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NEW GENERATING FUNCTION RELATIONS FOR A FAMILY OF MULTUVARIABLE POLYNOMIALS

Nejla Özmen

Abstract. In this paper, we get bilateral generating functions between the multvariable polynomials $\Phi_n^{(\alpha)}$ defined in [Srivastava, H.M. and Daoust, M.C., Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wet., Proc., Ser. A.*, **72**(1969) 449-457.] and the generalized Srivastava and Daoust functions. Furthermore, we discuss some applications of the our recent results obtained in [N. Özmen and E. Erkuş-Duman. Some results for a family of multivariable polynomials. *AIP Conf. Proc.*, **1558** (2013), 1124–1127] and find a partial differential equation including these multivariable polynomials $\Phi_n^{(\alpha)}$.

1. Introduction

The main motivation of this work relies on the paper [**13**]. Firstly, we recall some concepts used in this paper. The following multivariate polynomials $\Phi_n^{(\alpha)}$ have been defined in [**4**] and studied in [**13**]:

$$\Phi_n^{(\alpha)}(x_1, ..., x_r) = \sum_{n_1+...+n_r=n} (\alpha)_{n_1} \frac{x_1^{n_1}}{n_1!} ... \frac{x_r^{n_r}}{n_r!}, \tag{1.1}$$

whose generating function is given by the relation

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x_1, ..., x_r) z^n = (1 - x_1 z)^{-\alpha} e^{(x_2+...+x_r)z}, \tag{1.2}$$

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where $|z|<|x_1|^{-1}$. Observe that $\Phi_n^{(\alpha)}$ satisfies the following limit relation:

$$\Phi_n^{(\alpha_1)}(x_1,...,x_r):=\lim_{\min\{|\alpha_2|,...,|\alpha_r|\}\to\infty}\left\{g_n^{(\alpha_1,...,\alpha_r)}(x_1,\frac{x_2}{\alpha_2},...,\frac{x_r}{\alpha_r})\right\},$$

where $g_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ denotes the Chan-Chyan-Srivastava polynomials defined in [**1**].

Srivastava and Daoust [**11**] defined the following multivariable function which generalizes the familiar Kampé de Fériet hypergeometric function:

$$F_{C:D^{(1)};...;D^{(n)}}^{A:B^{(1)};...;B^{(n)}}\left(\begin{array}{l}\left[(a):\theta^{(1)},...,\theta^{(n)}\right]:\quad\left[(b^{(1)}):\phi^{(1)}\right];...;\quad\left[(b^{(n)}):\phi^{(n)}\right];\\ \left[(c):\psi^{(1)},...,\psi^{(n)}\right]:\quad\left[(d^{(1)}):\delta^{(1)}\right];...;\quad\left[(d^{(n)}):\delta^{(n)}\right];\end{array}\quad z_1,...,z_n\right)$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\Omega(m_1,...,m_n)\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},$$

where

$$\Omega(m_1,...,m_n):=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+...+m_n\psi_j^{(n)}}}\frac{\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}}...\frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}},$$

and the coefficients

$$\theta_j^{(k)}\ (j=1,...,A;\ k=1,...,n),\ \text{and}\ \phi_j^{(k)}\ (j=1,...,B^{(k)};\ k=1,...,n),$$

$$\psi_j^{(k)}\ (j=1,...,C;\ k=1,...,n),\quad\text{and}\ \delta_j^{(k)}\ (j=1,...,D^{(k)};\ k=1,...,n)$$

are real constants, and $\left(b_{B^{(k)}}^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}\ (j=1,...,B^{(k)};\ k=1,...,n)$$

with similar interpretations for other sets of parameters.

In the next section, we derive various families of bilateral generating functions between the multivariable functions $\Phi_n^{(\alpha)}$ and $F_{C:D^{(1)};...;D^{(n)}}^{A:B^{(1)};...;B^{(n)}}$. In the third section, we deal with some applications of the results regarding $\Phi_n^{(\alpha)}$ obtained in [**13**] and find a partial differential equation including $\Phi_n^{(\alpha)}$.

