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UNIQUE COMMON FIXED POINTS IN METRIC AND COMPACT METRIC SPACES

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Abstract. The aim of the present paper is to obtain common fixed point theorems under strict contractive inequalities using a few conditions. Our results improve the results of Popa [**14**], Aliouche [**1**], Imdad and Javid [**6**], and others ([**5**], [**8**], [**17**], [**18**]).

1. Introduction

Generalizing the concept of commuting mappings, Sessa [**15**] introduced the concept of weakly commuting mappings.

Further, in 1986, Jungck [**7**] gave more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept of compatible mappings was frequently used to prove existence theorems in common fixed point theory.

Afterwards, Jungck, Murthy and Cho [**9**] generalized the concept of compatible mappings by introducing compatible mappings of type (A), which is equivalent to the concept of compatible mappings under some conditions and examples are given to show that the two notions are independent. The same authors proved a common fixed point theorem for compatible mappings of type (A) in a metric space.

In [**12**], the concept of compatible mappings of type (P) was introduced and compared with compatible mappings of type (A) and compatible mappings.

In [**10**], Jungck and Rhoades defined weakly compatible mappings and showed that compatible mappings are weakly compatible but the converse need not be true.

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All of these concepts of commutativity have been used by many authors to prove fixed point theorems for contractive and expansive type conditions.

It is well known, that the above compatibility notions imply the weakly compatibility. However, as we shall show in the examples below, weakly compatible mappings need not be neither compatible, nor compatible of type (A) (resp. of type (P)).

The subject of this paper is to prove some common fixed point theorems for a family of self-mappings in a metric and a compact metric space using a few conditions. These theorems improve the results of Popa [**14**], Aliouche [**1**], Imdad and Javid [**6**], and others.

2. Main results

2.1. First part. We start by some needed definitions.

Definition 2.1. ([**15**]) Self-mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be weakly commuting if

$$d(fgx, gfx) \leqslant d(gx, fx), \text{ for all } x \in \mathcal{X}.$$

Definition 2.2. ([**7**]) Let f and g be two self-mappings of a metric space $(\mathcal{X}, d)$. f and g are said to be compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but neither implication is reversible (see 10, Ex. [**7**]).

Definition 2.3. ([**9**]) Let f and g be mappings from a metric space $(\mathcal{X}, d)$ into itself. Mappings f and g are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Obviously, weakly commuting mappings are compatible of type (A). By ([**9**], Ex. 2.2) follows that the implication is not reversible and by ([**9**], Ex. 2.1 and 2.2) follows that the notions of compatible mappings and compatible mappings of type (A) are independent.

Definition 2.4. ([**12**]) Let f and g be mappings from a metric space $(\mathcal{X}, d)$ into itself. f and g are said to be compatible of type (P) if

$$\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

DEFINITION 2.5. ([**10**]) Let f and g be mappings from a metric space $(\mathcal{X}, d)$. f and g are said to be weakly compatible if $fx = gx$ implies $fgx = gfx$.

LEMMA 2.1 ([**14, 7, 9, 13**]). *Let f and g be compatible (resp. compatible of type (A), compatible of type (P)) self-mappings on a metric space $(\mathcal{X}, d)$. If $ft = gt$ for some $t \in \mathcal{X}$, then $fgt = gft$.*

By the above lemma, it follows that, if f and g are compatible (resp. compatible of type (A), compatible of type (P)), then, f and g are weakly compatible. As we said above, not necessarily conversely, as it is shown in the following examples.

