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# SOME ALGEBRAIC PROPERTIES OF PICTURE FUZZY SETS

**I. Silambarasan**

Abstract. The concept of Picture fuzzy set (PFS), which are direct extensions of the fuzzy sets and intuitionistic fuzzy sets. In this paper, we study some algebraic operations of the Picture fuzzy sets, such as intersection, union, complement, algebraic sum, algebraic product, scalar multiplication and exponentiation operations. We prove some basic algebraic properties of idempotency, commutativity, associativity, absorption, distributivity and De Morgan's laws over complement of Picture fuzzy sets. Furthermore, we define new concentration and dilation of PFSs and proved some theorems. Finally, we define a new operation (@)on Picture fuzzy sets and discuss distributive laws in the case where the operations of $\oplus, \otimes, \cup$ and $\cap$ are combined each other.

## 1. Introduction

The fuzzy set (FS) theory was first developed by Zadeh [**14**] in 1965. In this theory, Zadeh only discussed the positive membership degree of the function. The FS theory has been studied in many fields of the real world such as clustering analysis, decision making problems, medical diagnosis, and also pattern recognition. Unfortunately, the FS theory has been failed due to lack of basic information of the negative membership degree of the function. Therefore, the Atanassov covered these gaps by including the negative membership degree of the function in FS theory. The intuitionistic fuzzy set (IFS) theory has been developed by Atanassov [**1**], in 1986. The concept of the IFS theory is the extension of the FS theory. In this theory, he discussed both the negative membership degree of the function and the positive membership degree of the function of IFS. Hence, the sum of

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its positive membership function and negative membership function is equal to or less than Atanassov [**2**] defined some basic operations and relations of AIFSs, including intersection, union, complement, algebraic sum, and algebraic product, etc., and proved the equality relation between IFSs [**3**]. After the introduction of IFS theory, many researchers attempted the important role in IFS theory developed [**7, 11, 12, 13**]. In real life, there is some problem which could not be symbolized in IFS theory. For example, in the situation of voting system, human opinions including more answers of such types: yes, no, abstain, refusal. Therefore, Cuong covered these gaps by adding the neutral function in IFS theory. Cuong [**4, 5**] introduced the core idea of the PFS (picture fuzzy set) model, and the PFS notion is the extension of IFS model. In PFS theory, he basically added the neutral term along with the positive membership degree and negative membership degree of the IFS theory. The only constraint is that in PFS theory, the sum of the positive membership, neutral and negative membership degrees of the function is equal to or less than 1. In 2014, Phong et al. [**10**] developed some composition of PF relations. Cuong et al. [**6**], in 2015, gave the core idea about some fuzzy logic operations for PFS. Therefore, the focus of this paper is to develop some basic algebraic operations of PFSs and investigated their desirable properties.

The part of this paper is as follows. In *Preliminaries* section, we give some basic definitions of IFS and PFS. In *Some results on PFSs* section, we proved some algebraic properties of PFSs. In *New operation (@) on PFSs* section, we define a new operation (@) on PFSs and investigated their algebraic properties. we write the *Conclusion* of the paper in the last section.

## 2. Preliminaries

In this section, some basic concepts related to the intuitionistic fuzzy set (IFS), Picture fuzzy set(PFS) have been given.

Definition 2.1. ([**1**]) A intuitionistic fuzzy set A on a universe X is an object of the form

$$A = \{(x, \mu_A(\hat{x}), \nu_A(\hat{x}))\,|x \in X\}$$

where $\mu_A(\hat{x}) \in [0,1]$ is called the degree of membership of $x$ in $A$, $\nu_A(\hat{x}) \in [0,1]$ is called the degree of non-membership of $x$ in $A$, and where $\mu_A(\hat{x})$ and $\nu_A(\hat{x})$ satisfy the following condition:

$$0 \leqslant \mu_A(\hat{x}) + \nu_A(\hat{x}) \leqslant 1 \text{ for all } x \in X$$

Let $\pi_A(x) = 1 - \mu_A(\hat{x}) - \nu_A(\hat{x})$, then it is usually called the intuitionistic fuzzy index of $x \in A$, representing the degree of inderminancy or hesitation of $x$ to $A$. It is obvious that $0 \leqslant \pi_A(x) \leqslant 1$ for every $x \in X$.

Definition 2.2. ([**5**]) A Picture fuzzy set $A$ on a universe $X$ is an object of the form

$$A = \{(x, \mu_A(\hat{x}), \eta_A(\breve{x}), \nu_A(\hat{x}))\,|x \in X\}$$

where $\mu_A(\hat{x}) \in [0,1]$ is called the degree of positive membership of $x$ in $A$, $\eta_A(\breve{x}) \in [0,1]$ is called the degree of neutral membership of $x$ in $A$ and $\nu_A(\hat{x}) \in [0,1]$ is called the degree of negative membership of $x$ in $A$, and where $\mu_A(\hat{x})$, $\eta_A(\breve{x})$ and $\nu_A(\hat{x})$ satisfy the following condition:

$$0 \leqslant \mu_A(\hat{x}) + \eta_A(\breve{x}) + \nu_A(\hat{x}) \leqslant 1 \text{ for all } x \in X$$

Let $\pi_A(x) = 1 - \mu_A(\hat{x}) - \eta_A(\breve{x}) - \nu_A(\hat{x})$ could be called the degree of refusal membership of x in A.

Basically , picture fuzzy sets based models may be adequate in situations when we face human opinions involving more answers of type: yes, abstain, no , refusal. Voting can be a good example of such a situation as the human voters may be divided into four group of those who: vote for, abstain, vote against, refusal of the voting.

In this paper, let $PFS(X)$ denote the set of all the Picture fuzzy set on a universe $X$.