2. Bilateral generating functions between $\Phi_n^{(\alpha)}$ and Srivastava-Daoust functions

For a suitably bounded non-vanishing multiple sequence

$$\{\Omega(m_1;m_2,...,m_s)\}_{m_1,m_2,...,m_s\in\mathbb{N}_0}$$

of real or complex parameters, Liu et. al. (see [**12**]) defined a function

$$\Upsilon_n(u_1; u_2, ..., u_s)$$

of s (real or complex) variables $u_1, u_2, ..., u_s$ by

$$\begin{aligned}(2.1) \quad \Upsilon_n(u_1; u_2, ..., u_s) \quad : \quad &= \sum_{m_1=0}^{n} \sum_{m_2,...,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\ &\times \Omega\left(f(m_1, m_2, ..., m_s); m_2, ..., m_s\right) \frac{u_1^{m_1}}{m_1!} \cdots \frac{u_s^{m_s}}{m_s!}\end{aligned}$$

where

$$((b))_{m_1\phi} = \prod_{j=1}^{B} (b_j)_{m_1\phi_j} \text{ and } ((d))_{m_1\delta} = \prod_{j=1}^{D} (d_j)_{m_1\delta_j}.$$

We first need the following lemma that is proved in [**13**].

Lemma 2.1 ([**13**]). *For $\Phi_n^{(\alpha)}$, we get*

$$\begin{aligned}&\sum_{n=0}^{\infty} \binom{m+n}{n} \Phi_{m+n}^{(\alpha)}(x_1, ..., x_r) z^n \\ = \quad &(1 - x_1 z)^{-\alpha} e^{(x_2+...+x_r)z} \Phi_m^{(\alpha)} \left(\frac{x_1}{1 - x_1 z}, x_2, ..., x_r \right),\end{aligned}$$

where $|z| < |x_1|^{-1}$.

Then, we get the next result.

Theorem 2.1. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x_1, ..., x_r) \Upsilon_n(u_1; u_2, ..., u_s) z^n \\ = \quad &(1 - x_1 z)^{-\alpha} e^{(x_2+...+x_r)z} \sum_{(n_1),...,(n_r),m_2,...,m_s=0}^{\infty} \frac{((b))_{(n_1+...+n_r)\phi} (\alpha)_{n_1}}{((d))_{(n_1+...+n_r)\delta}} \\ &\times \Omega(f((n_1 + ... + n_r), m_2, ..., m_s); m_2, ..., m_s) \\ &\times \frac{(-u_1 x_1 z/1 - x_1 z)^{n_1}}{n_1!} \frac{(-u_1 x_2 z)^{n_2}}{n_2!} \cdots \frac{(-u_1 x_r z)^{n_r}}{n_r!} \frac{u_2^{m_2}}{m_2!} \cdots \frac{u_s^{m_s}}{m_s!}.\end{aligned}$$

PROOF. By using (2.1), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)\Upsilon_n(u_1;u_2,...,u_s)z^n\\
=&\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)\sum_{m_1=0}^{n}\sum_{m_2,...,m_s=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}\\
&\times\Omega\left(f(m_1,m_2,...,m_s);m_2,...,m_s\right)\frac{u_1^{m_1}}{m_1!}...\frac{u_s^{m_s}}{m_s!}z^n\\
=&\sum_{m_1,m_2,...,m_s=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,m_2,...,m_s);m_2,...,m_s\right)\\
&\times(-u_1z)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_s^{m_s}}{m_s!}\ (1-x_1z)^{-\alpha}\ e^{(x_2+...+x_r)z}\Phi_{m_1}^{(\alpha)}(\frac{x_1}{1-x_1z},x_2,...,x_r).
\end{aligned}$$