EXAMPLE 2.1. Let $\mathcal{X} = [1, 20]$ with the usual metric. Define

$$fx = \begin{cases} 1 \text{ if } x = 1 \\ 3 \text{ if } 1 < x \leqslant 7 \\ x - 6 \text{ if } 7 < x \leqslant 20; \end{cases} \quad gx = \begin{cases} 1 \text{ if } x \in \{1\} \cup (7, 20] \\ 2 \text{ if } 1 < x \leqslant 7. \end{cases}$$

We have $f(1) = 1 = g(1)$; $fg(1) = 1 = gf(1)$. Clearly, f and g are weakly compatible mappings, since they commute at their coincidence point $x = 1$. To see that f and g are not compatible (resp. compatible of type (A), (P)), let us consider the sequence $\{x_n\}$ defined by $x_n = 7 + \frac{1}{n}, n = 1, 2, \ldots$. Then,

$$fx_n = x_n - 6 \to 1; gx_n = 1 \to 1 \text{ as } n \to \infty.$$

We have

$$\lim_{n\to\infty} |fgx_n - gfx_n| = 1 \neq 0,$$

and so, f and g are not compatible. We also have

$$\lim_{n\to\infty} |gfx_n - f^2x_n| = 1 \neq 0,$$

hence, f and g are not compatible of type (A). Further,

$$\lim_{n\to\infty} |f^2x_n - g^2x_n| = 2 \neq 0,$$

thus, f and g are not compatible of type (P).

EXAMPLE 2.2. Endow $\mathcal{X} = [0, 20]$ with the usual metric d and define

$$fx = \begin{cases} 0 \text{ if } x = 0 \\ x + 11 \text{ if } 0 < x \leqslant 9 \\ x - 9 \text{ if } 9 < x \leqslant 20; \end{cases} \quad gx = \begin{cases} 0 \text{ if } x \in \{0\} \cup (9, 20] \\ 10 \text{ if } 0 < x \leqslant 9. \end{cases}$$

Let $\{x_n\}$ be a sequence defined by $x_n = 9 + \frac{1}{n}$ for all $n = 1, 2, \ldots$. Then,

$$\begin{aligned} \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} (x_n - 9) = 0 \text{ and } \lim_{n\to\infty} gx_n = 0, \\ f(0) &= 0 = g(0) \text{ and } fg(0) = gf(0) = 0. \end{aligned}$$

Clearly, f and g are weakly compatible mappings, since they commute at their coincidence point $x = 0$. On the other hand, we have

$$\begin{aligned} fgx_n &= f(0) = 0 \to 0 \text{ as } n \to \infty, \\ gfx_n &= g(x_n - 9) = 10 \to 10 \text{ as } n \to \infty, \\ f^2x_n &= f(x_n - 9) = x_n + 2 \to 11 \text{ as } n \to \infty, \\ g^2x_n &= g(0) = 0 \to 0 \text{ as } n \to \infty. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} |fgx_n - gfx_n| = 10 \neq 0,$$

that is, f and g are not compatible. We have

$$\lim_{n\to\infty} |gfx_n - f^2x_n| = 1 \neq 0,$$

thus, f and g are not compatible of type (A). Further,

$$\lim_{n\to\infty} |f^2x_n - g^2x_n| = 11 \neq 0,$$

which tells that f and g are not compatible of type (P).

LEMMA 2.2 ([**14**]). *Two continuous self-mappings of a compact metric space are compatible (resp. compatible of type (A), compatible of type (P)) if and only if they are weakly compatible.*

Now, we give some implicit relations.

Implicit relations: Like in [**14**], we denote by $\mathcal{F}$ the set of real functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

- F_1: F is non increasing in variables t_5 and t_6,
- F_2: for every $u \geqslant 0, v > 0$
  (1) F_a: $F(u, v, v, u, u + v, 0) < 0$ or
  (2) F_b: $F(u, v, u, v, 0, u + v) < 0$
  we have $u < v$.
- F_3: $F(u, u, 0, 0, u, u) \geqslant 0, \forall u > 0$.

The next examples of functions in $\mathcal{F}$ are given in [**14**].

EXAMPLE 2.3. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$.

EXAMPLE 2.4.

$$\begin{aligned} F(t_1, t_2, t_3, t_4, t_5, t_6) &= t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3t_5, t_4t_6\} \\ &\quad -c_3t_5t_6, \end{aligned}$$

where c_1, c_2, $c_3 \geqslant 0$, $c_1 + 2c_2 \leqslant 1$ and $c_1 + c_3 \leqslant 1$.