Definition 2.3. ([**5**]) Let $X$ be a nonempty set and $I$ be the unit interval [0,1]. A picture fuzzy set A and B of the form, $A = \{(x, \mu_A(\hat{x}), \eta_A(\breve{x}), \nu_A(\hat{x}))\,|x \in X\}$ and $B = \{(x, \mu_B(\hat{x}), \eta_B(\breve{x}), \nu_B(\hat{x}))\,|x \in X\}$. Then

•$A \subset B$ if and omlu if for all $x \in X$

$$\mu_A(\hat{x}) \leqslant \mu_B(\hat{x}), \eta_A(\breve{x}) \leqslant \eta_B(\breve{x}) \text{ or } \eta_A(\breve{x}) \geqslant \eta_B(\breve{x}), \nu_A(\hat{x}) \geqslant \nu_B(\hat{x})$$

•$A^C = \{(x, \nu_A(\hat{x}), \eta_A(\breve{x}), \mu_A(\hat{x}))\,|x \in X\}$

•$A \cup B = \{(x, \max(\mu_A(\hat{x}), \mu_B(\hat{x})), \min(\eta_A(\breve{x}), \eta_B(\breve{x})), \min(\nu_A(\hat{x}), \nu_B(\hat{x})))\,|x \in X\}$

•$A \cap B = \{(x, \min(\mu_A(\hat{x}), \mu_B(\hat{x})), \min(\eta_A(\breve{x}), \eta_B(\breve{x})), \max(\nu_A(\hat{x}), \nu_B(\hat{x})))\,|x \in X\}$

•$A \oplus B = \{(x, \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x}))\,|x \in X\}$

•$A \otimes B = \Big[(x, \mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x}),$
$\nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x}))|x \in X\Big].$

Definition 2.4. [**5**] The scalar multiplication operation over PFSs $A$ of the universe $X$ is denoted by $nA$ and is defined by

$$nA = \{(x, 1 - [1 - \mu_A(\hat{x})]^n, [\eta_A(\breve{x})]^n, [\nu_A(\hat{x})]^n)\,|x \in X\}$$

Definition 2.5. ([**5**]) The exponentiation operation over PFSs $A$ of the universe $X$ is denoted by $nA$ and is defined by

$$A^n = \{(x, [\mu_A(\hat{x})]^n, 1 - [1 - \eta_A(\breve{x})]^n, 1 - [1 - \nu_A(\hat{x})]^n)\,|x \in X\}.$$

Lemma 2.1 ([**9**]). *Let $x, y, z$ be real numbers. Then the following equalities hold:*

$(i)$ $x - \min(y, z) = \max(x - y, x - z)$
$(ii)$ $x - \max(y, z) = \min(x - y, x - z)$
$(iii)$ $\min(x, y) - z = \min(x - z, y - z)$
$(iv)$ $\max(x, y) - z = \max(x - z, y - z)$.

Lemma 2.2 ([**9**]). $(i)$ *If $a, b, c$ are real numbers with $a \geqslant 0$ then the following holds:*

$$a \cdot \max(b, c) = \max(ab, ac),$$
$$a \cdot \min(b, c) = \min(ab, ac).$$

*(ii) For real numbers $a, b, c$ with $a \geqslant 0$ then addition distributes over the maximum operation and also over the minimum operation:*

$$\begin{aligned} a + \max(b, c) &= \max(a + b, a + c), \\ a + \min(b, c) &= \min(a + b, a + c). \end{aligned}$$

*(iii) For real numbers, the maximum operation is distributive over the minimum operation and vice versa:*

$$\begin{aligned} \max(a, \min(b, c)) &= \min(\max(a, b), \max(a, c)), \\ \min(a, \max(b, c)) &= \max(\min(a, b), \min(a, c)). \end{aligned}$$

## 3. Some results on Picture fuzzy sets

Cuong [**5**] and Garg [**8**], defined some basic operations and relations of PFSs, including intersection, union, complement, algebraic sum, and algebraic product, etc. In this section, we prove some basic algebraic peoperties of Picture fuzzy sets, such as idempotency, commutativity, associativity, absorption, distributivity and De Morgan's laws over complement.

The following theorem relation between algebraic sum, and algebraic product.

THEOREM 3.1. *For every $A, B \in PFS(X)$, the following holds*

$$A \otimes B \subseteq A \oplus B.$$

PROOF. Let $A \oplus B = \{\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\}$
$A \otimes B = \{\mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x})\}$
Assume that,

$$\mu_A(\hat{x})\mu_B(\hat{x}) \leqslant \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})$$

$(i.e)$ $\quad \mu_A(\hat{x})\mu_B(\hat{x}) - \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}) \geqslant 0$

$(i.e)$ $\quad \mu_A(\hat{x})(1 - \mu_B(\hat{x})) + \mu_B(\hat{x})(1 - \mu_A(\hat{x})) \geqslant 0$

which is true as $0 \leqslant \mu_A(\hat{x}) \leqslant 1$ and $0 \leqslant \mu_B(\hat{x}) \leqslant 1$ and

$$\eta_A(\breve{x})\eta_B(\breve{x}) \leqslant \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x})$$

$(i.e)$ $\quad \eta_A(\breve{x})\eta_B(\breve{x}) - \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x}) \geqslant 0$

$(i.e)$ $\quad \eta_A(\breve{x})(1 - \eta_B(\breve{x})) + \eta_B(\breve{x})(1 - \eta_A(\breve{x})) \geqslant 0$

which is true as $0 \leqslant \eta_A(\breve{x}) \leqslant 1$ and $0 \leqslant \eta_B(\breve{x}) \leqslant 1$ and

$$\nu_A(\hat{x})\nu_B(\hat{x}) \leqslant \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x})$$

$(i.e)$ $\quad \nu_A(\hat{x})\nu_B(\hat{x}) - \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x}) \geqslant 0$

$(i.e)$ $\quad \nu_A(\hat{x})(1 - \nu_B(\hat{x})) + \nu_B(\hat{x})(1 - \nu_A(\hat{x})) \geqslant 0$

which is true as $0 \leqslant \nu_A(\hat{x}) \leqslant 1$ and $0 \leqslant \nu_B(\hat{x}) \leqslant 1$. Hence $A \otimes B \subseteq A \oplus B$. □

THEOREM 3.2. *For every $A \in PFS(X)$, the following holds*
*(i) $A \oplus A \supseteq A$,*
*(ii) $A \otimes A \subseteq A$.*

PROOF. (i) $A \oplus A = \left\{\left(x, 2\mu_A(\hat{x}) - (\mu_A(\hat{x}))^2, (\eta_A(\breve{x}))^2, (\nu_A(\hat{x}))^2\right) |x \in X\right\}$

$$2\mu_A(\hat{x}) - (\mu_A(\hat{x}))^2 = \mu_A(\hat{x}) + \mu_A(\hat{x})(1 - \mu_A(\hat{x})) \geqslant \mu_A(\hat{x}) \text{ for all } x \in X$$

and $(\eta_A(\breve{x}))^2 \leqslant \eta_A(\breve{x})$ for all $x \in X$

and $(\nu_A(\hat{x}))^2 \leqslant \nu_A(\hat{x})$ for all $x \in X$.