By applying Lemma 2.1 as well as the explicit representation (1.1) on the right-hand of the last equation and doing some calculations, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)\Upsilon_n(u_1;u_2,...,u_s)z^n\\
=&\ (1-x_1z)^{-\alpha}\ e^{(x_2+...+x_r)z}\sum_{m_1,m_2,...,m_s=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,m_2,...,m_s);m_2,...,m_s\right)\\
&\times(-u_1z)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_s^{m_s}}{m_s!}\sum_{n_1+...+n_r=m_1}^{\infty}(\alpha)_{n_1}\ \frac{(x_1/1-x_1z)^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}...\frac{x_r^{n_r}}{n_r!}\\
=&\ (1-x_1z)^{-\alpha}\ e^{(x_2+...+x_r)z}\sum_{(n_1),...,(n_r),m_2,...,m_s=0}^{\infty}\frac{((b))_{(n_1+...+n_r)\phi}}{((d))_{(n_1+...+n_r)\delta}}\\
&\times\Omega\left(f((n_1+...+n_r),m_2,...,m_s);m_2,...,m_s\right)(-u_1z)^{(n_1+...+n_r)}\frac{u_2^{m_2}}{m_2!}...\frac{u_s^{m_s}}{m_s!}\\
&\times(\alpha)_{n_1}\ \frac{(x_1/1-x_1z)^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}...\frac{x_r^{n_r}}{n_r!},
\end{aligned}$$

and hence we obtain the desired result. □

Some appropriate choices of $\Omega(m_1,m_2,...,m_s)$ in Theorem 2.1 gives the following four possible cases:

Case I. If we take

$$\begin{aligned}
&\Omega\left(f(m_1,m_2,...,m_s);m_2,...,m_s\right)\\
=&\ \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_s\theta_j^{(s)}}}{\prod_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+...+m_s\psi_j^{(s)}}}\frac{\prod_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}}...\frac{\prod_{j=1}^{B^{(s)}}(b_j^{(s)})_{m_s\phi_j^{(s)}}}{\prod_{j=1}^{D^{(s)}}(d_j^{(s)})_{m_s\delta_j^{(s)}}}
\end{aligned}$$

in Theorem 2.1, we obtain the next result, immediately.

Corollary 2.1. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)F^{A:B+1;B^{(2)};...;B^{(S)}}_{E:D;D^{(2)};...;D^{(S)}}\left(\begin{array}{l}\left[(a):\theta^{(1)},...,\theta^{(s)}\right]: \quad [-n:1],\\ \left[(c):\psi^{(1)},...,\psi^{(s)}\right]:\end{array}\right.$$

$$\left.\begin{array}{l}[(b):\phi];[(b^{(2)}):\phi^{(2)}];...;[(b^{(s)}):\phi^{(s)}];\\ [(d):\delta];[(d^{(2)}):\delta^{(2)}];...;[(d^{(s)}):\delta^{(s)}];\end{array} u_1,...,u_s\right) z^n$$

$$= \quad (1-x_1z)^{-\alpha}e^{(x_2+...+x_r)z}F^{A+B:1;0;...;0;B^{(2)};...;B^{(S)}}_{E+D:0;0;...;0;D^{(2)};...;D^{(S)}}\left(\begin{array}{l}\left[(e):\varphi^{(1)},...,\varphi^{(r+s-1)}\right]:\\ \left[(f):\Theta^{(1)},...,\Theta^{(r+s-1)}\right]:\end{array}\right.$$

$$\begin{array}{l}[\alpha:1];\ -;\ ...;\ -;[(b^{(2)}):\phi^{(2)}];...;[(b^{(s)}):\phi^{(s)}];\\ -\ \ ;\ -;\ ...\ ;-;[(d^{(2)}):\delta^{(2)}];...;[(d^{(s)}):\delta^{(s)}];\end{array}$$

$$\left.\frac{x_1u_1z}{x_1z-1},-x_2u_1z,...,-x_ru_1z,u_2,...,u_s\right),$$

where the coefficients $e_j, f_j, \varphi_j^{(k)}$ *and* $\Theta_j^{(k)}$ *are given by*

