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ON VAGUE SUBBISEMIRINGS OF BISEMIRINGS

M. Palanikumar and K. Arulmozhi

Abstract. We discuss the notion of vague subbisemiring, level sets of vague subbisemirings and vague normal subbisemiring of a bisemiring. Also, we investigate some properties related to these subbisemirings. The homomorphic image of vague subbisemirings is also vague subbisemirings. The homomorphic preimage of vague subbisemirings is also vague subbisemirings. To illustrate our results and examples are given.

1. Introduction

L. A. Zadeh [**17**] proposed by fuzzy set theory in 1965. In 1993, J. Ahsan, K. Saifullah, and F. Khan [**1**] introduced the notion of fuzzy semirings. In 2001, M. K Sen and S. Ghosh was introduced in bisemirings. W. L. Gau and D. J. Buehrer [**10**] proposed the theory of vague sets as an improvement of theory of fuzzy sets in approximating the real life situations. R. Biswas [**7**] initiated the study of vague algebra by introducing the concepts of vague groups, vague cuts, vague normal groups. A bisemiring $(\mathbb{S}, +, \circ, \times)$ is an algebraic structure in which $(\mathbb{S}, +, \circ)$ and $(\mathbb{S}, \circ, \times)$ are semirings in which $(\mathbb{S}, +), (\mathbb{S}, \circ)$ and $(\mathbb{S}, \times)$ are semigroups such that ([**16**])

(i) $s_1 \circ (s_2 + s_3) = (s_1 \circ s_2) + (s_1 \circ s_3)$,
(ii) $(s_2 + s_3) \circ s_1 = (s_2 \circ s_1) + (s_3 \circ s_1)$
(iii) $s_1 \times (s_2 \circ s_3) = (s_1 \times s_2) \circ (s_1 \times s_3)$, and
(iv) $(s_2 \circ s_3) \times s_1 = (s_2 \times s_1) \circ (s_3 \times s_1), \forall\ s_1, s_2, s_3 \in \mathbb{S}$.

A non-empty subset A of $\mathbb{S}$ is a subbisemiring if and only if $s_1 + s_2, s_1 \circ s_2, s_1 \times s_2 \in A$ for all $s_1, s_2 \in A$ [**9**].

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2. Preliminaries

Notations: (i) vague subsemirings, vague subbisemirings, vague normal subsemirings, vague normal subbisemirings shortly VSS, VSBS, VNSS, VNSBS respectively.

(ii) fuzzy subsemirings, fuzzy normal subsemirings, fuzzy subbisemirings, subbisemirings shortly FSS, FNSS, FSBS, SBS respectively.

Definition 2.1. Let $(\mathbb{S}, +, \cdot)$ be semiring.
(i) A fuzzy subset L of $\mathbb{S}$ is said to be FSS if
(a) $\mu_L(s_1 + s_2) \geqslant \min\{\mu_L(s_1), \mu_L(s_2)\}$
(b) $\mu_L(s_1 \cdot s_2) \geqslant \min\{\mu_L(s_1), \mu_L(s_2)\}$ for all $s_1, s_2 \in \mathbb{S}$.
(ii) A fuzzy subset L of $\mathbb{S}$ is said to be a FNSS of $\mathbb{S}$ if
(a) $\mu_L(s_1 + s_2) = \mu_L(s_2 + s_1)$
(b) $\mu_L(s_1 \cdot s_2) = \mu_L(s_2 \cdot s_1)$ for all $s_1, s_2 \in \mathbb{S}$.

Definition 2.2. Let L and M be fuzzy subset of G and H respectively. The product of L and M denoted by $L \times M$ is defined as
$$L \times M = \{\mu_{L\times M}(s_1, s_2) |\ for\ all\ s_1 \in G\ and\ s_2 \in H\},$$
where $\mu_{L\times M}(s_1, s_2) = \min\{\mu_L(s_1), \mu_M(s_2)\}$.

