number* and is denoted by $\gamma_h(G)$. Any hop dominating set of order $\gamma_h(G)$ is called γ_h-*set* of G. In this paper necessary and sufficient conditions for the hop domination number to be 2, $n-1$ and n are given.

1. Introduction

For notation and graph theory terminology we in general, follow [**4**]. Specifically, let $G = (V, E)$ be a graph with vertex set V of *order* $n = |V|$ and edge set E of *size* $m = |E|$. Let v be a vertex in $V(G)$. Then the *open neighborhood* of v is the set $N(v) = \{u \in V(G)|uv \in E\}$, and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. The *degree* of a vertex v is $deg(v) = |N(v)|$. If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then e is called a *pendant edge* or *end edge*, u is a *leaf* or *end vertex* and v is a *support vertex* of u. A vertex of degree $n-1$ is called a *universal vertex.*

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. A u-v path of length $d(u, v)$ is called a u-v *geodesic.* A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. $e(v) = max\{d(v, u) : u \in V(G)\}$. The minimum eccentricity among the vertices of G is the *radius*, $radG$ or $r(G)$ and the maximum eccentricity is its *diameter*, $diamG$. We denote $rad(G)$ by r and $diamG$ by d.

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The center of a graph is the set of all vertices of minimum eccentricity, that is, the set of all vertices u where the greatest d to other vertices v is minimal. A quadrilateral book consists of r quadrilaterals sharing a common edge uv. That is, it is a cartesian product of a star and a single edge. It is denoted by $Q_{r,2}$.

A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex $v \in V(G) \setminus D$ is adjacent to some vertex in D. A dominating set D is said to be *minimal* if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. The domination number of a graph was studied in [**5**]. A set $S \subseteq V(G)$ of a graph G is a *hop dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called the *hop domination number* and is denoted by $\gamma_h(G)$. Any hop dominating set of order $\gamma_h(G)$ is called γ_h-*set* of G. The hop domination number of a graph was studied in [**1, 2, 3, 6, 7, 8, 9, 10**]. Hop domination has applications in social networks. In a cable network transport system, each person will stand with each vertex with a rope. Cable vehicle will travel, through that network. If a person want to travel in that vehicle, he/she want to stand at a distance two. If each person in the network want to travel in the cable vehicle the minimum hop dominating set is the desired set.

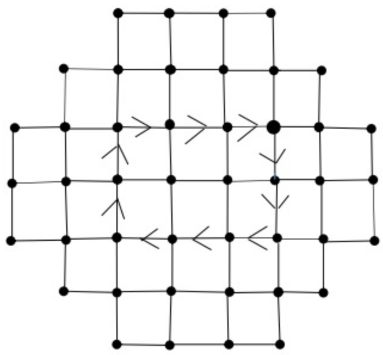

FIGURE 1

In this paper we have given some necessary and sufficient condition for the the hop domination to be 2, $n-1$ and n.

2. Graphs with small and large hop domination numbers

THEOREM 2.1. *Let G be a connected graph with $\gamma_h(G) = 2$ and $S = \{u, v\}$ be a γ_h-set of G. If*

(i) *$uv \in E(G)$, then $d \leqslant 5$, and*

(ii) *$uv \notin E(G)$,*

then $d \leqslant 4$.

Proof. (i) Suppose $uv \in E(G)$. Let $S = \{u, v\}$ be a γ_h-set of G. If $G = K_n$ ($n \geqslant 2$), then $n = 2$. Hence $d = 1$ so we assume that G is non-complete. Let $P : u_0, u_1, u_2, ..., u_d$ be any shortest path in G such that either u or v or both belongs to $V(P)$.