Hence $A \oplus A \supseteq A$.

Similarly, we can prove that (ii) $A \otimes A \subseteq A$. $\square$

THEOREM 3.3. *For every $A, B$ and $C \in PFS(X)$, the following holds*
*(i)* $A \oplus B = B \oplus A$,
*(ii)* $A \otimes B = B \otimes B$,
*(iii)* $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
*(iv)* $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

PROOF. (i) $A \oplus B = \{\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\}$
$= \{\mu_B(\hat{x}) + \mu_A(\hat{x}) - \mu_B(\hat{x})\mu_A(\hat{x}), \eta_B(\breve{x})\eta_A(\breve{x}), \nu_B(\hat{x})\nu_A(\hat{x})\}$
$= B \oplus A$.

(ii) $A \otimes B$
$= \{\mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x})\}$
$= \{\mu_B(\hat{x})\mu_A(\hat{x}), \eta_B(\breve{x}) + \eta_A(\breve{x}) - \eta_B(\breve{x})\eta_A(\breve{x}), \nu_B(\hat{x}) + \nu_A(\hat{x}) - \nu_B(\hat{x})\nu_A(\hat{x})\}$
$= B \otimes A$.

(iii) Let $(A \oplus B) \oplus C$

$$= \Big[x, (\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})) + \mu_C(\hat{x}) - (\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}))\mu_C(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\nu_C(\hat{x}) | x \in X\Big]$$

$$= \Big[x, \mu_A(\hat{x}) + \mu_B(\hat{x}) + \mu_C(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})\mu_C(\hat{x}) - \mu_A(\hat{x})\mu_C(\hat{x}) - \mu_B(\hat{x})\mu_C(\hat{x}) + \mu_A(\hat{x})\mu_B(\hat{x})\mu_C(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\nu_C(\hat{x}) | x \in X\Big]$$

$$= \Big[x, \mu_A(\hat{x}) + \mu_B(\hat{x}) + \mu_C(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}) - \mu_A(\hat{x})\mu_C(\hat{x}) - \mu_B(\hat{x})\mu_C(\hat{x}) + \mu_A(\hat{x})\mu_B(\hat{x})\mu_C(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\nu_C(\hat{x}) | x \in X\Big]$$

On the other hand, we have
$A \oplus (B \oplus C)$

$$= \Big[x, \mu_A(\hat{x}) + (\mu_B(\hat{x}) + \mu_C(\hat{x}) - \mu_B(\hat{x})\mu_C(\hat{x})) - \mu_A(\hat{x})(\mu_B(\hat{x}) + \mu_C(\hat{x}) - \mu_B(\hat{x})\mu_C(\hat{x})), \eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\nu_C(\hat{x}) | x \in X\Big]$$

$$= \Big[x, \mu_A(\hat{x}) + \mu_B(\hat{x}) + \mu_C(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}) - \mu_A(\hat{x})\mu_C(\hat{x}) - \mu_B(\hat{x})\mu_C(\hat{x}) + \mu_A(\hat{x})\mu_B(\hat{x})\mu_C(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x})\eta_C(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\nu_C(\hat{x}) | x \in X\Big]$$

Hence $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Similarly, we can prove that (iv) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$. $\square$

THEOREM 3.4. *For every $A, B \in PFS(X)$, the following holds*
*(i)* $A \oplus (A \otimes B) \supseteq A$,
*(ii)* $A \otimes (A \oplus B) \subseteq A$.

PROOF. (i) Let $A \oplus (A \otimes B)$

$$= \Big[x, \mu_A(\hat{x}) + \mu_A(\hat{x})\mu_B(\hat{x}) - \mu_A(\hat{x})[\mu_A(\hat{x})\mu_B(\hat{x})], \eta_A(\breve{x})[\eta_A(\breve{x}) + \eta_B(\breve{x})$$

$$- \eta_A(\breve{x})\eta_B(\breve{x})], \nu_A(\hat{x})[\nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x})] | x \in X\Big]$$
$$= \Big[x, \mu_A(\hat{x}) + \mu_A(\hat{x}) + \mu_A(\hat{x})\mu_B(\hat{x})[1 - \mu_A(\hat{x})], \eta_A(\breve{x})(1 - [1 - \eta_A(\breve{x})]$$
$$[1 - \eta_B(\breve{x})]), \nu_A(\hat{x})\left(1 - [1 - \nu_A(\hat{x})][1 - \nu_B(\hat{x})]\right) | x \in X\Big]$$
$$\geqslant A.$$

Hence $A \oplus (A \otimes B) \supseteq A$.

Similarly, we can prove that (ii) $A \otimes (A \oplus B) \subseteq A$. □

The following theorem is obvious.

THEOREM 3.5. *For every $A, B \in PFS(X)$, the following holds*
*(i)* $A \cup B = B \cup A$,
*(ii)* $A \cap B = B \cap A$,

THEOREM 3.6. *For every $A, B, C \in PFS(X)$, the following holds*
*(i)* $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$,
*(ii)* $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$,
*(iii)* $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$,
*(iv)* $A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C)$.

PROOF. In the following, we shall peove $(i)$, and $(ii) - (iv)$ can be peoved analogously.

