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SOME NEW RESULTS ON THE ORTHODOX, STRONGLY π-INVERSE AND π-REGULARITY OF SOME MONOIDS

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Abstract. In this article we give some results about the regularity, orthodox and strongly π-inverse of the Schützenberger and crossed products for monoids.

1. Introduction

The Schützenberger, semidirect and crossed product of semigroups (monoids, group) have a venerable history in semigroup theory. They have played an importante role in many algebraic properties. In this direction the authors give some conditions for regularity, strongly π-inverse and orthodox properties of semidirect products of monoids in [**8, 9, 10**]. Also, by using smilar method as in these papers, after defining a new version of the Schützenberger product for two monoids, the author gave some results about the regularity of this new version in [**3**]. Then in [**7**] the authors gave the necessary and sufficient conditions for this new version to be strongly π-inverse. Morever in [**5**] the authors defined a new monoid constructions under crossed products and gave some results about its regularity.

By using similar methods as in this above papers, we give some results on the regularity of crossed products for monoids in Section 2. In Section 3, we work on the strongly π-inverse of the Schützenberger product for monoids and we examined the orthodox properties of the new version of the Schützenberger product which is defined in [**3**].

The reader is referred to [**5, 4**] for more details.

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2. Crossed Product

A crossed system of monoids is a quadruple (A, B, α, β), where A and B are two monoids, and $f : B \times B \longrightarrow A$ and $\alpha : B \to End(A)$ ($End(A)$ denotes the collection of endomorphism of A) are two maps such that the following conditions hold,

$$\alpha_{b_1}\left(\alpha_{b_2}(a)\right) f\left(b_1, b_2\right) = f\left(b_1, b_2\right) \alpha_{b_1 b_2}(a) \tag{2.1}$$

$$f\left(b_1, b_2\right) f\left(b_1 b_2, b_3\right) = \alpha_{b_1}\left(f\left(b_2, b_3\right)\right) f\left(b_1, b_2 b_3\right) \tag{2.2}$$

for all $b_1, b_2, b_3 \in B, a \in A$. The crossed system (A, B, α, f) is called normalized if $f(1_B, 1_B) = 1_A$. The map $\alpha : B \to End(A)$ is called weak action and $f : B \times B \to A$ is called an $\alpha-$ cocycle.

If (A, B, α, f) is a normalized crossed system then we have

$$f(1_B, b) = f(b, 1_B) = 1_A \text{ and } \alpha_{1_B}(a) = a.$$

Let A and B be monoids, $f : B \times B \to A$ and $\alpha : B \to End(A)$ two maps. Let $A\#_{\alpha}^{f}B := A \times B$ as a set with a binary operation defined by the formula :

$$\left(a_1, b_1\right)\left(a_2, b_2\right) := \left(a_1\left(\alpha_{b_1}\left(a_2\right)\right) f\left(b_1, b_2\right), b_1 b_2\right)$$

for all $b_1, b_2 \in B, a_1, a_2 \in A$. Then $\left(A\#_{\alpha}^{f}B, .\right)$ is a monoid with unit $1_{A\#_{\alpha}^{f}B} = (1_A, 1_B)$ if and only if (A, B, α, f) is a normalized crossed system. In this case the monoid $A\#_{\alpha}^{f}B$ is called the crossed product of A and B associated to the crossed system (A, B, α, f). The reader is referred to [**1**] and [**2**] for more details.

For an element a in a monoid M. Let us take a^{-1} for the set of inverses of a in M, that is, $a^{-1} = \{b \in B : aba = a \text{ and } bab = b\}$. Hence M is regular if and only if, for all $a \in M$, that is, a^{-1} is not equal to the empty set.

Now we can give following theorem as the one of the main results of this paper.

Theorem 2.1. *Let A and B be any monoids. Also for $b \in d^{-1}$, let us have*

$$a \in A\alpha_b(\alpha_d(a))$$

such that $f(b, d)\alpha_{bd}(a)f(bd, b) = \alpha_b(\alpha_d(a))$ and $\alpha_d(a) f(d, b) f(db, d) = \alpha_d(a)$. Then the product $A\#_{\alpha}^{f}B$ is regular if and only if A and B regular.