EXAMPLE 2.5.

$$\begin{aligned} F(t_1, t_2, t_3, t_4, t_5, t_6) &= (1 + pt_2)t_1 - p \max\{t_3t_4, t_5t_6\} \\ &\quad - \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}, \end{aligned}$$

where $p > 0$.

EXAMPLE 2.6. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6), b\sqrt{t_5t_6}\}$, where $0<b<1$.

EXAMPLE 2.7. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$, where a, b, c, $d \geqslant 0$ and $a+b+c+d<1$.

EXAMPLE 2.8. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - c\dfrac{t_3^2t_4^2 + t_5^2t_6^2}{1+t_2+t_3+t_4}$, where $c \in (0,1)$.

Now, we state our first main results.

THEOREM 2.1. *Let f, g, h, k be self-mappings of a compact metric space $(\mathcal{X}, d)$ such that $f(\mathcal{X}) \subset k(\mathcal{X})$ and $g(\mathcal{X}) \subset h(\mathcal{X})$,*

$$F(d(fx,gy), d(hx,ky), d(hx,fx), d(ky,gy), d(hx,gy), d(ky,fx)) < 0 \tag{2.1}$$

for all x, y in $\mathcal{X}$ for which one of $d(hx,ky)$, $d(hx,fx)$, $d(ky,gy)$ is positive, where $F \in \mathcal{F}$, f and h as well as g and k are weakly compatible, and f and h or g and k are continuous. Then, f, g, h and k have a unique common fixed point u in $\mathcal{X}$.

PROOF. Suppose that g and k are continuous. Let

$$m = \inf\{d(gx,kx) : x \in \mathcal{X}\}.$$

Since $\mathcal{X}$ is compact, there is a convergent sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} d(kx_n, gx_n) = m.$$

Since

$$d(kx_0,gx_0) \leqslant d(kx_0,kx_n) + d(kx_n,gx_n) + d(gx_n,gx_0),$$

then, by continuity of k and g and $\lim_{n\to\infty} x_n = x_0$, we get

$$d(kx_0,gx_0) \leqslant m$$

and thus,

$$d(kx_0,gx_0) = m.$$

Since $g(\mathcal{X}) \subset h(\mathcal{X})$, there exists a point y_0 in $\mathcal{X}$ such that $hy_0 = gx_0$ and thus, $d(kx_0,hy_0) = m$. Suppose that $m>0$. Then, by (2.1), we have

$$F(d(fy_0,gx_0), d(hy_0,kx_0), d(hy_0,fy_0), d(kx_0,gx_0), d(hy_0,gx_0), d(kx_0,fy_0)) < 0.$$

Since F is non increasing in variable t_6, we get

$$\begin{aligned} & F(d(fy_0,hy_0), m, d(hy_0,fy_0), m, 0, d(kx_0,hy_0) + d(hy_0,fy_0)) \\ = \; & F(d(fy_0,hy_0), m, d(hy_0,fy_0), m, 0, m + d(hy_0,fy_0)) < 0. \end{aligned}$$

By property F_b follows that

$$d(fy_0,hy_0) < m.$$

Since $f(\mathcal{X}) \subset k(\mathcal{X})$, then, there is a point z_0 in $\mathcal{X}$ such that $kz_0 = fy_0$ and thus, $d(kz_0,hy_0) < m$. Since $d(kz_0,gz_0) \geqslant m > 0$, by (2.1), we have

$$F(d(fy_0,gz_0), d(hy_0,kz_0), d(hy_0,fy_0), d(kz_0,gz_0), d(hy_0,gz_0), d(kz_0,fy_0)) < 0.$$

By F_1 and since F is non increasing in variable t_5, we obtain

$$\begin{gathered}F(d(kz_0,gz_0),d(hy_0,fy_0),d(hy_0,fy_0),\\ d(kz_0,gz_0),d(hy_0,fy_0)+d(fy_0,gz_0),0)\\ =F(d(kz_0,gz_0),d(hy_0,fy_0),d(hy_0,fy_0),\\ d(kz_0,gz_0),d(hy_0,fy_0)+d(kz_0,gz_0),0)<0.\end{gathered}$$