Case (1): $u, v \in V(P)$. Since $uv \notin E(G)$ for any $w \in V - S$, $d(u, w) = d(v, w) = 2$, it follows that $d(u_0, u) \leqslant 2$ and $d(u, u_d) \leqslant 2$. Now

$$\begin{aligned} d(u_0, u_d) &\leqslant d(u_0, u) + d(u, u_d) \\ &\leqslant d(u_0, u) + d(u, v) + d(v, u_d) \\ &= 2 + 1 + 2 = 5 \end{aligned}$$

Therefore $d(u_0, u_d) \leqslant 5$. Since this is true for every shortest path P in G, we have that $d \leqslant 5$.

Case (2): $u \in V(P)$ and $v \notin V(P)$. Since for any $w \in V - S$, $d(u, w) = 2$, it follows that $d(u_0, u) \leqslant 2$ and $d(u_0, u_d) \leqslant 2$. Now

$$\begin{aligned} d(u_0, u_d) &\leqslant d(u_0, u) + d(u, u_d) \\ &= 2 + 2 = 4 \leqslant 5. \end{aligned}$$

Therefore $d(u_0, u_d) \leqslant 5$. Since this is true for every shortest path P in G, we have that $d \leqslant 5$.

Case (3): $v \in V(P)$ and $u \notin V(P)$. By the similar argument as in Case (2), we can prove that $d \leqslant 5$.

(ii) $uv \notin E(G)$.

Case (1): $u, v \in V(P)$. Since $uv \notin E(G)$, both u and v are end vertices of P. Without loss of generality, let us assume that $u = u_0$ and $v = u_d$. By definition of hop dominating set, there must be only two internal vertices in P. Hence it follows that $d(u, v) = 3$. Since this is true for all shortest path in G,we have that $d = 3$.

Case(2): $u \in V(P)$ and $v \notin V(P)$.

Case(2a): u is an end vertex of P. Without loss of generality, let us assume that $u = u_0$. Then by definition of hop dominating set $u_d = u_2$. Since this is true for all the shortest path in G. We have that $d = 2$.

Case(2b): u is not an end vertex of P. Then it follows that u is either u_1 or u_2, or u is either u_{d-1} or u_{d-2}. Suppose that $u = u_1$, $d(u_0, v) = d(u_2, v) = 2$. Also since $u = u_1$, $d(u_1, u_3) = 2$. Hence it follows that $u_3 = u_d$. Therefore

$$\begin{aligned} d(u_0, u_d) &= d(u_0, u_1) + d(u_1, u_d) \\ &= 1+2 = 3 \end{aligned}$$

Suppose that $u = u_2$. Then $d(v, u_1) = d(v, u_3) = 2$. Also since $u = u_2$, $d(u_0, u_2) = d(u_2, u_4) = 2$. Hence it follows that $u_4 = u_d$. Therefore

$$\begin{aligned} d(u_0, u_d) &= d(u_0, u_2) + d(u_2, u_d) \\ &= 2+2 = 4. \end{aligned}$$

Case(3): $v \in V(P)$ and $u \notin V(P)$. By the similar argument as in Case 2, we can prove that $d \leqslant 4$. □

Theorem 2.2. *Let G be a connected graph with order $n \geqslant 3$. Then $\gamma_h(G) = 2$ if and only if any one the following conditions hold.*

(i) *If $\langle S \rangle$ is connected, then $\langle S \rangle$ contains no triangles and $d \leqslant 5$, where S is a γ_h-set of G.*

(ii) *If $\langle S\rangle$ is independent, then $\langle S\rangle$ contains no triangles and $d \leqslant 4$, where S is a γ_h-set of G.*