Proof. Let us suppose that $A\#_{\alpha}^{f}B$ is regular, then there exist $(c, d) \in A\#_{\alpha}^{f}B$ for $(a, 1_B) \in A\#_{\alpha}^{f}B$ such that

$$\begin{aligned}
(a, 1_B) = (a, 1_B)(c, d)(a, 1_B) &= (a\alpha_{1_B}(c) f(1_A, d), 1_B d)(a, 1_B) \\
&= (ac, d)(a, 1_B) = (ac\alpha_d(a) f(d, 1_B), d1_B) \\
&= (ac\alpha_d(a), d)
\end{aligned}$$

and

$$\begin{aligned}
(c, d) = (c, d)(a, 1_B)(c, d) &= (c\alpha_d(a) f(d, 1_B), d1_B)(c, d) \\
&= (c\alpha_d(a), d)(c, d) = (c\alpha_d(a)\alpha_d(c) f(d, d), d.d) \\
&= (c\alpha_d(ac) f(d, d), d.d).
\end{aligned}$$

Thus we have $d = 1_B$. This give us that $a = aca$ and $c = cac$. Hence A is regular. By using the similar argument, for $(1_A, b) \in A\#_\alpha^f B$ there exist (c, d) such that

$$\begin{aligned} (1_A, b) = (1_A, b)(c, d)(1_A, b) &= (\alpha_b(c) f(b, d), bd)(1_A, b) \\ &= (\alpha_b(c) f(b, d) \alpha_{bd}(1_A) f(bd, b), bdb) \\ &= (\alpha_b(c) f(b, d) f(bd, b), bdb) \end{aligned}$$

and

$$\begin{aligned} (c, d) = (c, d)(1_A, b)(c, d) &= (c\alpha_d(1_A) f(d, b), db)(c, d) \\ &= (cf(d, b) \alpha_{db}(c) f(db, d), dbd) \end{aligned}$$

Thus we have $b = bdb$ and $d = dbd$ which give us B is regular.

Conversely suppose that A and B are regular. Then by assumption we have for $b \in d^{-1}, a \in A\alpha_b(\alpha_d(a))$ such that $f(b,d)\alpha_{bd}(a)f(bd,b) = \alpha_b(\alpha_d(a))$ and $\alpha_d(a) f(d, b) f(db, d) = \alpha_d(a)$. Then there are some $u \in A, a = u\alpha_b(\alpha_d(a))$ and also for $v \in a^{-1}$, there exist $c = \alpha_d(v)$. So we get

$$\begin{aligned} a\alpha_b(c) f(b, d) \alpha_{bd}(a) f(bd, b) &= u\alpha_b(\alpha_d(a))\alpha_b(\alpha_d(v)) f(b, d) \alpha_{bd}(a) f(bd, b) \\ &= u\alpha_b(\alpha_d(a))\alpha_b(\alpha_d(v)) \alpha_b(\alpha_d(a))f(b, d)f(bd, b) \\ &= u\alpha_b(\alpha_d(a)\alpha_d(v)\alpha_d(a))f(b, d)f(bd, b) \\ &= u\alpha_b(\alpha_d(ava))f(b, d)f(bd, b) \\ &= u\alpha_b(\alpha_d(a))f(b, d)f(bd, b) \\ &= uf(b, d)\alpha_{bd}(a)f(bd, b) \\ &= u\alpha_b(\alpha_d(a)) = a \end{aligned}$$

and

$$\begin{aligned} c\alpha_d(a) f(d, b) \alpha_{db}(c) f(db, d) &= \alpha_d(v) \alpha_d(a) f(d, b) \alpha_{db}(\alpha_d(v)) f(db, d) \\ &= \alpha_d(v) \alpha_d(a) f(d, b) f(db, d) \alpha_{dbd}(v) \\ &= \alpha_d(v) \alpha_d(a) f(d, b) f(db, d) \alpha_d(v) \\ &= \alpha_d(v) \alpha_d(a) \alpha_d(v) \\ &= \alpha_d(vav) \\ &= \alpha_d(v) = c \end{aligned}$$