By F_a follows that

$$d(kz_0,gz_0)<d(hy_0,fy_0).$$

Then, we obtain

$$m\leqslant d(kz_0,gz_0)<d(hy_0,fy_0)<m.$$

This contradiction demands that $m=0$. Therefore, we have

$$kx_0=gx_0=hy_0.$$

If $d(hy_0,fy_0)>0$, then, using (2.1), we have successively

$$\begin{gathered}F(d(fy_0,gx_0),d(hy_0,kx_0),d(hy_0,fy_0),\\ d(kx_0,gx_0),d(hy_0,gx_0),d(kx_0,fy_0))\\ =F(d(fy_0,hy_0),0,d(hy_0,fy_0),0,0,d(hy_0,fy_0))<0\end{gathered}$$

which implies by F_b that $d(fy_0,hy_0)<0$, but this contradicts $d(hy_0,fy_0)>0$. Thus, $d(hy_0,fy_0)=0$, which implies that $hy_0=fy_0$. Therefore,

$$kx_0=gx_0=hy_0=fy_0.$$

Since f and h are weakly compatible and $hy_0=fy_0$, we get

$$h^2y_0=hfy_0=fhy_0=f^2y_0.$$

Suppose that $f^2y_0\neq fy_0$, then, inequality (2.1) gives

$$\begin{aligned}&F(d(f^2y_0,gx_0),d(hfy_0,kx_0),d(hfy_0,f^2y_0),\\ &d(kx_0,gx_0),d(hfy_0,gx_0),d(kx_0,f^2y_0))\\ =\;&F(d(f^2y_0,fy_0),d(f^2y_0,fy_0),0,0,d(f^2y_0,fy_0),d(fy_0,f^2y_0))<0\end{aligned}$$

which contradicts F_3, then, we have $f^2y_0=fy_0$. Hence,

$$fy_0=f^2y_0=hfy_0.$$

Similarly, since g and k are weakly compatible and $kx_0=gx_0$, we get

$$k^2x_0=kgx_0=gkx_0=g^2x_0.$$

If $g^2x_0\neq gx_0$, then, again by (2.1) we have successively

$$\begin{aligned}&F(d(fy_0,g^2x_0),d(hy_0,kgx_0),d(hy_0,fy_0),\\ &d(kgx_0,g^2x_0),d(hy_0,g^2x_0),d(kgx_0,fy_0))\\ =\;&F(d(gx_0,g^2x_0),d(gx_0,g^2x_0),0,0,d(gx_0,g^2x_0),d(g^2x_0,gx_0))<0\end{aligned}$$

a contradiction of F_3. Therefore, $gx_0=g^2x_0$. Hence,

$$gx_0=g^2x_0=kgx_0.$$

Let $u=fy_0=gx_0$. Then,

$$fu=hu=u=gu=ku$$

and u is a common fixed point of f, g, h and k.

Suppose that g and k have another common fixed point, then, $d(u,v) \neq 0$ and the use of (2.1) gives

$$\begin{aligned} & F(d(fu,gv), d(hu,kv), d(hu,fu), d(kv,gv), d(hu,gv), d(kv,fu)) \\ = \; & F(d(u,v), d(u,v), 0, 0, d(u,v), d(v,u)) < 0 \end{aligned}$$

which is a contradiction with F_3. Thus, $v = u$.

Similarly, u is the unique common fixed point of f and h. Indeed, suppose that h and f have another common fixed point, then, $d(u,t) \neq 0$ and using condition (2.1), one may get

$$\begin{aligned} & F(d(ft,gu), d(ht,ku), d(ht,ft), d(ku,gu), d(ht,gu), d(ku,ft)) \\ = \; & F(d(t,u), d(t,u), 0, 0, d(t,u), d(u,t)) < 0 \end{aligned}$$

contradicts F_3. Therefore, $t = u$. Hence u is the unique common fixed point of f and h and g and k.