Proof. (i) Let $\gamma_h(G) = 2$. Let $S = \{u, v\}$ be a γ_h-set of G. Let $\langle S\rangle$ be connected. Then $uv \in E(G)$. By Theorem 2.1, $d \leqslant 5$. Since S contains two elements, S has no triangles. Conversely, let $\langle S\rangle$ be connected and $d \leqslant 5$, $\langle S\rangle$ contains no triangles. We prove that $|S| = 2$. On the contrary, suppose that $\mid S \mid\geqslant 3$. Since $\langle S\rangle$ is connected, let $u, v, w \in S$ such that $\langle u, v, w\rangle$ is connected. Then there exists an edge u_1u_2 such that $u_2u \in E(G)$ and $u_1u \notin E(G)$. Also there exists end edge w_1w_2 such that $w_2w \in E(G)$ and $w_1w \notin E(G)$. Since $\langle u, v, w\rangle$ is a path and $\mid S \mid\geqslant 3$, we have that $d(u_1, v) \geqslant 3$ and $d(w_1, v) \geqslant 3$. This implies $d(u_1, w_1) \geqslant 6$. Hence it follows that $d \geqslant 6$, which is a contradiction. Hence $\mid S \mid= 2$.

(ii) Let $\gamma_h(G) = 2$. Let $S = \{u, v\}$ be a γ_h-set of G. Let $\langle S\rangle$ be independent. Then $uv \notin E(G)$. By Theorem 2.1, $d \leqslant 4$. Since $\langle S\rangle$ contains two elements, $\langle S\rangle$ has no triangles. Conversely, let $\langle S\rangle$ be independent, $d \leqslant 4$ and $\langle S\rangle$ contains no triangles. We prove that $|S| = 2$. On the contrary, suppose that $|S| \geqslant 3$. Since S is independent, let $u, v, w \in S$ such that u, v and w is independent, let $u, v, w \in S$ such that u, v and w are independent. Since S is a γ_h-set of G, $d(u, v) = d(v, w) = 3$ and $d(u, w) \geqslant 5$. Therefore u-w is a shortest path. Then $d \geqslant 5$, which is a contradiction. Therefore $\gamma_h(G) = 2$. □

Corollary 2.1. *For the complete bipartite graph $G = K_{m,n}$ $(2 \leqslant m \leqslant n)$, the following holds $\gamma_h(G) = 2$.*

Corollary 2.2. *For the graph $G = K_{1,n-1} + e$ $(n \geqslant 4)$, holds $\gamma_h(G) = 2$.*

Corollary 2.3. *Let $H = \overline{K}_2 + K_{n-2}$ $(n \geqslant 4)$ and $V(K_2) = \{u, v\}$*

(i) *Then $\gamma_h(G) = 2$.*

(ii) *Let $H^{'}$ be a graph obtained from H by attaching end vertices in u or v or both. Then $\gamma_h(H^{'}) = 2$.*

(iii) *Let G be the graph obtained from $H^{'}$ by joining u and v. Then $\gamma_h(G) = 2$.*

Corollary 2.4. *Let $H = Q_{r,2}$ be the book graph with centre (u, v).*

(i) *Then $\gamma_h(H) = 2$.*

(ii) *Let G be the graph obtained from H by attaching end vertices in u or v or both. Then $\gamma_h(G) = 2$.*

Corollary 2.5. *Let $V(K_2) = \{u, v\}$. Let H be the graph obtained from K_2 by attaching end vertices in u or v or both.*

(i) *Then $\gamma_h(H) = 2$.*

(ii) *Let $H^{'}$ be the graph from H by attaching triangles in u or v or both. Then $\gamma_h(H^{'}) = 2$.*

(iii) *Let G be the graph obtained from $H^{'}$ by attaching end vertices to any triangles of $H^{'}$. Then $\gamma_h(G) = 2$.*

(iv) *Let $G^{'}$ be the graph obtained from H by attaching end vertices to any end edge of H. Then $\gamma_h(G^{'}) = 2$.*

(v) *Let K be the graph obtained from $H^{'}$ by attaching end vertices in u and v or both. Then $\gamma_h(K) = 2$.*

(vi) *Let* $K^{'}$ *be the graph obtained from* K *by attaching end vertices to end edge of* K*. Then* $\gamma_h(K^{'}) = 2$*.*

(vii) *Let* I *be the graph obtained from* G *by attaching end vertices* u *or* v *or both. Then* $\gamma_h(I) = 2$*.*