$$e_j=\begin{cases} a_j & (1\leqslant j\leqslant A)\\ b_{j-A} & (A<j\leqslant A+B)\end{cases},$$

$$f_j=\begin{cases} c_j & (1\leqslant j\leqslant E)\\ d_{j-E} & (E<j\leqslant E+D)\end{cases},$$

$$\varphi_j^{(k)}=\begin{cases} \theta_j^{(1)} & (1\leqslant j\leqslant A;\ 1\leqslant k\leqslant r)\\ \theta_j^{(k-r+1)} & (1\leqslant j\leqslant A;\ r<k\leqslant r+s-1)\\ \phi_{j-A} & (A<j\leqslant A+B;\ 1\leqslant k\leqslant r)\\ 0 & (A<j\leqslant A+B;\ r<k\leqslant r+s-1)\end{cases},$$

and

$$\Theta_j^{(k)}=\begin{cases} \psi_j^{(1)} & (1\leqslant j\leqslant E;\ 1\leqslant k\leqslant r)\\ \psi_j^{(k-r+1)} & (1\leqslant j\leqslant E;\ r<k\leqslant r+s-1)\\ \delta_{j-E} & (E<j\leqslant E+D;\ 1\leqslant k\leqslant r)\\ 0 & (E<j\leqslant E+D;\ r<k\leqslant r+s-1)\end{cases},$$

respectively.

Case II. Upon setting

$$\Omega\left(f(m_1,m_2,...,m_s);m_2,...,m_s\right)=\frac{(a)_{m_1+...+m_s}(b_2)_{m_2}...(b_s)_{m_s}}{(c_1)_{m_1}...(c_s)_{m_s}}$$

and

$$\phi=\delta=0 \ \ (\text{that is, } \phi_1=...=\phi_B=\delta_1=...=\delta_D=0)$$

in Theorem 2.1, we get the following result.

Corollary 2.2. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x_1,...,x_r) F_A^{(s)}\left[a,-n,b_2,...,b_s;c_1,...,c_s;u_1,u_2,...,u_s\right] z^n$$
$$= (1-x_1z)^{-\alpha} e^{(x_2+...+x_r)z} F^{1:1;0;...;0;1;...;1}_{1:0;0;...;0;1;...;1}\left(\begin{array}{c} [(a):1,...,1]: \quad [\alpha:1];\ -;\ ...;\ -;\ [b_2:1];\ ...;\ [b_s:1]; \\ \left[(c_1):\psi^{(1)},...,\psi^{(r+s-1)}\right]: \quad -;\ -;\ ...;\ -;\ [c_2:1];\ ...;\ [c_s:1]; \end{array} \left(\frac{u_1x_1z}{x_1z-1}\right),(-u_1x_2z),...,(-u_1x_rz),u_2,...,u_s\right),$$

where $F_A^{(s)}$ *denotes the first Lauricella function in* s *variables and the coefficients* $\psi^{(k)}$ *are given by*

$$\psi^{(k)} = \begin{cases} 1 & (1 \leqslant k \leqslant r) \\ 0 & (r < k \leqslant r+s-1 \end{cases}.$$

Case III. If we put

$$\Omega\left(f(m_1,m_2,...,m_s);m_2,...,m_s\right) = \frac{(a_1^{(1)})_{m_2}...(a_1^{(s-1)})_{m_s}(a_2^{(1)})_{m_2}...(a_2^{(s-1)})_{m_s}}{(c)_{m_1+...+m_s}}$$

and

$$B=1, b_1=b, \phi_1=1 \text{ and } \delta=0$$

in Theorem 2.1,we obtain the following corollary.