$(i)$ Let $A \oplus (B \cup C)$
$$= \Big[x, \mu_A(\hat{x}) + \max\left(\mu_B(\hat{x}), \mu_C(\hat{x})\right) - \mu_A(\hat{x}).\max\left(\mu_B(\hat{x}), \mu_C(\hat{x})\right),$$
$$\eta_A(\breve{x}).\max\left(\eta_B(\breve{x}), \eta_C(\breve{x})\right), \nu_A(\hat{x}).\max\left(\nu_B(\hat{x}), \nu_C(\hat{x})\right) | x \in X\Big]$$
$$= \Big[x, \max\left(\mu_A(\hat{x}) + \mu_B(\hat{x}), \mu_A(\hat{x}) + \mu_C(\hat{x})\right) - \max\left(\mu_A(\hat{x})\mu_B(\hat{x}), \mu_A(\hat{x})\mu_C(\hat{x})\right),$$
$$\min\left(\eta_A(\breve{x})\eta_B(\breve{x}), \eta_A(\breve{x})\eta_C(\breve{x})\right), \min\left(\nu_A(\hat{x})\nu_B(\hat{x}), \nu_A(\hat{x})\nu_C(\hat{x})\right) | x \in X\Big]$$
by Lemma 2.2.
$$= \Big[x, \max\left(\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}), \mu_A(\hat{x}) + \mu_C(\hat{x}) - \mu_A(\hat{x})\mu_C(\hat{x})\right),$$
$$\min\left(\eta_A(\breve{x})\eta_B(\breve{x}), \eta_A(\breve{x})\eta_C(\breve{x})\right), \min\left(\nu_A(\hat{x})\nu_B(\hat{x}), \nu_A(\hat{x})\nu_C(\hat{x})\right) | x \in X\Big]$$
$$= (A \oplus B) \cup (A \oplus C).$$ □

THEOREM 3.7. *For every $A, B \in PFS(X)$, the folloholds*
*(i)* $(A \cap B) \oplus (A \cup B) = A \oplus B$,
*(ii)* $(A \cap B) \otimes (A \cup B) = A \otimes B$,
*(iii)* $(A \oplus B) \cap (A \otimes B) = A \otimes B$,
*(iv)* $(A \oplus B) \cup (A \otimes B) = A \oplus B$.

PROOF. In the following, we shall prove $(i)$, and $(ii) - (iv)$ can be proved analogously.

$(i)$ Let $(A \cap B) \oplus (A \cup B)$
$$= \Big[x, \min\left(\mu_A(\hat{x}), \mu_B(\hat{x})\right) + \max\left(\mu_A(\hat{x}), \mu_B(\hat{x})\right) - \min\left(\mu_A(\hat{x}), \mu_B(\hat{x})\right).$$
$$\max\left(\mu_A(\hat{x}), \mu_B(\hat{x})\right), \max\left(\eta_A(\breve{x}), \eta_B(\breve{x})\right).\min\left(\eta_A(\breve{x}), \eta_B(\breve{x})\right),$$

$\max(\nu_A(\hat{x}), \nu_B(\hat{x})) . \min(\nu_A(\hat{x}), \nu_B(\hat{x})) \,|x \in X\Big]$
$= \{(x, \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x}))\,|x \in X\}$
$= A \oplus B.$ □

In the following theorems, the operator complement obey th De Morgan's laws for the operation $\oplus, \otimes, \cup, \cap$.

Theorem 3.8. *For every $A, B \in PFS(X)$, the the following holds*
$(i)$ $(A \oplus B)^C = A^C \otimes B^C$,
$(ii)$ $(A \otimes B)^C = A^C \oplus B^C$,
$(iii)$ $(A \oplus B)^C \subseteq A^C \oplus B^C$,
$(iv)$ $(A \otimes B)^C \supseteq A^C \otimes B^C$.

Proof. We shall prove $(iii), (iv)$, and $(i), (ii)$ are straightfforward.
(iii) $(A \oplus B)^C$
$= \{\nu_A(\hat{x})\nu_B(\hat{x}), \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x}), \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})\}$.
$A^C \oplus B^C = \{(x, \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \mu_A(\hat{x})\mu_B(\hat{x}))\,|x \in X\}$.
Since $\nu_A(\hat{x})\nu_B(\hat{x}) \leqslant \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x})$
$\eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x}) \geqslant \eta_A(\breve{x})\eta_B(\breve{x})$
$\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}) \geqslant \mu_A(\hat{x})\mu_B(\hat{x})$
Hence $(A \oplus B)^C \subseteq A^C \oplus B^C$.
(iv) $(A \otimes B)^C$
$= \{(x, \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \mu_A(\hat{x})\mu_B(\hat{x}))\,|x \in X\}$.
$A^C \otimes B^C$
$= \{\nu_A(\hat{x})\nu_B(\hat{x}), \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x}), \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})\}$.
Since $\nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x}) \geqslant \nu_A(\hat{x})\nu_B(\hat{x})$
$\eta_A(\breve{x})\eta_B(\breve{x}) \leqslant \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x})$
$\mu_A(\hat{x})\mu_B(\hat{x}) \leqslant \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})$
Hence $(A \otimes B)^C \supseteq A^C \otimes B^C$. □

Theorem 3.9. *For every $A, B, C \in PFS(X)$, the the following holds*
$(i)$ $(A^C)^C = A$,
$(ii)$ $(A \cup B)^C = A^C \cap B^C$,
$(iii)$ $(A \cap B)^C = A^C \cup B^C$.