(viii) *Let* $I^{'}$ *be the graph obtained from* I *by attaching end vertices to any end edge of* I*. Then* $\gamma_h(I^{'}) = 2$*.*

Corollary 2.6. *Let* $P_3 : v_1, v_2, v_3$*. Let* H *be the graph obtained from* P_3 *and* $\overline{K}_r$ *by joining each vertex of* $\overline{K}_r$ *with* v_1 *and* v_3*.*

(i) *Then* $\gamma_h(H) = 2$*, where* $r = n - 3$*.*

(ii) *Let* $H^{'}$ *be a graph obtained from* H *by attaching end vertices in* v_1 *or* v_3 *or both. Then* $\gamma_h(H^{'}) = 2$*.*

(iii) *Let* $H^{''}$ *be the graph obtained from* H *and* $\overline{K}_s$ *where joining each vertex of* $\overline{K}_s$ *with* v_1 *and* v_2*. Then* $\gamma_h(H^{''}) = 2$*, where* $r + s = n - 3$*.*

(iv) *Let* $H^{'''}$ *be the graph obtained from* $H^{''}$ *by attaching end vertices on* v_1 *or* v_3 *or both. Then* $\gamma_h(H^{'''}) = 2$*.*

(v) *Let* K *be the graph obtained from* $H^{''}$ *by introducing the edges* u_2u_i*,* $1 \leqslant i \leqslant r$*. Then* $\gamma_h(K) = 2$*.*

(vi) *Let* $K^{'}$ *be the graph obtained from* K *by attaching end vertices on* v_1 *or* v_3*. Then* $\gamma_h(K^{'}) = 2$*.*

Corollary 2.7. *Let* $V(K_1) = u$*. Let* $P_i : u_i, v_i, w_i$ $(1 \leqslant i \leqslant r)$ *be a copy on three vertices. Let* G *be the graph obtained from* K_1, P_i $(1 \leqslant i \leqslant r)$ *and* $\overline{K}_s$ *by joining* u *with each* u_i *and* w_i $(1 \leqslant i \leqslant r)$ *and joining* u *with each element of* $\overline{K}_s$*. Then* $\gamma_h(H) = 2$*, where* $r + s = n - 1$*.*

Corollary 2.8. *Let* $C_4 : v_1, v_2, v_3, v_4$*. Let* G *be the graph obtained from* $C_4, \overline{K}_{m_1}, \overline{K}_{m_2}, \overline{K}_{m_3}, \overline{K}_{m_4}$ *and* $\overline{K}_{m_5}$ *by joining each element of* $\overline{K}_{m_1}$ *with* v_1 *and* v_2*, each element of* $\overline{K}_{m_2}$ *with* v_2 *and* v_3*, each element of* $\overline{K}_{m_4}$ *with* v_1 *and* v_3*, each element of* $\overline{K}_{m_5}$ *with* v_2 *and* v_4*. Then* $\gamma_h(G) = 2$*, where* $m_1 + m_2 + m_3 + m_4 + m_5 = n - 4$*.*

Theorem 2.3. *Let* G *be a connected graph having diameter* $d = 2$ *with a cut vertex. Then* $\gamma_h(G) = 2$ *if and only if* $\delta(G) = 1$*.*

Proof. Let v be a cut vertex of G. Let $G_1, G_2, ..., G_r$ $(r \geqslant 2)$ be the components of $G - v$. First assume that $\delta(G) = 1$. Then G contains at least one end vertex. Let x be an end vertex of G and $S = \{v, x\}$. Then for every $y \in V - S$, $d(x, y) = 2$ and so S is a hop dominating set of G so that $\gamma_h(G) = 2$.