Consequantly, for every $(a, b) \in A\#_\alpha^f B$, there exist $(c, d) \in A\#_\alpha^f B$, such that;

$$\begin{aligned} (a, b)(c, d)(a, b) &= (a\alpha_b(c) f(b, d) \alpha_{bd}(a) f(bd, b), bdb) = (a, b) \\ (c, d)(a, b)(c, d) &= (c\alpha_d(a) f(d, b) \alpha_{db}(c) f(db, d), bdb) = (c, d). \end{aligned}$$

Hence the result. □

Let us think the monoids A , B and α, f given in Section 2. It is known that if we take f trivial map, then the crossed product becomes semidirect product. If we take α trivial action, then $Im(f) \subseteq Z(A)$ and $f : B \times B \to Z(A)$ is a $2 - cocycle$, where $Z(A)$ is central of A. The crossed product $A\#_\alpha^f B$ associated to this crossed system will be denoted by $A \times^f B$. It is called the twisted product of A and B

associated to the $2-cocycle \quad f : B \times B \to Z(A)$ in [7]. In fact, the multiplication of a twisted product of monoids $A \times^f B$ is given by the formula:

$$(a_1, b_1)(a_2, b_2) := (a_1 a_2 f(b_1, b_2), b_1 b_2)$$

for all $a_1, a_2 \in A, b_1, b_2 \in B$.

Then we can give the following corollaries:

Corollary 2.1. *For $b \in d^{-1}$, let us have $a \in A$ such that; $f(b,d) a f(bd, b) = a$ and $a f(d, b) f(db, d) = a$. Then the twisted product $A \times^f B$ is regular if and only if A and B regular.*

The following corollary can be found also in [**8**].

Corollary 2.2. *For $b \in d^{-1}$, let us have $a \in A\alpha_{bd}(a)$. Then the semidirect product $A \times_\alpha B$ is regular if and only if A and B regular.*

3. The Schützenberger Product

Let A and B be monoids. For a subset P of $A \times B$ and $a \in A$, $b \in B$, we define $Pb = \{(c, db) : (c, d) \in P\}$ and $aP = \{(ac, d) : (c, d) \in P\}$. Then the Schützenberger product of A and B, denoted by $A \diamondsuit B$, is the set $A \times \wp(A \times B) \times B$ with the multiplication

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 a_2, P_1 b_2 \cup a_1 P_2, b_1 b_2).$$

It is known that $A \diamondsuit B$ is a monoid with the identity $(1_A, \emptyset, 1_B)$, [**6**].

Let $E(S)$ and $RegS$ be the set of idempotent and regular elements, respectively, for a semigroup S. Here, S is called π- regular if, for every s in S, there is an $m \in N$ such that $s^m \in RegS$. If S is π- regular and the set $E(S)$ is a commutative subsemigroup of S, then S is called a strongly π- inverse semigroup.

In the following theorem we aim to give necessary and sufficient conditions for $A \diamondsuit B$ to be strongly π -inverse monoid for the given monoids A and B.

Theorem 3.1. *Let A and B be two monoids. Then $A \diamondsuit B$ is strongly $\pi-$ inverse monoid if and only if for every $(a, P, b) \in A \diamondsuit B$, there exist $m \in N, x \in A, y \in B$ such that:*

(i) *$a^m x = 1_A$ and $y b^m = 1_B$*
(ii) *$E(A)$ and $E(B)$ are commutative and $P_1 f_2 \cup e_1 P_2 = P_2 f_1 \cup e_2 P_1$ for every (e_1, P_1, f_1), $(e_2, P_2, f_2) \in E(A \diamondsuit B)$.*

Proof. Let us suppose that $A \diamondsuit B$ is strongly $\pi-$ inverse monoid. Thus for every $(a, \{(1_A, 1_B)\}, b) \in A \diamondsuit B$, there exist $m \in N$ and (x, P, y) such that;

$$(a, \{(1_A, 1_B)\}, b)^m = (a, \{(1_A, 1_B)\}, b)^m (x, P, y)(a, \{(1_A, 1_B)\}, b)^m.$$