Proof. We shall prove $(ii)$ only, $(i)$ is obvious.
$A \cup B = \{(x, \max(\mu_A(\hat{x}), \mu_B(\hat{x})), \min(\eta_A(\breve{x}), \eta_B(\breve{x})), \min(\nu_A(\hat{x}), \nu_B(\hat{x})))\,|x \in X\}$
$(A \cup B)^C$
$= \{(x, \min(\nu_A(\hat{x}), \nu_B(\hat{x})), \min(\eta_A(\breve{x}), \eta_B(\breve{x})), \max(\mu_A(\hat{x}), \mu_B(\hat{x})))\,|x \in X\}$
$\Rightarrow A^C = \{(x, \nu_A(\hat{x}), \eta_A(\breve{x}), \mu_A(\hat{x}))\,|x \in X\}$
$B^C = \{(x, \nu_B(\hat{x}), \eta_B(\breve{x}), \mu_B(\hat{x}))\,|x \in X\}$
$\Rightarrow A^C \cap B^C$
$= \{(x, \min(\nu_A(\hat{x}), \nu_B(\hat{x})), \min(\eta_A(\breve{x}), \eta_B(\breve{x})), \max(\mu_A(\hat{x}), \mu_B(\hat{x})))\,|x \in X\}$
Hence $(A \cup B)^C = A^C \cap B^C$,
Similarly, we can prove that $(iii)$ $(A \cap B)^C = A^C \cup B^C$. □

Based on the Definition 2.4 and Definition 2.5., we shall next prove the algebraic properties of Picture fuzzy sets under the operations of scalar multiplication and exponentiation.

THEOREM 3.10. *For every $A, B \in PFS(X)$ and for any positive number $n$, the following holds*
$(i)$ $n(A + B) = nA \oplus nB, n > 0,$
$(ii)$ $n_1 A \oplus n_2 A = (n_1 + n_2)A, n_1, n_2 > 0,$
$(iii)$ $(A \otimes B)^n = A^n \otimes B^n, n > 0,$
$(iv)$ $A_1^n \otimes A_2^n = A^{(n_1+n_2)}, n_1, n_2 > 0.$

PROOF. For the three PFSs $A$ and $B$, and $n, n_1, n_2 > 0$, according to definition, we can obtain

(i) $n(A \oplus B)$
$= n\{\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x})\}$
$= \{1 - [1 - \mu_A(\hat{x})]^n[1 - \mu_A(\hat{x})]^n, [\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\nu_A(\hat{x})\nu_B(\hat{x})]^n\}$
$= \{1 - [1 - \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})]^n, [\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\nu_A(\hat{x})\nu_B(\hat{x})]^n\}$

$nA \oplus nB$
$= \{(1 - [1 - \mu_A(\hat{x})]^n, [\eta_A(\breve{x})]^n, [\nu_A(\hat{x})]^n) \oplus (1 - [1 - \mu_B(\hat{x})]^n, [\eta_B(\breve{x})]^n, [\nu_B(\hat{x})]^n)\}$

$$= \Big[ (1 - [1 - \mu_A(\hat{x})]^n + 1 - [1 - \mu_A(\hat{x})]^n) - (1 - [1 - \mu_A(\hat{x})]^n) \\ (1 - [1 - \mu_B(\hat{x})]^n), [\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\nu_A(\hat{x})\nu_B(\hat{x})]^n \Big]$$

$= \{1 - [1 - \mu_A(\hat{x})]^n[1 - \mu_B(\hat{x})]^n, [\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\nu_A(\hat{x})\nu_B(\hat{x})]^n\}$
$= \{1 - [1 - \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})]^n, [\eta_A(\breve{x})\eta_B(\breve{x})]^n, [\nu_A(\hat{x})\nu_B(\hat{x})]^n\}$
$= n(A \oplus B).$

(ii) $n_1 A \oplus n_2 B$

$$= \Big[1 - [1 - \mu_A(\hat{x})]^{n_1} + 1 - [1 - \mu_A(\hat{x})]^{n_2} - (1 - [1 - \mu_A(\hat{x})]^{n_1})(1 - [1 - \mu_A(\hat{x})]^{n_2}), \\ [\eta_A(\breve{x})]^{n_1}[\eta_A(\breve{x})]^{n_2}, [\nu_A(\hat{x})]^{n_1}[\nu_A(\hat{x})]^{n_2}\Big]$$

$= \{1 - [1 - \mu_A(\hat{x})]^{n_1+n_2}, [\eta_A(\breve{x})]^{n_1+n_2}, [\nu_A(\hat{x})]^{n_1+n_2}\}$
$= (n_1 + n_2)A.$

(iii) $(A \otimes B)^n$

$$= \Big[ (\mu_A(\hat{x})\mu_B(\hat{x}))^n, 1 - [1 - \eta_A(\breve{x}) + \eta_B(\breve{x}) - \eta_A(\breve{x})\eta_B(\breve{x})]^n, \\ 1 - [1 - \nu_A(\hat{x}) + \nu_B(\hat{x}) - \nu_A(\hat{x})\nu_B(\hat{x})]^n \Big]$$

$$= \Big[ (\mu_A(\hat{x})\mu_B(\hat{x}))^n, 1 - [1 - \eta_A(\breve{x})]^n[1 - \eta_B(\breve{x})]^n, \\ 1 - [1 - \nu_A(\hat{x})]^n[1 - \nu_B(\hat{x})]^n \Big]$$

$A^n \otimes B^n$

$$= \Big[ (\mu_A(\hat{x})\mu_B(\hat{x}))^n, 1 - [1 - \eta_A(\breve{x})]^n + 1 - [1 - \eta_B(\breve{x})]^n - (1 - [1 - \eta_A(\breve{x})]^n)\, 1 - [1 - \\ \eta_B(\breve{x})]^n, 1 - [1 - \nu_A(\hat{x})]^n + 1 - [1 - \nu_B(\hat{x})]^n - (1 - [1 - \nu_A(\hat{x})]^n)(1 - [1 - \nu_B(\hat{x})]^n) \Big]$$

$= \{(\mu_A(\hat{x})\mu_B(\hat{x}))^n, 1 - [1 - \eta_A(\breve{x})]^n[1 - \eta_B(\breve{x})]^n, 1 - [1 - \nu_A(\hat{x})]^n[1 - \nu_B(\hat{x})]^n\} = (A \otimes B)^n.$

