BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(3)(2021), 443-449
DOI: 10.7251/BIMVI2103443R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

THE THIRD ISOMORPHISM THEOREM FOR IMPLICATIVE SEMIGROUP WITH APARTNESS

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ABSTRACT. Implicative semigroups with apartness have been introduced in 2016 by this author who then analyzed them in several papers. In this paper, a form of the third isomorphism theorem for this type of semigroups is shown, which has no counterpart in the classical semigroup theory.

1. Introduction

The notion of implicative semigroup was introduced by Chan and Shum [**7**]. Ordered ideals and filters play an important role in the theory of implicative semilattices. For this reason Chan and Shum [**7**] established some elementary properties and constructed a quotient structure of implicative semigroups via ordered filters, whereas Jun and Kim [**10**].

In the setting of Bishops constructive mathematics, the notion of implicative semigroups with tight apartness was introduced in [**12**], following the ideas of above mentioned authors, and some fundamental characterizations of these semigroups were given. Then the author continued his research on implicative semigroups with apartness using co-order relations instead of partial order. In particular strongly extensional homomorphisms between implicative semigroups with apartness are discussed in [**13**]; co-filters and co-ideals in such semigroups were considered in [**15, 16, 17, 18**]. An interested reader can find in paper [**14**] more information on co-quasiorder and co-order relations and their applications in algebraic structures having sets with apartness as carriers. Some forms of the first isomorphism theorem

2010 *Mathematics Subject Classification.* 03F65, 20M12, 06F05, 06A99.
Key words and phrases. Bishop's constructive algebra, implicative semigroup with apartness, co-congruence on implicative semigroup with apartness, co-quasiorder on implicative semigroup with apartness, quotient structures, isomorphism theorem.
Communicated by Duško Bogdanić.

for this class of algebraic structures can be found in [**20**]. In particular, the paper [**19**] provides a recapitulation of previous research.

In this article, as a continuation of previous research, a form of the third isomorphism theorem for implicative semigroups with apartness is presented, which does not have its dual in the classical semigroup theory:

Let σ and τ be co-quasiorder relations on an implicative semigroup with apartness $((S, =_S, \neq_S), \cdot, \alpha, \otimes)$ such that $\sigma \subseteq \tau \subseteq \alpha$. Then there is a unique injective, embedding and surjective strongly extensional mapping

$$[[S : \tau \cup \tau^{-1}] : [\sigma : \tau] \cup [\sigma : \tau]^{-1}] \longrightarrow [S : \sigma \cup \sigma^{-1}].$$

In addition, in this paper an interested reader can get acquainted with the techniques used in the analysis of such algebraic structures. This paper is a direct continuation of articles [**19**] and [**20**].

2. Preliminaries

In this section, we recall from [**6, 8, 9, 12, 13, 15, 16, 17**] some concepts and processes necessary in the sequel of this paper and the reader is referred to [**12, 13, 14, 15, 16, 17, 19, 18**] for undefined notions and notations. This investigation is in Bishops constructive algebra in the sense of papers [**8, 9, 6, 14**] and books [**1, 2, 3, 4, 5**] and Chapter 8: Algebra of [**21**].

2.1. Set with apartness. Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). A diversity relation "$\neq$" satisfying conditions

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z$$

is called apartness. In this paper, we assume that the apartness is tight, i.e. it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \Longrightarrow x = y).$$

A subset X of S is called a strongly extensional subset of S if and only if $(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S)$. Let X, Y be subsets of S. According with Bridge and Vita definition (see for instance [**5**]), we say that X is set-set apartned from Y (denoted $X \bowtie Y$) if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \lhd Y$ and $x \neq y$, instead of $\{x\} \bowtie Y$ and $\{x\} \bowtie \{y\}$ respectively. With $X^{\lhd} = \{x \in S : x \lhd X\}$ we denote the apartness complement of X.