Corollary 2.3. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x_1,...,x_r) F_B^{(s)}\left[-n,a_1^{(1)},...,a_1^{(s-1)},b,a_2^{(1)},...,a_2^{(s-1)};c;u_1,u_2,...,u_s\right] z^n$$
$$= (1-x_1z)^{-\alpha} e^{(x_2+...+x_r)z} F^{1:1;0;...;0;2;...;2}_{1:0;0;...;0;0;...;0}\left(\begin{array}{c} \left[b:\theta^{(1)},...,\theta^{(r+s-1)}\right]: \quad [\alpha:1];\ -;\ ...;\ -;\ \left[a^{(1)}:1\right];\ ...;\ \left[a^{(s-1)}:1\right]; \\ [(c):1,...,1]: \quad -;\ -;\ ...;\ -;\ -;\ ...;\ -; \end{array} \left(\frac{u_1x_1z}{x_1z-1}\right),(-u_1x_2z),...,(-u_1x_rz),u_2,...,u_s\right),$$

where $F_B^{(s)}$ *denotes the second Lauricella function in* s *variables and the coefficients* $\theta^{(k)}$ *are given by*

$$\theta^{(k)} = \begin{cases} 1 & (1 \leqslant k \leqslant r) \\ 0 & (r < k \leqslant r+s-1 \end{cases}.$$

Case IV. By letting

$$\Omega\left(f(m_1,m_2,...,m_s);m_2,...,m_s\right) = \frac{(a)_{m_1+...+m_s}(b_2)_{m_2}...(b_s)_{m_s}}{(c)_{m_1+...+m_s}}$$

and

$$\phi = \delta = 0,$$

in Theorem 2.1, we have the following result.

COROLLARY 2.4. *The following bilateral generating function holds true:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x_1, ..., x_r) F_D^{(s)}\left[a, -n, b_2, ..., b_s; c; u_1, u_2, ..., u_s\right] z^n \\
= \quad & (1 - x_1 z)^{-\alpha} e^{(x_2 + ... + x_r) z} F^{1:1;0;...;0;1;...;1}_{1:0;0;...;0;0;...;0} \\
&\left(\begin{array}{l} \left[(a) : \theta^{(1)}, ..., \theta^{(r+s-1)}\right] : \quad [\alpha : 1]; \quad -; \quad ...; \quad -; \quad [b_2 : 1]; \quad ...; \quad [b_s : 1]; \\ \left[(c) : \psi^{(1)}, ..., \psi^{(r+s-1)}\right] : \quad -; \quad -; \quad ...; \quad -; \quad -; \quad ...; \quad -; \end{array} \right. \\
&\qquad\qquad \left. \left(\frac{u_1 x_1 z}{x_1 z - 1}\right), (-u_1 x_2 z), ..., (-u_1 x_r z), u_2, ..., u_s \right),
\end{aligned}$$

where $F_D^{(s)}$ denotes the fourth Lauricella function in s variables and

$$\theta^{(1)} = ... = \theta^{(r+s-1)} = \psi^{(1)} = ... = \psi^{(r+s-1)} = 1.$$

3. Further properties of $\Phi_n^{(\alpha)}$ and some applications

In [**13**], we obtained the next theorem. In this section, we derive several families of bilinear and bilateral generating functions for the multivariate polynomials $\Phi_n^{(\alpha)}(x, ..., x_r)$ defined by (1.1) using the similar method considered in (see, [**3, 13**]).

THEOREM 3.1 ([**13**]). *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_s)$ of complex variables $y_1, ..., y_s$ $(s \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_s) z^k \quad (a_k \neq 0,\ \psi \in \mathbb{C}),$$

and

$$\Theta_{n,p}^{\mu,\psi}(x_1, ..., x_r; y_1, ..., y_s; \zeta) := \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha)}(x_1, ..., x_r) \Omega_{\mu+\psi k}(y_1, ..., y_s) \zeta^k$$

where $a_k \neq 0$, $n, p \in \mathbb{N}$. We get the result immediately

(3.1)

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}(x_1, ..., x_r; y_1, ..., y_s; \frac{\eta}{t^p}) t^n = (1 - x_1 t)^{-\alpha} e^{(x_2 + ... + x_r) t} \Lambda_{\mu,\Psi}(y_1, ..., y_s; \eta)$$

provided that each member of (3.1) exists.