Here we have

$$\begin{aligned}
(a,(1_A,1_B),b)^m &= (a^m,\{(1_A,1_B)\}b^{m-1}\cup a\{(1_A,1_B)\}b^{m-2}\\
&\cup \ldots\cup a^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}\\
&\cup a^{m-1}\{(1_A,1_B)\}b^{m-m},b^m)(x,P,y)(a^m,\{(1_A,1_B)\}b^{m-1}\\
&\cup a\{(1_A,1_B)\}b^{m-2}\cup ..\cup a^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}\\
&\cup a^{m-1}\{(1_A,1_B)\},b^m)\\
&= (a^m x,\{(1_A,1_B)\}b^{m-1}y\cup a\{(1_A,1_B)\}b^{m-2}y\\
&\cup \ldots\cup a^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}y\cup a^{m-1}\{(1_A,1_B)\}y\\
&\cup a^m P,b^m y)(a^m,\{(1_A,1_B)\}b^{m-1}\cup a\{(1_A,1_B)\}b^{m-2}\\
&\cup \ldots\cup a^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}\cup a^{m-1}\{(1_A,1_B)\},b^m)\\
&= (a^m x a^m,\{(1_A,1_B)\}b^{m-1}yb^m\cup a\{(1_A,1_B)\}b^{m-2}yb^m\\
&\cup \ldots\cup a^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}yb^m\cup a^{m-1}\{(1_A,1_B)\}yb^m\\
&\cup a^m Pb^m\cup a^m x\{(1_A,1_B)\}b^{m-1}\cup a^m xa\{(1_A,1_B)\}b^{m-2}\\
&\cup \ldots\cup a^m xa^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}\\
&\cup a^m xa^{m-1}\{(1_A,1_B)\},b^m yb^m)
\end{aligned}$$

So we get that

$$\begin{aligned}
\{(1_A,1_B)\}b^{m-1}yb^m &\cup a\{(1_A,1_B)\}b^{m-2}yb^m\cup\ldots\\
&\cup a^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}yb^m\cup a^{m-1}\{(1_A,1_B)\}yb^m\\
&\cup a^m Pb^m\cup a^m x\{(1_A,1_B)\}b^{m-1}\\
&\cup a^m xa\{(1_A,1_B)\}b^{m-2}\cup\ldots\\
&\cup a^m xa^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}\cup a^m xa^{m-1}\{(1_A,1_B)\}\\
&= \{(1_A,1_B)\}b^{m-1}\cup a\{(1_A,1_B)\}b^{m-2}\\
&\cup \ldots\cup a^{m-2}\{(1_A,1_B)\}b^{m-(m-1)}\cup a^{m-1}\{(1_A,1_B)\}.
\end{aligned}$$

This gives us

$$\begin{aligned}
\left\{\left(1_A,b^{m-1}yb^m\right)\right\} &\cup \left\{\left(a,b^{m-2}yb^m\right)\right\}\\
&\cup \ldots\cup\left\{\left(a^{m-2},b^{m-(m-1)}yb^m\right)\right\}\cup\left\{\left(a^{m-1},yb^m\right)\right\}\\
&\cup a^m Pb^m\cup\left\{\left(a^m x,b^{m-1}\right)\right\}\cup\left\{\left(a^m xa,b^{m-2}\right)\right\}\\
&\cup \ldots\cup\left\{\left(a^m xa^{m-2},b^{m-(m-1)}\right)\right\}\cup\left\{\left(a^m xa^{m-1},1_B\right)\right\}\\
&= \left\{\left(1_A,b^{m-1}\right)\right\}\cup\left\{\left(a,b^{m-2}\right)\right\}\\
&\cup \ldots\cup\left\{\left(a^{m-2},b^{m-(m-1)}\right)\right\}\cup\left\{\left(a^{m-1},1_B\right)\right\}.
\end{aligned}$$

Thus in order to get this above equation, for every $(a,P,b)\in A\diamondsuit B$, we must have $a^m x=1_A$ and $yb^m=1_B$. This implies that (i) must hold.