2.2. Implicative semigroups with apartness. Let $((S, =, \neq), \cdot)$ be a semigroup with apartness. We recall we recall that its binary operation '$\cdot$' has to be extensional and strongly extensional, i.e '$\cdot$' is a function from $S \times S$ into S such that

$$(\forall a, b, u, v \in S)((a, b) = (u, v) \implies ab = uv),$$
$$(\forall a, b, u, v \in S)(ab \neq uv \implies (a, b) \neq (u, v))$$

whereby we sent ab instead of $a \cdot b$.

By a *negatively co-ordered* semigroup (briefly, n.a-o. semigroup) we mean a semigroup with apartness S with a co-order α such that for all $x, y, z \in S$ the following hold:

(1) $(xy)z = x(yz)$,

(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and

(3) $(xy, x) \lhd \alpha$ and $(xy, y) \lhd \alpha$.

In such case α we will be called a negative co-order relation on S.

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that the following is true

(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha$

for any elements x, y, z of S.

In addition, let us recall that the internal binary operation '$\otimes$' must satisfy the following implications:

$$(\forall a, b, u, v \in S)((a, b) = (u, v) \implies a \otimes b = u \otimes v),$$
$$(\forall a, b, u, v \in S)(a \otimes b \neq u \otimes v \implies (a, b) \neq (u, v)).$$

The operation $\otimes$ is called *implication.* From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup.*

In any implicative semigroup S there exists a special element of S, the biggest element in $(S, \alpha^{\lhd})$, which is the left neutral element in $(S, \cdot)$.

Let $S = ((S, =, \neq), \cdot, \alpha, \otimes)$ and $T = ((T, =, \neq), \cdot, \beta, \otimes)$ be two implicative semigroups and let $f : S \longrightarrow T$ be a strongly extensional mapping from S in T. As the usual procedure in the construction of a mathematical system, for mapping f we say that it is a *homomorphism* ([**13**]) between implicative semigroups S and T if

$$(\forall x, y \in S)(f(x \otimes y) = f(x) \otimes f(y))$$

holds. In this case, f is said to be *se-homomorphism.* If f is surjective, then f is a semigroup homomorphism, that ([**13**], Theorem 3.1)

$$(\forall x, y \in S)(f(xy) = f(x)f(y))$$

is valid. A surjective se-homomorphism is *se-endomorphism* and an injective se-homomorphism is *se-monomorphism.*

2.3. The first isomorphism theorem. A relation $\sigma \subseteq S \times S$ is a co-quasiorder on S if it is consistent and co-transitive. In the following, we assume that σ is compatible with the operations in S and the following $\sigma \subseteq \alpha$ holds. The relation $q = \sigma \cup \sigma^{-1}$ is a co-congruence on S.

We can construct the family $[S : q] = \{xq : x \in S\}$, where

$$xq =_2 yq \iff (x, y) \lhd q, \;\; xq \neq_2 yq \iff (x, y) \in q.$$

If we define the operations '$\cdot_2$' and '$\otimes_2$' and the co-order Θ on $[S : q]$ as follows

$$xq \cdot yq =_2 (x \cdot y)q, \;\; xq \otimes_2 yq =_2 (x \otimes y)q, \;\; (xq, yq) \in \Theta \iff (x, y) \in \sigma$$

it can be verified:

THEOREM 2.1 ([**20**]). *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness such that the co-quasiorder $\sigma \subseteq \alpha$ satisfies condition (4). Let $q = \sigma \cup \sigma^{-1}$. Then $(([S : q], =_2, \neq_2), \cdot_2, \Theta, \otimes_2)$ is an implicative semigroup with apartnes and there exists a unique reverse isotone se-epimorphism $\vartheta : S \longrightarrow [S : q]$.*

It should be emphasized here that the semigroup [S: q] has no counterpart in the classical theory of implicit semigroups. However, there is a strong link between the semigroup $S/(q^{\triangleleft}, q)$ and the semigroup $[S : q]$.

The following theorem can be viewed as the First Isomorphism Theorem for implicative semigroups with apartness.