Definition 2.3. ([**9**]) Let $(\mathbb{S}_1, +, \cdot, \times)$ and $(\mathbb{S}_2, \oplus, \circ, \otimes)$ be two bisemirings. A function $\theta : \mathbb{S}_1 \to \mathbb{S}_2$ is said to be a homomorphism if
(i) $\theta(s_1 + s_2) = \theta(s_1) \oplus \theta(s_2)$,
(ii) $\theta(s_1 \cdot s_2) = \theta(s_1) \circ \theta(s_2)$,
(iii) $\theta(s_1 \times s_2) = \theta(s_1) \otimes \theta(s_2)$
for all $s_1, s_2 \in \mathbb{S}_1$.

Definition 2.4. A vague set $L = (\mathcal{T}_L, \mathcal{F}_L)$ of $\mathbb{S}$ is said to be vague semiring if the following conditions are true: For all $s_1, s_2 \in \mathbb{S}$,
(i) $V_L(s_1 + s_2) \geqslant min\{V_L(s_1), V_L(s_2)\}$ and
(ii) $V_L(s_1 \cdot s_2) \geqslant min\{V_L(s_1), V_L(s_2)\}$.
That is
$$\begin{cases} \mathcal{T}_L(s_1 + s_2) \geqslant min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} \\ \mathcal{T}_L(s_1 \cdot s_2) \geqslant min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} \end{cases}$$
and
$$\begin{cases} 1 - \mathcal{F}_L(s_1 + s_2) \geqslant min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\} \\ 1 - \mathcal{F}_L(s_1 \cdot s_2) \geqslant min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\} \end{cases}$$

Definition 2.5. ([**7**]) (i) A vague set L in the universe U is a pair $(\mathcal{T}_L, \mathcal{F}_L)$, where $\mathcal{T}_L : U \to [0, 1], \mathcal{F}_L : U \to [0, 1]$ are mappings such that $\mathcal{T}_L(u) + \mathcal{F}_L(u) \leqslant 1$, for all $u \in U$. The functions $\mathcal{T}_L$ and $\mathcal{F}_L$ are called true membership function and false membership function respectively.

(ii) The interval $[\mathcal{T}_L(u), 1 - \mathcal{F}_L(u)]$ is called the vague value of u in L and it is denoted by $V_L(u)$, i.e., $V_L(u) = [\mathcal{T}_L(u), 1 - \mathcal{F}_L(u)]$.

Definition 2.6. ([**7**]) (i) A vague set L is contained in the other vague set M, $L \subseteq M$ if and only if $V_L(u) \leqslant V_M(u)$, i.e. $\mathcal{T}_L(u) \leqslant \mathcal{T}_M(u)$ and $1 - \mathcal{F}_L(u) \leqslant 1 - \mathcal{F}_M(u)$, for all $u \in U$.

(ii) The union of two vague sets L and M, as $N = L \cup M$, $\mathcal{T}_N = max\{\mathcal{T}_L, \mathcal{T}_M\}$ and $1 - \mathcal{F}_N = max\{1 - \mathcal{F}_L, 1 - \mathcal{F}_M\} = 1 - min\{\mathcal{F}_L, \mathcal{F}_M\}$.

(iii) The intersection of two vague sets L and M as $N = L \cap M$, $\mathcal{T}_N = min\{\mathcal{T}_L, \mathcal{T}_M\}$ and $1 - \mathcal{F}_N = min\{1 - \mathcal{F}_L, 1 - \mathcal{F}_M\} = 1 - max\{\mathcal{F}_L, \mathcal{F}_M\}$.

Definition 2.7. ([**7**]) A vague set L of a set U, for all $u \in U$ with (i) $\mathcal{T}_L(u) = 0$ and $\mathcal{F}_L(u) = 1$ is called zero vague set of U.

(ii) $\mathcal{T}_L(u) = 1$ and $\mathcal{F}_L(u) = 0$ is called unit vague set of U.