Now let us take $(e_1,P_1,f_1),(e_2,P_2,f_2)\in E(A\diamondsuit B)$. Since $A\diamondsuit B$ is a Strongly $\pi-$ inverse monoid, we have $(e_1,P_1,f_1)(e_2,P_2,f_2)=(e_2,P_2,f_2)(e_1,P_1,f_1)$ and so $(e_1e_2, P_1f_2 \cup e_1P_2, f_1f_2) = (e_2e_1, P_2f_1 \cup e_2P_1, f_2f_1)$. Therefore we obtain that $P_1f_2 \cup e_1P_2 = P_2f_1 \cup e_2P_1$. Now let us think that $e_1, e_2 \in E(A)$. Since $A\diamondsuit B$ is strongly π- inverse monoids, we say that $E(A\diamondsuit B)$ is commutative. Then since we have $(e_1, \{1_A, 1_B\}, 1_B), (e_2, \{1_A, 1_B\}, 1_B) \in E(A\diamondsuit B)$, we say that $(e_1, \{1_A, 1_B\}, 1_B)(e_2, \{1_A, 1_B\}, 1_B) = (e_2, \{1_A, 1_B\}, 1_B)(e_1, \{1_A, 1_B\}, 1_B)$. So we get that $e_1e_2 = e_2e_1$. This says that $E(A)$ is commutative. Smilarly, we get that $E(B)$ is commutative. Which this implies us that (ii) must hold.

Conversely suppose that there exist $m \in N$ and $x \in A, y \in B$ such that the monoids A and B both satisfies conditions (i) and (ii). For each $(a, P, b) \in A\diamondsuit B$ and $n = 1, 2, ..., m-1$, let us think $\big(x, xa^nPb^{m-(n+1)}y, y\big) \in A\diamondsuit B$. Also let us say $F = (a, P, b)^m(x, xa^nPb^{m-(n+1)}y, y)(a, P, b)^m$. Then we have

$$\begin{aligned} F = (a^m, Pb^{m-1} &\cup aPb^{m-2} \cup a^2Pb^{m-3} \cup ... \cup a^{m-2}Pb^{m-(m-1)} \\ &\cup a^{m-1}Pb^{m-m}, b^m)(x, xa^nPb^{m-(n+1)}y, y)(a^m, Pb^{m-1} \\ &\cup aPb^{m-2} \cup a^2Pb^{m-3} \cup ... \\ &\cup a^{m-2}Pb^{m-(m-1)} \cup a^{m-1}Pb^{m-m}, b^m) = (a^mx, Pb^{m-1}y \\ &\cup aPb^{m-2}y \cup a^2Pb^{m-3}y \cup ... \cup a^{m-2}Pb^{m-(m-1)}y \\ &\cup a^{m-1}Py \cup a^mxa^nPb^{m-(n+1)}y, b^my)(a^m, Pb^{m-1} \cup aPb^{m-2} \\ &\cup a^2Pb^{m-3} \cup ... \cup a^{m-2}Pb^{m-(m-1)} \cup a^{m-1}Pb^{m-m}, b^m) \end{aligned}$$

So we get that

$$\begin{aligned} F = (a^mxa^m, Pb^{m-1}yb^m &\cup aPb^{m-2}yb^m \\ &\cup a^2Pb^{m-3}yb^m \cup ... \cup a^{m-2}Pb^{m-(m-1)}yb^m \\ &\cup a^{m-1}Pyb^m \cup a^mxa^nPb^{m-(n+1)}yb^m \cup a^mxPb^{m-1} \\ &\cup a^mxaPb^{m-2} \cup a^mxa^2Pb^{m-3} \cup ... \\ &\cup a^mxa^{m-2}Pb^{m-(m-1)} \cup a^mxa^{m-1}Pb^{m-m}, b^myb^m) \end{aligned}$$