THEOREM 2.2 ([**20**]). *Let $f : S \longrightarrow T$ be a reverse isotone se-epimorphism from implicative semigroups $((S, =, \neq), \cdot, \alpha, \otimes)$ onto implicative semigroup $((T, =, \neq), \cdot, \beta, \otimes)$. Then there exist a unique $f_2 : [S : q_f] \longrightarrow T$ of embedding, injective and surjective se-homomorphism such that*

$$f = f_2 \circ \vartheta.$$

3. The Main Result

The following theorem is one of the specifics of the Bishop aspect on algebraic structures with apartness ordered under a co-order relation and does not have its counterpart in the classical theory.

Let us consider an implicative semigroup $((S, =_S, \neq_S), \cdot, \alpha, \otimes)$ ordered under a co-order α and let σ and τ be co-quasiorder relations on S such that $\sigma \subseteq \tau \subseteq \alpha$. It is known that co-congruences $q_\sigma = \sigma \cup \sigma^{-1}$ and $q_\tau = \tau \cup \tau^{-1}$ on S such that $q_\sigma \subseteq q_\tau$ can be designed. Further, by Theorem 2.1, this allows us to construct implicative semigroups with apartness

$$(([S : q_\sigma], =_\sigma, \neq_\sigma), \cdot_\sigma, \Theta_\sigma, \otimes_\sigma)$$

and

$$(([S : q_\tau], =_\tau, \neq_\tau), \cdot_\tau, \Theta_\tau, \otimes_\tau)$$

which are ordered by co-order relations Θ_σ and Θ_τ respectfully as follows

$$(\forall x, y \in S)((xq_\sigma, yq_\sigma) \in \Theta_\sigma \iff (x, y) \in \sigma),$$

$$(\forall x, y \in S)((xq_\tau, yq_\tau) \in \Theta_\tau \iff (x, y) \in \tau).$$

The operations are defined as follows

$$(\forall x, y \in S)(xq_\sigma \cdot_\sigma yq_\sigma :=_\sigma (x \cdot y)q_\sigma \text{ and } xq_\sigma \otimes_\sigma yq_\sigma :=_\sigma (x \otimes y)q_\sigma)$$

$$(\forall x, y \in S)(xq_\tau \cdot_\tau yq_\tau :=_\tau (x \cdot y)q_\tau \text{ and } xq_\tau \otimes_\tau yq_\tau :=_\tau (x \otimes y)q_\tau).$$

Let us define the relation $[\sigma : \tau]$ on the implicative semigroup with apartness $[S : q_\tau]$ as follows

$$(\forall x, y \in S)((xq_\tau, yq_\tau) \in [\sigma : \tau] \iff (x, y) \in \sigma).$$

LEMMA 3.1. *Let σ and τ be co-quasiorder on an implicative semigroup with apartness $((S, =_S, \neq_S), \cdot, \alpha, \otimes)$ such that $\sigma \subseteq \tau \subseteq \alpha$. Then $[\sigma : \tau]$ is a co-quasiorder relation on $[S : q_\tau]$.*

PROOF. Ler $x, y, z \in S$ be arbitrary elements. Then:

$$\begin{aligned}(xq_\tau, yq_\tau) \in [\sigma : \tau] &\Longrightarrow (x, y) \in \sigma \subseteq \tau \subseteq q_\tau \\ &\Longrightarrow xq_\tau \neq_\tau yq_\tau; \\ (xq_\tau, zq_\tau) \in [\sigma : \tau] &\Longleftrightarrow (x, z) \in \sigma \\ &\Longrightarrow (x, y) \in \sigma \vee (y, z) \in \sigma \\ &\Longrightarrow (xq_\tau, yq_\tau) \in [\sigma : \tau] \vee (yq_\tau, zq_\tau) \in [\sigma : \tau].\end{aligned}$$