Definition 2.8. ([**7**]) Let L be a vague set of a universe U with true membership function $\mathcal{T}_L$ and false membership function $\mathcal{F}_L$. For $\alpha, \beta \in [0, 1]$ with $\alpha \leqslant \beta$, the (α, β)-cut or vague cut of a vague set L is the crisp subset of U is given by $L_{(\alpha,\beta)} = \{u \in U | V_L(u) \geqslant [\alpha, \beta]\}$. That is,

$$L_{(\alpha,\beta)} = \{u \in U | \mathcal{T}_L(u) \geqslant \alpha,\ \ 1 - \mathcal{F}_L(u) \geqslant \beta]\}.$$

Definition 2.9. Let L and M be any two vague set in U.
(i) $L \cap M = \left\{ \langle u, \min\{\mathcal{T}_L(u), \mathcal{T}_M(u)\}, \min\{1 - \mathcal{F}_L(u), 1 - \mathcal{F}_M(u)\}\rangle \right\}$
(ii) $L \cup M = \left\{ \langle u, \max\{\mathcal{T}_L(u), \mathcal{T}_M(u)\}, \max\{1 - \mathcal{F}_L(u), 1 - \mathcal{F}_M(u)\}\rangle \right\}$
(iii) $\Box L = \left\{ \langle u, \mathcal{T}_L(u), 1 - \mathcal{T}_L(u)\rangle | u \in U \right\}$
(iv) $\Diamond L = \left\{ \langle u, 1 - \mathcal{F}_L(u), \mathcal{F}_L(u)\rangle | u \in U \right\}$ for all $u \in U$.

Definition 2.10. ([**12**]) A fuzzy subset L of $\mathbb{S}$ is said to be a FSBS if

$$\begin{cases} \mu_L(s_1 *_1 s_2) \geqslant \min\{\mu_L(s_1), \mu_L(s_2)\} \\ \mu_L(s_1 *_2 s_2) \geqslant \min\{\mu_L(s_1), \mu_L(s_2)\} \\ \mu_L(s_1 *_3 s_2) \geqslant \min\{\mu_L(s_1), \mu_L(s_2)\} \end{cases}$$

for all $s_1, s_2 \in \mathbb{S}$.

Definition 2.11. ([**12**]) A fuzzy subset L of $\mathbb{S}$ is said to be a FNSBS if

$$\begin{cases} \mu_L(s_1 *_1 s_2) = \mu_L(s_2 *_1 s_1) \\ \mu_L(s_1 *_2 s_2) = \mu_L(s_2 *_2 s_1) \\ \mu_L(s_1 *_3 s_2) = \mu_L(s_2 *_3 s_1) \end{cases}$$

for all $s_1, s_2 \in \mathbb{S}$.

3. Vague Subbisemirings

Here $\mathbb{S}$ denotes a bisemiring unless otherwise stated.

Definition 3.1. A vague set $L = (\mathcal{T}_L, \mathcal{F}_L)$ of $\mathbb{S}$ is said to be vague subbisemiring (shortly VSBS) if the following conditions are true: For all $s_1, s_2 \in \mathbb{S}$,

$$\begin{cases} V_L(s_1 *_1 s_2) \geqslant min\{V_L(s_1), V_L(s_2)\} \\ V_L(s_1 *_2 s_2) \geqslant min\{V_L(s_1), V_L(s_2)\} \\ V_L(s_1 *_3 s_2) \geqslant min\{V_L(s_1), V_L(s_2)\} \end{cases}$$

That is,

$$\begin{cases} \mathcal{T}_L(s_1 *_1 s_2) \geqslant min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} \\ \mathcal{T}_L(s_1 *_2 s_2) \geqslant min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} \\ \mathcal{T}_L(s_1 *_3 s_2) \geqslant min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} \end{cases}$$

$$\begin{cases} 1-\mathcal{F}_L(s_1 *_1 s_2) \geqslant min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\} \\ 1-\mathcal{F}_L(s_1 *_2 s_2) \geqslant min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\} \\ 1-\mathcal{F}_L(s_1 *_3 s_2) \geqslant min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\} \end{cases}$$

EXAMPLE 3.1. Let $\mathbb{N}$ be the set of natural numbers. Then $(\mathbb{N}, min, max, \cdot)$ is a subbisemirings.