From (i) by taking $a^mx = 1_A$ and $yb^m = 1_B$ then we get that, $a^mxa^m = a^m$ and $b^myb^m = b^m$ and

$$\begin{aligned} Pb^{m-1}yb^m \cup aPb^{m-2}yb^m &\cup a^2Pb^{m-3}yb^m \cup ... \cup a^{m-2}Pb^{m-(m-1)}yb^m \\ &\cup a^{m-1}Pyb^m \cup a^mxa^nPb^{m-(n+1)}yb^m \cup a^mxPb^{m-1} \\ &\cup a^mxaPb^{m-2} \cup a^mxa^2Pb^{m-3} \cup ... \\ &\cup a^mxa^{m-2}Pb^{m-(m-1)} \cup a^mxa^{m-1}Pb^{m-m} \\ &= Pb^{m-1} \cup aPb^{m-2} \cup a^2Pb^{m-3} \\ &\cup ... \cup a^{m-2}Pb^{m-(m-1)} \cup a^{m-1}P \cup a^nPb^{m-(n+1)} \end{aligned}$$

So we get that $a^nPb^{m-(n+1)}$ is equal to aPb^{m-2}, if $n = 1$ or a^2Pb^{m-3} if $n = 2$ or a^3Pb^{m-4} if $n = 3$ or $\ldots$ or $a^{m-1}Pb^{m-m} = a^{m-1}P$ if $n = m-1$. Then we have

$$\begin{aligned} Pb^{m-1} &\cup aPb^{m-2} \cup a^2Pb^{m-3} \cup ... \cup a^{m-2}Pb^{m-(m-1)} \cup a^{m-1}P \cup a^nPb^{m-(n+1)} \\ &= Pb^{m-1} \cup aPb^{m-2} \cup a^2Pb^{m-3} \cup ... \\ &\cup a^{m-1}Pb^{m-(m-2)} \cup a^mPb^{m-(m-1)} \cup Pb^{m-1}. \end{aligned}$$

Thus we deduced that

$$(a,P,b)^m\left(x,xa^nPb^{m-(n+1)}y,y\right)(a,P,b)^m=(a,P,b)^m$$

where $n=1,2,...,m-1$. This gives the $\pi-$ regularity of $A\diamondsuit B$.

Now we need to show that $E\left(A\diamondsuit B\right)$ is commutative. For $(e_1,P_1,f_1),(e_2,P_2,f_2)\in E\left(A\diamondsuit B\right)$, we then have

$$(e_1,P_1,f_1)^2=(e_1,P_1,f_1) \text{ and } (e_2,P_2,f_2)^2=(e_2,P_2,f_2).$$

So we get $\left(e_1^2,P_1f_1\cup e_1P_1,f_1^2\right)=(e_1,P_1,f_1)$ and $\left(e_2^2,P_2f\cup e_2P_2,f_2^2\right)=(e_2,P_2,f_2)$. We get $e_1^2=e_1,f_1^2=f_1,e_2^2=e_2$ and $f_2^2=f_2$. This means that $e_1,e_2\in E\left(A\right)$ and $f_1,f_2\in E\left(B\right)$. By using the condition ii), we get that

$$\begin{aligned}(e_1,P_1,f_1)(e_2,P_2,f_2) &= (e_1e_2,P_1f_2\cup e_1P_2,f_1f_2)\\ &= (e_2e_1,P_2f_1\cup e_2P_1,f_2f_1)\\ &= (e_2,P_2,f_2)(e_1,P_1,f_1)\end{aligned}$$

Hence the result. □

Now by considering above theorem, we can give the following corollary which gives necessary and sufficient conditions for $A\diamondsuit B$ to be an $\pi-$ regular monoid.

COROLLARY 3.1. *Let A and B be two monoids. The product $A\diamondsuit B$ is π - regular monoid if and only if for every $(a,P,b)\in A\diamondsuit B$, there exist $m\in N,x\in A$ and $y\in B$ such that $a^mx=1_A$ and $yb^m=1_B$.*

Let A and B be two monoids and α is a monoid homomorphism from B to $End(A)$ such that, for every $a\in A$, $b_1,b_2\in B$,

$$\alpha_{b_1b_2}(a)=\alpha_{b_1}(\alpha_{b_2}(a)).$$

For a subset P of $A\times B$ and $a\in A$, $b\in B$, let us define

$$Pb=\{(a,db):(a,d)\in P\}$$

In [**3**], the author defined a new monoid construction which relationship between semidirect and the schützenberger product, denoted by $A\diamondsuit_{sv}B$, is the set $A\times\wp(A\times B)\times B$ with the multiplication

$$(a_1,P_1,b_1)(a_2,P_2,b_2)=(a_1\alpha_{b_1}(a_2),P_1b_2\cup P_2,b_1b_2).$$

It is known that a regular semigroup S is orthodox if $E(S)$ forms a subsemigroup of S. Thus we can give the fallowing theorem which gives us necessary and sufficient conditions for $A\diamondsuit_{sv}B$ to be orthodox monoid.