Now we discuss some applications of Theorem 3.1.

If we set $s = r$ and $\Omega_{\mu+\psi k}(y_1, ..., y_r) = \Phi_{\mu+\psi k}^{(\alpha)}(y_1, ..., y_r)$ in Theorem 3.1, then we have the following result which provides a class of bilinear generating functions for the multivariable polynomials $\Phi_n^{(\alpha)}(x_1, ..., x_r)$ given by (1.1).

Corollary 3.1. *If*

$$\Lambda_{\mu,\psi}\left[y_1,...,y_r;z\right] \quad : \quad =\sum_{k=0}^{\infty} a_k \Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r) z^k$$
$$(a_k \neq 0,\ \mu,\psi \in \mathbb{C})$$

and

$$\Theta_{n,p}^{\mu,\psi}(x_1,...,x_r;y_1,...,y_r;\zeta)$$
$$: \quad =\sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha)}(x_1,...,x_r)\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)\zeta^k$$
$$(n \in \mathbb{N}_0,\ p \in \mathbb{N})$$

then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}(x_1,...,x_r;y_1,...,y_r;\frac{\eta}{t^p})t^n = (1-x_1t)^{-\alpha} e^{(x_2+...+x_r)t}\Lambda_{\mu,\psi}\left(y_1,...,y_r;\eta\right) \tag{3.2}$$

provided that each member of (3.2) exists.

Remark 3.1. Using (1.2) and taking

$$a_k = 1\ (k\in\mathbb{N}_0),\ \mu = 0 \text{ and } \psi = 1,$$

in Corollary 3.1, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \Phi_{n-pk}^{(\alpha)}(x_1,...,x_r)\Phi_{k}^{(\alpha)}\left(y_1,...,y_r\right)\eta^k t^{n-pk}$$
$$= (1-x_1t)^{-\alpha} e^{(x_2+...+x_r)t}\left(1-y_1\eta\right)^{-\alpha} e^{(y_2+...+y_r)\eta}$$
$$\left(\ |t| < |x_1|^{-1} \text{ and } |\eta| < |y_1|^{-1}\right).$$

If we choose

$$s = r \quad\text{and}\quad \Omega_{\mu+\psi k}(y_1,...,y_r) = h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$$

in Theorem 3.1 , where $h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ denotes the multivariable Lagrange-Hermite polynomials introduced in [**6, 7**], then we obtain the following result which gives a class of bilateral generating functions for the multivariable Lagrange-Hermite polynomials $h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}$ and the multivariable polynomials $\Phi_n^{(\alpha)}$.

Corollary 3.2. *If*

$$\Lambda_{\mu,\psi}\left[y_1,...,y_r;z\right] \quad : \quad =\sum_{k=0}^{\infty} a_k h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r) z^k$$
$$(a_k \neq 0,\ \mu,\psi \in \mathbb{C})$$

and

$$
\begin{aligned}
& \Theta_{n,p}^{\mu,\psi}(x_1,...,x_r;y_1,...,y_r;\zeta) \\
: \quad & = \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha)}(x_1,...,x_r) h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\zeta^k \\
& (n \in \mathbb{N}_0, \ p \in \mathbb{N}),
\end{aligned}
$$

then

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}(x_1,...,x_r;y_1,...,y_r;\frac{\eta}{t^p})t^n \\
= \quad & (1-x_1t)^{-\alpha} \, e^{(x_2+...+x_r)t} \Lambda_{\mu,\psi}(y_1,...,y_r;\eta)
\end{aligned}
\tag{3.3}
$$

provided that each member of (3.3) exists.