Similarly, one can obtain this conclusion by supposing f and h are continuous. □

Truly, the above theorem improves the result of Popa [**14**] and others. Indeed, by replacing F of our theorem by any function satisfying conditions F_1, F_2 and F_3 we can obtain the following corollaries:

COROLLARY 2.1. *Let f, g, h and k be as in theorem 2.1. If the inequality*

$$\begin{aligned} d(fx,gy) \; < \; & \max\{d(hx,ky), d(hx,fx), d(ky,gy), \\ & \tfrac{1}{2}(d(hx,gy) + d(ky,fx))\Big\} \end{aligned}$$

holds for all x, y in $\mathcal{X}$ for which the right hand side of the above inequality is positive. Then, f, g, h and k have a unique common fixed point in $\mathcal{X}$.

PROOF. It follows from Theorem 2.1 and Example 2.3. □

COROLLARY 2.2. *If in the hypotheses of theorem 2.1, we have in lieu of the condition*

$$\begin{aligned} (1 + pd(hx,ky))d(fx,gy) \; < \; & p\max\{d(hx,fx)d(ky,gy), d(hx,gy)d(ky,fx)\} \\ & + \max\{d(hx,ky), d(hx,fx), d(ky,gy), \\ & \tfrac{1}{2}(d(hx,gy) + d(ky,fx))\Big\} \end{aligned}$$

for all x, $y \in \mathcal{X}$ for which the right hand side of the above inequality is positive, where $p > 0$. Then, f, g, h and k have a unique common fixed point in $\mathcal{X}$.

PROOF. Use Theorem 2.1 and Example 2.5. □

In a similar way as in corollaries 2.1 and 2.2, one can obtain much more corollaries by using the examples given above.

REMARK 2.1. We can get much more corollaries if we let in Theorem 2.1 and its corollaries

(1) $h = k = I_{\mathcal{X}}$ (the identity mapping on $\mathcal{X}$) with f or g is continuous,
(2) $f = g$ and $h = k$, and also
(3) $f = g$ and $h = k = I_{\mathcal{X}}$.

Now, using the recurrence on n, we can give the next result:

Theorem 2.2. *Let h, k and $\{F_n\}_{n=1,2,\ldots}$ be mappings from a compact metric space into itself having the following conditions:*

(1) *$F_n(\mathcal{X}) \subset k(\mathcal{X})$ and $F_{n+1}(\mathcal{X}) \subset h(\mathcal{X})$,*
(2) *h and $\{F_n\}_{n=1,2,\ldots}$ are weakly compatible as well as k and $\{F_{n+1}\}_{n=1,2,\ldots}$,*
(3) *mappings $\{F_n\}_{n=1,2,\ldots}$ and h or $\{F_{n+1}\}_{n=1,2,\ldots}$ and k are continuous,*
(4) *the inequality*

$$F(d(F_nx, F_{n+1}y), d(hx, ky), d(hx, F_nx),\\ d(ky, F_{n+1}y), d(hx, F_{n+1}y), d(ky, F_nx)) < 0$$

holds for all x, y in $\mathcal{X}$, for all $n = 1, 2, \ldots$, for which one of $d(hx, ky), d(hx, F_nx)$ and $d(ky, F_{n+1}y)$ is positive, where $F \in \mathcal{F}$.

Then, h, k and $\{F_n\}_{n=1,2,\ldots}$ have a unique common fixed point in $\mathcal{X}$.

2.2. Second part. In 2008, Al-Thagafi and Shahzad [**2**] introduced a generalization of weakly compatible mappings by giving the notion of occasionally weakly compatible mappings.

Definition 2.6. ([**2**]) Let f and g be self-mappings of a subset D of a metric space $(\mathcal{X}, d)$. Then f and g are called occasionally weakly compatible if $fgx = gfx$ for some $x \in C(f, g)$ where $C(f, g)$ is the set of coincidence points of f and g.

In [**3**] (the old version) and [**4**] (the new version), we introduced the notion of subcompatible mappings which is a significant enriched generalization of occasionally weakly compatible mappings given by Al-Thagafi and Shahzad [**2**]. Several authors used our notion to prove some fixed point theorems in various settings.