(iv) $A^{n_1} \otimes A^{n_2}$

$= \Big[ (\mu_A(\hat{x}))^{n_1+n_2}, 1 - [1-\eta_A(\check{x})]^{n_1} + 1 - [1-\eta_A(\check{x})]^{n_2} - (1-[1-\eta_A(\check{x})]^{n_1})$
$\left(1-[1-\eta_A(\check{x})]^{n-2}\right), 1-[1-\nu_A(\hat{x})]^{n_1} + 1 - [1-\nu_A(\hat{x})]^{n-2} - (1-[1-\nu_A(\hat{x})]^{n_1})$
$(1-[1-\nu_A(\hat{x})]^{n_2}) \Big]$
$= \Big\{ (\mu_A(\hat{x}))^{n_1+n_2}, 1-[1-\eta_A(\check{x})]^{n_1+n_2}, 1-[1-\nu_A(\hat{x})]^{n_1+n_2} \Big\}$
$= A^{(n_1+n_2)}$.
which complete the proof of the theorem. ☐

THEOREM 3.11. *For every $A, B \in PFS(X)$ and for any positive number $n$, the following holds*
*(i)* $nA \subseteq nB$,
*(ii)* $A^n \subseteq B^n$.

PROOF. *(i)* Let $A \subseteq B$
$\Rightarrow \mu_A(\hat{x}) \leqslant \mu_B(\hat{x})$ and $\eta_A(\check{x}) \geqslant \eta_B(\check{x})$ and $\nu_A(\hat{x}) \geqslant \nu_B(\hat{x})$ for all $x \in X$.
$\Rightarrow 1-[1-\mu_A(\hat{x})]^n \leqslant 1-[1-\mu_B(\hat{x})]^n$,
$[\eta_A(\check{x})]^n \geqslant [\eta_B(\check{x})]^n$ and
$[\nu_A(\hat{x})]^n \geqslant [\nu_B(\hat{x})]^n$. for all $x \in X$.

(ii) Also, $[\mu_A(\hat{x})]^n \geqslant [\mu_B(\hat{x})]^n$,
$1-[1-\eta_A(\check{x})]^n \leqslant 1-[1-\eta_B(\check{x})]^n$,
$1-[1-\nu_A(\hat{x})]^n \leqslant 1-[1-\nu_B(\hat{x})]^n$, for all $x \in X$. ☐

THEOREM 3.12. *For every $A, B \in PFS(X)$, and for any positive number $n$, the following holds*
*(i)* $n(A \cap B) = nA \cap nB$,
*(ii)* $n(A \cup B) = nA \cup nB$.

PROOF. (i) Let $n(A \cap B)$

$$= \Big[x, 1-[1-\min(\mu_A(\hat{x}), \mu_B(\hat{x}))]^n, \max([\eta_A(\check{x})]^n, [\eta_B(\check{x})]^n), \\ \max([\nu_A(\hat{x})]^n, [\nu_B(\hat{x})]^n) \,|x \in X\Big]$$

$$= \Big[x, 1-[\max(1-\mu_A(\hat{x}), 1-\mu_B(\hat{x}))]^n, \max([\eta_A(\check{x})]^n, [\eta_B(\check{x})]^n), \\ \max([\nu_A(\hat{x})]^n, [\nu_B(\hat{x})]^n) \,|x \in X\Big] \text{ by Lemma 2.1.}$$

$$= \Big[x, 1-(\max([1-\mu_A(\hat{x})]^n, [1-\mu_B(\hat{x})]^n)), \max([\eta_A(\check{x})]^n, [\eta_B(\check{x})]^n), \\ \max([\nu_A(\hat{x})]^n, [\nu_B(\hat{x})]^n) \,|x \in X\Big]$$

$$= \Big[x, \max(1-[1-\mu_A(\hat{x})]^n, 1-[1-\mu_B(\hat{x})]^n), \max([\eta_A(\check{x})]^n, [\eta_B(\check{x})]^n), \\ \max([\nu_A(\hat{x})]^n, [\nu_B(\hat{x})]^n) \,|x \in X\Big]$$

$= nA \cap nB$. Hence $n(A \cap B) = nA \cap nB$,
Similarly, we can prove that $(ii) n(A \cup B) = nA \cup nB$. ☐

THEOREM 3.13. *For every $A, B \in PFS(X)$ and for any positive number $n$, the following holds*

$(i)$ $(A \cap B)^n = A^n \cap B^n$,
$(ii)$ $(A \cup B)^n = A^n \cup B^n$.

PROOF. (i) Let

$$(A \cap B)^n = \Big[x, \min\left([\mu_A(\hat{x})]^n, [\mu_B(\hat{x})]^n\right), 1 - [\max\left(1-\eta_A(\breve{x}), 1-\eta_B(\breve{x})\right)]^n,$$
$$1 - [\max\left(1-\nu_A(\hat{x}), 1-\nu_B(\hat{x})\right)]^n | x \in X\Big]$$
$$= \Big[x, \min\left([\mu_A(\hat{x})]^n, [\mu_B(\hat{x})]^n\right), 1 - \left(\min\left([1-\eta_A(\breve{x})]^n, [1-\eta_B(\breve{x})]^n\right)\right),$$
$$1 - \left(\min\left([1-\nu_A(\hat{x})]^n, [1-\nu_B(\hat{x})]^n\right)\right) | x \in X\Big]$$
$$= \Big[x, \min\left([\mu_A(\hat{x})]^n, [\mu_B(\hat{x})]^n\right), \max\left(1-[1-\eta_A(\breve{x})]^n, 1-[1-\eta_B(\breve{x})]^n\right),$$
$$\max\left(1-[1-\nu_A(\hat{x})]^n, 1-[1-\nu_B(\hat{x})]^n\right) | x \in X\Big]$$
$$A^n \cap B^n = \Big[x, \left([\mu_A(\hat{x})]^n, 1-[1-\eta_A(\breve{x})]^n, 1-[1-\nu_A(\hat{x})]^n\right) \bigcap$$
$$\left([\mu_B(\hat{x})]^n, 1-[1-\eta_B(\breve{x})]^n, 1-[1-\nu_B(\hat{x})]^n\right) | x \in X\Big]$$
$$= \Big[x, \min\left([\mu_A(\hat{x})]^n, [\mu_B(\hat{x})]^n\right), \max\left(1-[1-\eta_A(\breve{x})]^n, 1-[1-\eta_B(\breve{x})]^n\right),$$
$$\max\left(1-[1-\nu_A(\hat{x})]^n, 1-[1-\nu_B(\hat{x})]^n\right) | x \in X\Big]$$
$$= (A \cap B)^n.$$