THEOREM 3.2. *Let A and B be two monoids. Then the monoid $A\diamondsuit_{sv}B$ is orthodox if and only if A is orthodox and B is group.*

PROOF. Let us suppose that $A\diamondsuit_{sv}B$ is orthodox. Then $A\diamondsuit_{sv}B$ is regular. Thus, for $(a,\{(1_A,1_B)\},1_B)\in A\diamondsuit_{sv}B$, there exist (x,P,y) such that

$$\begin{aligned}(a,\{(1_A,1_B)\},1_B) &= (a,\{(1_A,1_B)\},1_B)(x,P,y)\left(a,\{(1_A,1_B)\},1_B\right)\\ &= \left(ax,\{(1_A,1_B)\}\,y\cup P,y\right)\left(a,\{(1_A,1_B)\},1_B\right)\\ &= \left(ax\alpha_y\left(a\right),\{(1_A,1_B)\}\,y\cup P\cup\{(1_A,1_B)\},y\right)\end{aligned}$$

and

$$\begin{aligned}(x,P,y) &= (x,P,y)\left(a,\{(1_A,1_B)\},1_B\right)(x,P,y)\\ &= \left(x\alpha_y\left(a\right),P\cup\{(1_A,1_B)\},y\right)(x,P,y)\\ &= \left(x\alpha_y\left(a\right)\alpha_y\left(x\right),Py\cup\{(1_A,1_B)\}\,y\cup P,y^2\right).\end{aligned}$$

Thus we have $y=1_B$. This gives that $a=axa$ and $x=xax$. Hence A is regular. By using the similar argument, for $(1_A,\{(1_A,1_B)\},b)\in A\diamondsuit_{sv}B$, there exists (x,P,y) such that

$$\begin{aligned}(1_A,\{(1_A,1_B)\},b) &= (1_A,\{(1_A,1_B)\},b)(x,P,y)(1_A,\{(1_A,1_B)\},b)\\ &= \left(\alpha_b\left(x\right),\{(1_A,1_B)\}\,y\cup P,by\right)(1_A,\{(1_A,1_B)\},b)\\ &= \left(\alpha_b\left(x\right),\{(1_A,1_B)\}\,yb\cup Pb\cup\{(1_A,1_B)\},byb\right)\end{aligned}$$

and

$$\begin{aligned}(x,P,y) &= (x,P,y)(1_A,\{(1_A,1_B)\},b)(x,P,y)\\ &= \left(x,Pb\cup\{(1_A,1_B)\},yb\right)(x,P,y)\\ &= \left(x\alpha_{yb}\left(x\right),Pby\cup\{(1_A,1_B)\}\,y\cup P,yby\right).\end{aligned}$$

Here since we have

$$\begin{aligned}\{(1_A,1_B)\} &= \{(1_A,1_B)\}\,yb\cup Pb\cup\{(1_A,1_B)\}\\ P &= Pby\cup\{(1_A,1_B)\}\,y\cup P\end{aligned}$$

and in partical $(1_A,yb)=(1_A,1_B)$ and $Pby=P$, we get $yb=by=1_B$. This says us that B is group.