Definition 2.7. ([**3, 4**]) Let f and g be two self-mappings of a metric space $(\mathcal{X}, d)$. f and g are subcompatible if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.

In his paper [**11**], Pant introduced the concept of reciprocally continuous mappings as follows:

Definition 2.8. ([**11**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$.

In [**3, 4**], we introduced the notion of subsequentially continuous mappings which weakens the concepts of continuity and reciprocally continuity.

Definition 2.9. ([**3, 4**]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be subsequentially continuous if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$ and satisfy $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$.

Now, we are ready to present and prove our second result which improves the main result of [**1**] and Theorem 2 of [**6**] because we removed continuity, compactness, inclusions, and so on.

THEOREM 2.3. *Let f, g, h, k be self-mappings of a metric space $(\mathcal{X}, d)$ such that the pairs (f, h) and (g, k) are subcompatible and reciprocally continuous or compatible and subsequentially continuous. Suppose that the four mappings satisfy the inequality*

$$\varphi(d(fx,gy),d(hx,ky),d(hx,fx),d(ky,gy),d(hx,gy),d(ky,fx))<0 \tag{2.2}$$

for all x, y in $\mathcal{X}$, where φ is upper semi-continuous and $\varphi(t,t,0,0,t,t) > t$ for all $t > 0$. Then, f, g, h and k have a unique common fixed point t in $\mathcal{X}$.

PROOF. Since the pairs (f, h) and (g, k) are subcompatible and reciprocally continuous, then, there exist two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = t \text{ for some } t \in \mathcal{X}$$

and which satisfy

$$\lim_{n\to\infty} d(fhx_n, hfx_n) = d(ft, ht) = 0;$$
$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ky_n = z$$

for some $z \in \mathcal{X}$ and which satisfy

$$\lim_{n\to\infty} d(gky_n, kgy_n) = d(gz, kz) = 0.$$

Therefore $ft = ht$ and $gz = kz$; that is, t is a coincidence point of f and h and z is a coincidence point of g and k.

Now, we prove that $z = t$. Indeed, suppose that $d(t, z) > 0$, using inequality (2.2) we get

$$\varphi(d(fx_n,gy_n),d(hx_n,ky_n),d(hx_n,fx_n),d(ky_n,gy_n),d(hx_n,gy_n),d(ky_n,fx_n))<0.$$

Since φ is upper semi-continuous, we obtain at infinity

$$\varphi(d(t,z),d(t,z),0,0,d(t,z),d(z,t)) \leqslant 0$$

which is a contradiction, hence $z = t$.

Suppose that $ft \neq t$, the use of condition (2.2) gives

$$\varphi(d(ft,gy_n),d(ht,ky_n),d(ht,ft),d(ky_n,gy_n),d(ht,gy_n),d(ky_n,ft))<0$$

At infinity we obtain

$$\varphi(d(ft,t),d(ft,t),0,0,d(ft,t),d(t,ft)) \leqslant 0$$

this contradiction implies that $t = ft = ht$.

Similarly, If $gt \neq t$, using condition (2.2) we obtain

$$\begin{aligned} &\varphi(d(ft,gt),d(ht,kt),d(ht,ft),d(kt,gt),d(ht,gt),d(kt,ft)) \\ = \;&\varphi(d(t,gt),d(t,gt),0,0,d(t,gt),d(gt,t)) < 0 \end{aligned}$$

which is a contradiction, hence $t = gt = kt$.

For the uniqueness of the common fixed point t, let w be another common fixed point of f, g, h and k. Then using (2.2) we get

$$\begin{aligned} & \varphi(d(ft, gw), d(ht, kw), d(ht, ft), d(kw, gw), d(ht, gw), d(kw, ft)) \\ = \; & \varphi(d(t, w), d(t, w), 0, 0, d(t, w), d(w, t)) < 0 \end{aligned}$$

a contradiction, therefore $w = t$. □

The next theorem improves Theorem 4 of [**6**].