Hence $(A \cap B)^n = A^n \cap B^n$,

Similarly, we can prove that $(ii) (A \cup B)^n = A^n \cup B^n$. □

THEOREM 3.14. *For every $A, B \in PFS(X)$,then for any positive number $n$, $(A \oplus B)^n \neq A^n \oplus B^n$.*

PROOF. Let $(A \oplus B)^n$

$$= \Big[x, [\mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x})]^n, 1 - [1 - \eta_A(\breve{x})\eta_B(\breve{x})]^n,$$
$$1 - [1 - \nu_A(\hat{x})\nu_B(\hat{x})]^n | x \in X\Big]$$
$$A^n = \{(x, [\mu_A(\hat{x})]^n, 1 - [1-\eta_A(\breve{x})]^n, 1 - [1-\nu_A(\hat{x})]^n) | x \in X\}$$
$$B^n = \{(x, [\mu_B(\hat{x})]^n, 1 - [1-\eta_B(\breve{x})]^n, 1 - [1-\nu_B(\hat{x})]^n) | x \in X\}$$
$$A^n \oplus B^n = \Big[x, [\mu_A(\hat{x})]^n + [\mu_B(\hat{x})]^n - [\mu_A(\hat{x})]^n[\mu_B(\hat{x})]^n, [1-[1-\eta_A(\breve{x})]^n]^n.$$
$$[1-[1-\eta_B(\breve{x})]^n]^n, [1-[1-\nu_A(\hat{x})]^n]^n.[1-[1-\nu_A(\hat{x})]^n]^n | x \in X\Big]$$

Hence $(A \oplus B)^n \neq A^n \oplus B^n$. □

Based on the Definition 2.4 and Definition 2.5, scalar multiplication and exponentiation operations of PFSs, we can define new concentration and dilation of the PFS as follows.

DEFINITION 3.1. The concentration of a PFS $A$ in the universe $X$ is denoted by $CON(A)$ and is defined by

$$CON(A) = \left\{\left(x, \mu_{CON(A)}(\hat{x}), \eta_{CON(A)}(\breve{x}), \nu_{CON(A)}(\hat{x})\right) | x \in X\right\}$$

where

$$\mu_{CON(A)}(\hat{x}) = [\mu_A(\hat{x})]^2,\ \eta_{CON(A)}(\breve{x}) = 1 - [1 - \eta_A(\breve{x})]^2,$$
$$\nu_{CON(A)}(\hat{x}) = 1 - [1 - \nu_A(\hat{x})]^2.$$

In other words, concentration of a PFS is defined by $CON(A) = A^2$.

Definition 3.2. The dilation of a PFS $A$ in the universe $X$ is denoted by $DIL(A)$ and is defined by

$$DIL(A) = \left\{ \left(x, \mu_{DIL(A)}(x), \eta_{DIL(A)}(\breve{x}), \nu_{DIL(A)}(\hat{x})\right) |x \in X \right\}$$

where

$$\mu_{DIL(A)}(\hat{x}) = [\mu_A(\hat{x})]^{\frac{1}{2}},\ \eta_{DIL(A)}(\breve{x}) = 1 - [1 - \eta_A(\breve{x})]^{\frac{1}{2}},$$
$$\nu_{DIL(A)}(\hat{x}) = 1 - [1 - \nu_A(\hat{x})]^{\frac{1}{2}}.$$

In other words, dilation of a PFS is defined by $DIL(A) = A^{\frac{1}{2}}$.

The following theorem are now straightforward.

Theorem 3.15. *For every $A \in PFS(X)$, the following holds*
(i) $CON(A) \subseteq A \subseteq DIL(A)$,
(ii) *If $\pi_A(x) = 0$, then $\pi_{CON(A)}(x) = 0$,*
(iii) *If $\pi_A(x) = 0$, then $\pi_{DIL(A)}(x) = 0$.*

## 4. New operation on Picture fuzzy sets

In this section, we define a new operation (@) on Picture fuzzy sets and proved their algebraic properties. Further, we discuss the Disstributivity laws in the case where the operations of $\oplus, \otimes, \cup$ and $\cap$ combined each other.

Definition 4.1. Let $X$ be a nonempty set and I be the unit interval [0,1]. A picture fuzzy set A and B of the form, $A = \{(x, \mu_A(\hat{x}), \eta_A(\breve{x}), \nu_A(\hat{x}))\,|x \in X\}$ and $B = \{(x, \mu_B(\hat{x}), \eta_B(\breve{x}), \nu_B(\hat{x}))\,|x \in X\}$. Then

$$A@B = \left\{ \left( x, \frac{\mu_A(\hat{x}) + \mu_B(\hat{x})}{2}, \frac{\eta_A(\breve{x}) + \eta_B(\breve{x})}{2}, \frac{\nu_A(\hat{x}) + \nu_B(\hat{x})}{2} \right) |x \in X \right\}.$$

Remark 4.1. Obviously, for every two PFSs A and B, then A@B is a PFS. Simple illustration given: For A@B,

$$\begin{aligned} 0 &\leqslant \frac{\mu_A(\hat{x}) + \mu_B(\hat{x})}{2} + \frac{\eta_A(\breve{x}) + \eta_B(\breve{x})}{2} + \frac{\nu_A(\hat{x}) + \nu_B(\hat{x})}{2} \\ &\leqslant \frac{\mu_A(\hat{x}) + \eta_A(\breve{x}) + \nu_A(\hat{x})}{2} + \frac{\mu_B(\hat{x}) + \eta_B(\breve{x}) + \nu_B(\hat{x})}{2} \leqslant \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Theorem 4.1. *For every $A \in PFS(X)$, holds $A@A = A$.*