Let us show that $[\sigma, \tau]$ satisfies the condition (4). Let $x, y, z \in S$ be such that $(zq_\tau, xq_\tau \otimes_\tau yq_\tau) \in [\sigma : \tau]$. Then $(zq_\tau, (x \otimes y)q_\tau) \in [\sigma : \tau]$ by definition of '$\otimes_\tau$'. Thus $(z, x \otimes y) \in \sigma$ and $(x \cdot x, y) \in \sigma$ since the relation σ satisfies the condition (4). Hence $((z \cdot x)q_\tau, yq_\tau) \in [\sigma : \tau]$. So, we have $(zq_\tau \cdot_\tau xq_\tau, yq_\tau) \in [\sigma : \tau]$.

Finally, if $(xq_\tau, yq_\tau) \in [\sigma : \tau]$ for some $x, y \in S$, we have $(x, y) \in \sigma \subseteq \tau$. Therefore, $(xq_\tau, yq_\tau) \in \Theta_\tau$. □

For ease of writing, let's put

$$q := q_{[\sigma:\tau]} = [\sigma : \tau] \cup [\sigma : \tau]^{-1}.$$

Without major difficulties it can be verified that q is a co-congruence on the co-ordered set with apartness $([S, q_\tau], =_\tau, \neq_\tau, \nleq_\tau)$. Set with apartness

$$([[S : q_\tau] : q], =_3, \neq_3, \nleq_3)$$

can be designed, where is

$$(\forall x, y \in S)((xq_\tau)q =_3 (yq_\tau)q \Longleftrightarrow (xq_\tau, yq_\tau) \lhd q),$$

$$(\forall x, y \in S)((xq_\tau)q \neq_3 (yq_\tau)q \Longleftrightarrow (xq_\tau, yq_\tau) \in q).$$

The co-order relation $\nleq_3$ in $[[S : q_\tau] : q]$ is determined as follows

$$(\forall x, y \in S)((xq_\tau)q \nleq_3 (yq_\tau)q \Longleftrightarrow (xq_\tau, yq_\tau) \in [\sigma : \tau]).$$

Operations in this semigroup are determined as follows:

$$(\forall x, y \in S)((xq_\tau)q \cdot_3 (yq_\tau)q) =_3 ((xq_\tau) \cdot_\tau (yq_\tau))q =_3 ((x \cdot y)q_\tau)q),$$

$$(\forall x, y \in S)((xq_\tau)q \otimes_3 (yq_\tau)q) =_3 ((xq_\tau) \otimes_\tau (yq_\tau))q =_3 ((x \otimes y)q_\tau)q),$$

We can now design and prove the following theorem. Of course, this form of this theorem does not have its counterpart in the classical theory.

THEOREM 3.1. *Let σ and τ be co-quasiorder relations on co-ordered set with apartness $((S, =_S, \neq_S), \cdot, \alpha, \otimes)$ such that $\sigma \subseteq \tau \subseteq \alpha$. Then there is a unique injective, embedding and surjective strongly extensional mapping*

$$\gamma : [[S : q_\tau] : q] \longrightarrow [S : q_\sigma].$$

PROOF. Let us define γ by

$$(\forall (xq_\tau)q \in [[S : q_\tau] : q])(\gamma((xq_\tau)q) := xq_\sigma).$$

First, let us show that γ is a well-defined mapping. Assume $x, y, u, v \in S$ are such that $(xq_\tau)q =_3 (yq_\tau)q$ and $(u, v) \in q_\sigma$. then $(xq_\tau, yq_\tau) \lhd q$. On the other hand, from $(u, v) \in q_\sigma$ we get $(u, x) \in q_\sigma \vee (x, y) \in q_\sigma \vee (y, v) \in q_\sigma$. If we assume that