THEOREM 2.4. *Let f_n, $n = 1, 2, \ldots$, h and k be self-mappings of a metric space $(\mathcal{X}, d)$ such that the pairs (f_n, h) and (f_{n+1}, k) are subcompatible and reciprocally continuous or compatible and subsequentially continuous. Suppose that the four mappings satisfy the inequality*

$$\varphi(d(f_n x, f_{n+1} y), d(hx, ky), d(f_n x, hx), d(f_{n+1} y, ky), d(hx, f_{n+1} y), d(f_n x, ky)) < 0$$

for all x, y in $\mathcal{X}$, $n = 1, 2, \ldots$, where φ is upper semi-continuous and

$$\varphi(t, t, 0, 0, t, t) > 0 \text{ for all } t > 0.$$

Then, f_n, h and k have a unique common fixed point t in $\mathcal{X}$.

References

[1] A. Aliouche. A common fixed point theorem for weakly compatible mappings in compact metric spaces satisfying an implicit relation. *Sarajevo J. Math.*, **3**(15)(2007), 123–130.

[2] M. A. Al-Thagafi and N. Shahzad. Generalized I-nonexpansive selfmaps and invariant approximations. *Acta Math. Sin. (Engl. Ser.)*, **24**(5)(2008), 867–876.

[3] H. Bouhadjera and C. Godet-Thobie. Common fixed point theorems for pairs of subcompatible maps. arXiv: 0906.3159v1 [math. FA] 17 June 2009.

[4] H. Bouhadjera and C. Godet-Thobie. Common fixed point theorems for pairs of subcompatible maps. arXiv: 0906.3159v2 [math. FA] 23 May 2011.

[5] B. Fisher, M. Telci and K. Tas. A generalization of fixed point theorem of Nešic. *Doga-Tr. J. Math.*, **17**(1993), 247–250.

[6] M. Imdad and A. Javid. Remarks on a common fixed point theorem in compact metric spaces. *Math. Commun.*, 15(1)(2010), 107–116.

[7] G. Jungck. Compatible mappings and common fixed points. *Internat. J. Math. Math. Sci.*, **9**(1986), 771–779.

[8] G. Jungck. Common fixed points for commuting and compatible maps on compacta. *Proc. Amer. Math. Soc.*, **103**(3)(1988), 977–983.

[9] G. Jungck, P. P. Murthy and Y. J. Cho. Compatible mappings of type (A) and common fixed points. *Math. Japonica*, **36**(1993), 381–390.

[10] G. Jungck and B. E. Rhoades. Fixed point for valued functions without continuity. *Indian J. Pure Appl. Math.*, **29**(3)(1998), 227–238.

[11] R. P. Pant. A common fixed point theorem under a new condition. *Indian J. Pure Appl. Math.*, **30**(2)(1999), 147–152.

[12] H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang. Compatible mappings of type (P) and fixed point theorem in metric spaces and probabilistic metric spaces. *Novi Sad J. Math.*, **26**(2)(1996), 87–109.

[13] H. K. Pathak, Y. J. Cho, S. M. Kang and B. S. Lee. Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming. *Le Mathematiche*, **50**(1)(1995), 15–33.

[14] V. Popa. A general fixed point theorem for weakly compatible mappings in compact metric spaces. *Turk. J Math.*, **25** (2001), 465–474.

[15] S. Sessa. On a weak commutativity condition in fixed point considerations. *Publ. Inst. Math.*, **32**(46)(1986), 149–153.

[16] S. Sessa and B. Fisher. Common fixed points of weakly commuting mappings. *Bull. Polish Acad. Sci. Math.*, **36**(1987), 341–349.

[17] K. Tas, M. Telci and B. Fisher. Common fixed points theorems for compatible mappings. *Internat. J. Math. Math. Sci.*, **19**(3)(1996), 451–456.

[18] M. Telci, K. Tas and B. Fisher. A generalization of the fixed point theorem of Bhola and Sharma. *Math. Balkanica, New Series*, **9**(Fasc. 2-3)(1995), 113–116.

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