Conversely that $\gamma_h(G) = 2$. Suppose that $\delta(G) \geqslant 2$. Let u be a vertex of minimum degree. Therefore $deg(u) \geqslant 2$. Let us assume that $u \in V(G_s)$ for some s; $1 \leqslant s \leqslant r$. Therefore $|V(G_s)| \geqslant 2$. Then $S^{'} = V(G_s) \cup \{v\}$ is a hop dominating set and so $\gamma_h(G) \geqslant 3$, which is a contradiction. Therefore $\delta(G) = 1$. □

Corollary 2.9. *For the star graph* $G = K_{1,n-1}$ *holds* $\gamma_h(G) = 2$*.*

Corollary 2.10. *For the graph* $G = K_1 + (m_1K_1 \cup m_1K_1 \cup m_2K_2 \cup ... \cup m_rK_r)$, *where* $m_1 + m_2 + .. + m_r = n - 1$ *and* $r \geqslant 2$, *the following golds* $\gamma_h(G) = 2$ *if and if only* $m_1 \neq 0$.

Theorem 2.4. *Let* G *be a non-complete connected graph with order* $n \geqslant 3$. *Then* $\gamma_h(G) \leqslant n - 1$.

Proof. On the contrary, suppose that $\gamma_h(G) = n$. Hence it follows that $S = V(G)$ is the unique γ_h-set of G. Therefore vertex x in G is adjacent to all the elements of G. Hence it follows that $G = K_n$, which is a contradiction. Hence $\gamma_h(G) \leqslant n - 1$. □

Remark 2.1. The bound in Theorem 2.4, is sharp. For $G = P_3$, we have $\gamma_h(G) = 2 = n - 1$.

Corollary 2.11. *For a connected graph* G *of order* $n \geqslant 2$, *holds* $\gamma_h(G) = n$ *if and only if* $G = K_n$.

Proof. Let $\gamma_h(G) = n$. We prove that $G = K_n$. On the contrary, suppose that $G \neq K_n$. By Theorem 2.4, $\gamma_h(G) \leqslant n - 1$, which is a contradiction. Therefore $\gamma_h(G) = n$. Converse is clear. □

Theorem 2.5. *Let* G *be a connected graph of order* $n \geqslant 3$. *Then* $\gamma_h(G) = n-1$ *if and only if* $G = P_3$ *or* $G = K_n - \{e\}$.

Proof. Let $\gamma_h(G) = n - 1$. Let $n = 3$. Then G is either P_3 or K_3. If $G = K_3$ then by Corollary 2.11, $\gamma_h(G) = n$, which is a contradiction. If $G = P_3$, then $\gamma_h(G) = 2$. So, we have done. So let $n \geqslant 4$. Let $x \in V$ and $S = V - \{x\}$ be a γ_h-set of G. We prove that $G[S]$ is a clique. On the contrary, suppose that $G[S]$ is not a clique. Then there exist vertices $y, z \in G[S]$ such that $d_{G[S]}(y, z) \geqslant 2$. Let $S_1 = S - \{y\}$. Then S_1 is a hop dominating set of G so that $\gamma_h(G) \leqslant n - 2$, which is a contradiction. Therefore $G[S]$ is clique.

Next we prove that x is adjacent to $|G[S]| - 1$ vertices of $G[S]$. On the contrary, suppose that x is adjacent to all vertices of $G[S]$. Then $G = K_n$. Which implies $\gamma_h(G) = n$, which is a contradiction. Therefore x is adjacent to $n - 2$ vertices of G. Therefore $G = K_n - \{e\}$. Converse is clear. □

3. Conclusion

In this article we characterize connected graphs of order n with hop domination number 2 or $n - 1$ or n. We will make effort to characterize connected graphs of order n with hop domination number $n - 2$ in future study.

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D. Anusha: Research Scholar, Register Number-19223162092032, Department of Mathematics, Sree Devi Kumari Women's College,, Kuzhithurai-629 163,, India.

E-mail address: anushasenthil84@gmail.com

J. John: Department of Mathematics, Government College of Engineering, Tirunelveli 627 007, India.

E-mail address: john@gcetly.ac.in

S. Joseph Robin: Department of Mathematics, Scott Christian College, Nagercoil-629 003, India.

E-mail address: dr.robinscc@gmail.com