$(x,y) \in q_\sigma$ is valid, then we would have the following $(x,y) \in \sigma$ or $(y,x) \in \sigma$. It would follow from here

$$(x,y) \in \sigma \vee (y,x) \in \sigma \Longrightarrow (xq_\tau, yq_\tau) \in [\sigma : \tau] \subseteq q \vee (yq_t, xq_t) \in [\sigma : \tau] \subseteq q$$

which would contradict the hypothesis $(xq_\tau, yq_\tau) \lhd q$. So, it has to be $(u,x) \in q_\sigma$ or $(y,v) \in q_\sigma$. Thus $x \neq_S u$ or $y \neq_S v$. Therefore, $(x,y) \neq (u,v) \in q_\sigma$. This means $(x,y) \lhd q_\sigma$. Hence

$$\gamma((xq_\tau)q) := xq_\sigma =_\sigma yq_\sigma := \gamma((yq_\tau)q).$$

Second, let us show that γ is a strongly extensional mapping. Let $x,y \in S$ be such that

$$xq_\sigma = \gamma((xq_\tau)q) \neq_3 \gamma((yq_\tau)q) = yq_\sigma.$$

Then $(x,y) \in q_\sigma = \sigma \cup \sigma^{-1}$. Thus $((xq_\tau, yq_\tau) \in [\sigma : \tau] \cup [\sigma : \tau]^{-1} = q$. Hence

$$(xq_\tau)q \neq_3 (yq_\tau)q.$$

Let $x,y,u,v \in S$ be such that $xq_\sigma =_\sigma yq_\sigma$ and $(uq_\tau, vq_\tau) \in q$. Then $(x,y) \lhd q_\sigma = \sigma \cup \sigma^{-1}$ and $(uq_\tau, vq_\tau) \in q$. Thus $(uq_\tau, xq_\tau) \in q$ or $(xq_\tau, yq_\tau) \in q$ or $(yq_\tau, vq_{tau}) \in q$. The option $(xq_\tau, yq_\tau) \in q$ gives $(x,y) \in \sigma \cup \sigma^{-1}$ which is in contradiction with the hypothesis. So, it has to be $(uq_\tau, xq_\tau) \in q$ or $(yq_\tau, vq_\tau) \in q$. Thus $xq_\tau \neq_\tau uq_\tau$ or $yq_\tau \neq_\tau vq_\tau$. Hence $(xq_\tau, yq_\tau) \neq (uq_\tau, vq_\tau) \in q$. This means $(xq_\tau, yq_\tau) \lhd q$, i.e. $(xq_\tau)q =_3 (yq_\tau)q$. This shows that γ is an injective mapping.

It remains to show that γ is an embedding. Let $x,y \in S$ be such that $(xq_\tau)q \neq_3 (yq_\tau)q$. Then $(xq_\tau, yq_\tau) \in q = [\sigma : \tau] \cup [\sigma : \tau]^{-1}$. Thus $(x,y) \in \sigma \cup \sigma^{-1} = q_\sigma$. Hence $xq_\sigma \neq_\sigma yq_\sigma$.

Finally, it is obvious that γ is a surjective mapping.

It remains to be seen that γ is a homomorphism. Let $x,y \in S$ be arbitrary elements. Then

$$\begin{aligned}\gamma((xq_\tau)q \otimes_3 (yq_\tau)q) &=_3 \gamma((x \otimes y)q_\tau)q) := (x \otimes y)q_\sigma =_\sigma xq_\sigma \otimes_\sigma yq_\sigma \\ &:= \gamma((xq_\tau)q) \otimes_\sigma \gamma((yq_\tau)q)\end{aligned}$$

and

$$\begin{aligned}\gamma((xq_\tau)q \cdot_3 (yq_\tau)q) &=_3 \gamma((x \cdot y)q_\tau)q) := (x \cdot y)q_\sigma =_\sigma xq_\sigma \cdot_\sigma yq_\sigma \\ &:= \gamma((xq_\tau)q) \cdot_\sigma \gamma((yq_\tau)q).\end{aligned}$$

This proves the theorem. □

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Received by editors 10.01.2021; Revised version 10.03.2021; Available online 12.04.2021.

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