Remark 3.2. Using the multivariable Lagrange-Hermite polynomials [**6**]

$$h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$$

and setting

$$a_k = 1 \ (k \in \mathbb{N}_0), \ \mu = 0 \text{ and } \psi = 1,$$

in Corollary 3.2, we find that

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} \Phi_{n-pk}^{(\alpha)}(x_1,...,x_r) h_k^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\,\eta^k t^{n-pk} \\
= \quad & (1-x_1t)^{-\alpha} \, e^{(x_2+...+x_r)t} \prod_{j=1}^{r} \left\{ (1-y_j\eta^j)^{-\alpha_j} \right\} \\
& \left(\ |t| < |x_1|^{-1} \text{ and } |\eta| < \min_{j \in \{1,...,r\}} \left\{ |y_j|^{-1/j} \right\} \right).
\end{aligned}
$$

exists.

It is known from [**13**] that the recurrence relation

$$\sum_{j=1}^{r} x_j \Phi_{n-1}^{(\alpha)}(x_1,...,x_r) = n\Phi_n^{(\alpha)}(x_1,...,x_r) \tag{3.4}$$

holds. Finally, using (3.4), we find a partial differential equation for the product of two multivariable polynomials $\Phi_n^{(\alpha)}(x_1,...,x_r)$ given explicitly by (1.1).

Lee [**8**] gave a method for finding partial differential equations satisfied by the product of two polynomials in one variable (see also [**2, 10**]). In [**5, 9**], Erkuş-Duman et. al. extended this method to the multivariable case as follows:

Let us define two linear differential operators by

$$L = \sum_{i=1}^{r} a_i(x_i)\frac{\partial}{\partial x_i} \quad \text{and} \quad N = \sum_{j=1}^{s} b_j(y_j)\frac{\partial}{\partial y_j}.$$

Theorem 3.2 ([**5**]). *Let* $\{P_n(x_1,...,x_r)\}|_{n=0}^{\infty}$ *and* $\{Q_n(y_1,...,y_s)\}|_{n=0}^{\infty}$ *be polynomials satisfying*

$$L\left[P_n\right]=\lambda_n P_n=nP_n$$

and

$$N\left[Q_n\right]=\eta_n Q_n=nQ_n.$$

Then the product polynomial $\{S_n\}|_{n=0}^{\infty}=\{P_{n-k}(x_1,...,x_r)Q_k(y_1,...,y_s)\}|_{k=0,n=0}^{n,\infty}$ *holds the following partial differential equation:*

$$L\left[\omega\right]+N\left[\omega\right]=n\omega. \tag{3.5}$$

Now, we apply this theorem to the multivariable polynomials $\Phi_n^{(\alpha)}(x_1,...,x_r)$.

Corollary 3.3. *Let* $\{P_n(x_1,...,x_r)\}|_{n=0}^{\infty}$ *and* $\{Q_n(y_1,...,y_s)\}|_{n=0}^{\infty}$ *be the polynomials satisfying* $\Phi_n^{(\alpha)}$ *polynomials so that*

$$\{S_n\}|_{n=0}^{\infty}=\left\{\Phi_{n-k}^{(\alpha)}(x_1,...,x_r)\Phi_k^{(\alpha)}(y_1,...,y_s)\right\}\Big|_{k=0,n=0}^{n,\infty}.$$

Then, from(3.4),

$$L=\sum_{i=1}^{r}x_i\frac{\partial}{\partial x_i}\quad \text{and } N=\sum_{j=1}^{s}y_j\frac{\partial}{\partial y_j}.$$

Hence, the partial differential equation (3.5) becomes

$$\sum_{i=1}^{r}x_i\frac{\partial w}{\partial x_i}+\sum_{j=1}^{s}y_j\frac{\partial w}{\partial y_j}=n\omega.$$

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Faculty of Art and Science Department of Mathematics, Dzce University, Düzce, Turkey

E-mail address: nejlaozmen06@gmail.com, nejlaozmen@duzce.edu.tr