Now we need to show that A is orthodox. Let $a_1,a_2\in E\left(A\right)$. Then it is clear that $(a_1,\emptyset,1_B),(a_2,\emptyset,1_B)\in E\left(A\diamondsuit_{sv}B\right)$. In fact,

$$(a_1,\emptyset,1_B)^2=(a_1,\emptyset,1_B)(a_1,\emptyset,1_B)=(a_1a_1,\emptyset,1_B)=(a_1,\emptyset,1_B)$$

and

$$(a_2,\emptyset,1_B)^2=(a_2,\emptyset,1_B)(a_2,\emptyset,1_B)=(a_2a_2,\emptyset,1_B)=(a_2,\emptyset,1_B).$$

If $A\diamondsuit_{sv}B$ is orthodox, then

$$\left((a_1a_2)^2,\emptyset,1_B\right)=\left((a_1,\emptyset,1_B)(a_2,\emptyset,1_B)\right)^2=(a_1,\emptyset,1_B)(a_2,\emptyset,1_B)=(a_1a_2,\emptyset,1_B)$$

so that $(a_1a_2)^2=a_1a_2$. Consequantly A is orthodox monoids.

Conversely let us suppose that A is orthodox and B is group. Let us take $(a,P,b)\in A\diamondsuit_{sv}B$. Because of B is group, then there exist $y\in B$ such that $yb=by=1_B$. Since A is orthodox also A is regular. Since A is regular we can take $c=\alpha_y\left(v\right)$ for some $v\in a^{-1}$. Let us consider the following :

$a\alpha_b(c)\,\alpha_{by}(a) = a\alpha_b(\alpha_y(v))\,a = a\alpha_{by}(v)\,a = ava = a$

and

$c\alpha_y(a)\,\alpha_{yb}(c) = c\alpha_y(a)\,c = \alpha_y(v)\,\alpha_y(a)\,\alpha_y(v) = \alpha_y(vav) = \alpha_y(v) = c.$

Also, by choosing $P_2 = P_1 y \subseteq A \times B$, where $P_1 \subseteq A \times B$, we get $P_1 yb \cup P_2 b \cup P_1 = P_1 \cup P_1 yb \cup P_1 = P_1 \cup P_1 \cup P_1 = P_1$ and $P_2 by \cup P_1 y \cup P_2 = P_1 y \cup P_1 y \cup P_1 y = P_1 y = P_2$

Consequently, for every $(a, P_1, b) \in A\diamondsuit_{sv}B$, there exists $(c, P_2, y) \in A\diamondsuit_{sv}B$, such that

$$\begin{aligned}(a, P_1, b)(c, P_2, y)(a, P_1, b) &= (a\alpha_b(c)\,\alpha_{by}(a), P_1 yb \cup P_2 b \cup P_1, byb) \\ &= (a, P_1, b)(c, P_2, y)(a, P_1, b)(c, P_2, y) \\ &= (c\alpha_y(a)\,\alpha_{yb}(c), P_2 by \cup P_1 y \cup P_2, yby) = (c, P_2, y)\end{aligned}$$

where $P_2 = P_1 y, by = yb = 1_B$ and $c = \alpha_y(v)$ for some $v \in a^{-1}$ this gives us $A\diamondsuit_{sv}B$ is regular.

Now we need to show that $A\diamondsuit_{sv}B$ is orthodox. Let us take $(a, P_1, b), (c, P_2, y) \in E(A\diamondsuit_{sv}B)$ .By above the second part of proof we can take again the same argument, which is $c = \alpha_y(v)$ for some $v \in a^{-1}$ and $by = yb = 1_B$. Let us consider the following:

$$a\alpha_b(c) = a\alpha_b(\alpha_y(v)) = a\alpha_{by}(v) = av.$$

so

$$\begin{aligned}(a\alpha_b(c), P_1 y \cup P_2, by)^2 &= (av, P_1 y \cup P_2, 1_B)^2 \\ &= (av, P_1 y \cup P_2, 1_B)(av, P_1 y \cup P_2, 1_B) \\ &= (av\alpha_{1_B}(av), P_1 y \cup P_2 \cup P_1 y \cup P_2, 1_B 1_B) \\ &= (av, P_1 y \cup P, 1_B) \\ &= (a\alpha_b(c), P_1 y \cup P_2, by).\end{aligned}$$

That is, $(a\alpha_b(c), P_1 y \cup P_2, by) = (a, P_1, b)(c, P_2, y) \in E(A\diamondsuit_{sv}B)$,which is means that $A\diamondsuit_{sv}B$ is orthodox. Hence the result. □

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