Proof. Let $A@A$

$$= \left\{ \left( x, \frac{\mu_A(\hat{x}) + \mu_A(\hat{x})}{2}, \frac{\eta_A(\breve{x}) + \eta_A(\breve{x})}{2}, \frac{\nu_A(\hat{x}) + \nu_A(\hat{x})}{2} \right) |x \in X \right\}$$

$$= \left\{ \left( x, \frac{2\mu_A(\hat{x})}{2}, \frac{2\eta_A(\breve{x})}{2}, \frac{2\nu_A(\hat{x})}{2} \right) | x \in X \right\}$$
$$= \{(x, \mu_A(\hat{x}), \eta_A(\breve{x}), \nu_A(\hat{x})) \, | x \in X\}$$
$$= A. \qquad \square$$

THEOREM 4.2. *For every $A, B, C \in PFS(X)$, the following holds*
*(i)* $A@(B \cup C) = (A@B) \cup (A@C)$,
*(ii)* $A@(B \cup C) = (A@B) \cup (A@C)$.

PROOF. (i) Let $(A@B) \cup (A@C)$

$$= \left[ \max\left( \frac{\mu_A(\hat{x}) + \mu_B(\hat{x})}{2}, \frac{\mu_A(\hat{x}) + \mu_C(\hat{x})}{2} \right), \right.$$
$$\left. \min\left( \frac{\eta_A(\breve{x}) + \eta_B(\breve{x})}{2}, \frac{\eta_A(\breve{x}) + \eta_C(\breve{x})}{2} \right), \min\left( \frac{\nu_A(\hat{x}) + \nu_B(\hat{x})}{2}, \frac{\nu_A(\hat{x}) + \nu_C(\hat{x})}{2} \right) \right]$$

$A@(B \cup C)$

$$= \left[ x, \frac{\mu_A(\hat{x}) + \max(\mu_B(\hat{x}), \mu_C(\hat{x}))}{2}, \frac{\eta_A(\breve{x}) + \min(\eta_B(\breve{x}), \eta_C(\breve{x}))}{2}, \right.$$
$$\left. \frac{\nu_A(\hat{x}) + \min(\nu_B(\hat{x}), \nu_C(\hat{x}))}{2} | x \in X \right]$$
$$= \left[ \max\left( \frac{\mu_A(\hat{x}) + \mu_B(\hat{x})}{2}, \frac{\mu_A(\hat{x}) + \mu_C(\hat{x})}{2} \right), \right.$$
$$\left. \min\left( \frac{\eta_A(\breve{x}) + \eta_B(\breve{x})}{2}, \frac{\eta_A(\breve{x}) + \eta_C(\breve{x})}{2} \right), \min\left( \frac{\nu_A(\hat{x}) + \nu_B(\hat{x})}{2}, \frac{\nu_A(\hat{x}) + \nu_C(\hat{x})}{2} \right) \right]$$

by Lemma 2.2. Hence, $A@(B \cup C) = (A@B) \cup (A@C)$

(ii) It can be prove similarly, $A@(B \cup C) = (A@B) \cup (A@C)$. $\square$

REMARK 4.2. For $a, b \in [0, 1]$, holds

$$ab \leqslant \frac{a+b}{2} \text{ and } \frac{a+b}{2} \leqslant a + b - ab.$$

THEOREM 4.3. *For every $A, B \in PFS(X)$, the following holds*
*(i)* $(A \oplus B) \cup (A@B) = A \oplus B$,
*(ii)* $(A \otimes B) \cap (A@B) = A \otimes B$,
*(iii)* $(A \oplus B) \cap (A@B) = A@B$,
*(iv)* $(A \otimes B) \cup (A@B) = A@B$.

PROOF. In the following, we shall prove only (i),and (ii),(iii) and (iv) can be proved analogously.

(i) Let $(A \oplus B) \cup (A@B)$

$$= \left[ \max\left( \mu_A(\hat{x}) + \mu_B(\hat{x}) - \mu_A(\hat{x})\mu_B(\hat{x}), \frac{\mu_A(\hat{x}) + \mu_B(\hat{x})}{2} \right), \right.$$

$$\min\left(\eta_A(\breve{x})\eta_B(\breve{x}), \frac{\eta_A(\breve{x})+\eta_B(\breve{x})}{2}\right), \min\left(\nu_A(\hat{x})\nu_B(\hat{x}), \frac{\nu_A(\hat{x})+\nu_B(\hat{x})}{2}\right)\Bigg]$$
$$= \{(x, \mu_A(\hat{x})+\mu_B(\hat{x})-\mu_A(\hat{x})\mu_B(\hat{x}), \eta_A(\breve{x})\eta_B(\breve{x}), \nu_A(\hat{x})\nu_B(\hat{x}))\,|x \in X\}$$
$$= A \oplus B. \qquad \square$$

REMARK 4.3. The Picture fuzzy set forms a semilattice, associativity, commutativity, idempotency under the Picture fuzzy set operation of algebraic sum and algebraic product. The distributive law also holds for $\oplus, \otimes$ and $\wedge, \vee, @$ are combined each other.

## 5. Applications

The formation of Picture fuzzy semilattice structure, Picture fuzzy set and algebraic structure on this set, the results are applicable.

## 6. Conclusion

In this paper, we have established some algebraic properties of Picture fuzzy sets, such as idempotency, commutativity, associativity, absorption, distributivity and De Morgan's laws over complement. Furthermore, we have defined new concentration and dilation of PFSs and proved some theorems. Finally, we defined a new operation (@) on Picture fuzzy sets and discussed distributive laws in the case where the operations of $\oplus, \otimes, \cup$ and $\cap$ are combined each other. This result can be applied further application of Picture fuzzy set theory. For the development of Picture fuzzy semilattice and its algebraic property the results of this paper would be helpful. In the future, the application of the proposed aggregating operators of PFSs needs to be explored in the decision making, risk analysis and many other uncertain and fuzzy environment.

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Assistant Professor, Department of Mathematics, Sacred Heart Arts and Science College, Perani-605652, Tamilnadu, India
*E-mail address*: sksimbuking@gmail.com