Let $L = (\mathcal{T}_L, \mathcal{F}_L)$, where $\mathcal{T}_L : \mathbb{N} \to [0,1]$ and $\mathcal{F}_L : \mathbb{N} \to [0,1]$ defined by

$$\mathcal{T}_L(s) = \begin{cases} 0.40 & if \quad s = 1 \\ 0.60 & if \quad s = 2 \\ 0.80 & if \quad s \geqslant 3 \end{cases} \quad and \quad \mathcal{F}_L(s) = \begin{cases} 0.60 & if \quad s = 1 \\ 0.30 & if \quad s = 2 \\ 0.20 & if \quad s \geqslant 3 \end{cases}$$

Clearly L is a VSBS of $\mathbb{S}$.

EXAMPLE 3.2. Let $\mathbb{S} = \{z_1, z_2, z_3, z_4\}$ be the bisemirings with the Cayley tables:

$*_1$	z_1	z_2	z_3	z_4
z_1	z_1	z_2	z_3	z_4
z_2	z_2	z_2	z_2	z_2
z_3	z_3	z_2	z_3	z_4
z_4	z_4	z_2	z_4	z_4

$*_2$	z_1	z_2	z_3	z_4
z_1	z_1	z_1	z_1	z_1
z_2	z_1	z_2	z_2	z_2
z_3	z_1	z_3	z_3	z_3
z_4	z_1	z_2	z_2	z_2

$*_3$	z_1	z_2	z_3	z_4
z_1	z_1	z_1	z_1	z_1
z_2	z_1	z_1	z_1	z_3
z_3	z_1	z_1	z_1	z_3
z_4	z_1	z_1	z_1	z_3

$$\mathcal{T}_L(z) = \begin{cases} 0.70 & if \quad z = z_1 \\ 0.60 & if \quad z = z_2 \\ 0.50 & if \quad z = z_3 \\ 0.40 & if \quad z = z_4 \end{cases} \quad and \quad \mathcal{F}_L(z) = \begin{cases} 0.30 & if \quad z = z_1 \\ 0.35 & if \quad z = z_2 \\ 0.45 & if \quad z = z_3 \\ 0.60 & if \quad z = z_4 \end{cases}$$

Clearly L is a VSBS of $\mathbb{S}$.

THEOREM 3.1. *The intersection of a family of VSBS's is a VSBS of* $\mathbb{S}$.

PROOF. Let $\{W_i : i \in I\}$ be a family of VSBS and $L = \bigcap_{i \in I} W_i$. Let $s_1, s_2 \in \mathbb{S}$. Then

$$\begin{aligned} \mathcal{T}_L(s_1 *_1 s_2) &= \inf_{i \in I} \mathcal{T}_{W_i}(s_1 *_1 s_2) \\ &\geqslant \inf_{i \in I} \min\{\mathcal{T}_{W_i}(s_1), \mathcal{T}_{W_i}(s_2)\} \\ &= \min\left\{\inf_{i \in I} \mathcal{T}_{W_i}(s_1), \inf_{i \in I} \mathcal{T}_{W_i}(s_2)\right\} \\ &= \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} \end{aligned}$$

$$\begin{aligned} 1-\mathcal{F}_L(s_1 *_1 s_2) &= \inf_{i\in I}\ [1-\mathcal{F}_{W_i}(s_1 *_1 s_2)] \\ &\geqslant \inf_{i\in I}\ \min\{1-\mathcal{F}_{W_i}(s_1), 1-\mathcal{F}_{W_i}(s_2)\} \\ &= \min\left\{\inf_{i\in I} 1-\mathcal{F}_{W_i}(s_1), \inf_{i\in I} 1-\mathcal{F}_{W_i}(s_2)\right\} \\ &= \min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\} \end{aligned}$$

Thus, $V_L(s_1 *_1 s_2) \geqslant \min\{V_L(s_1), V_L(s_2)\}$. Similar manner we get

$$V_L(s_1 *_2 s_2) \geqslant \min\{V_L(s_1), V_L(s_2)\}, \quad V_L(s_1 *_3 s_2) \geqslant \min\{V_L(s_1), V_L(s_2)\}.$$

Hence L is a VSBS. □

THEOREM 3.2. *If L and M are any two VSBS's of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively, then $L \times M$ is a VSBS.*

PROOF. Let L and M be two VSBS's of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. Let $s_1, s_2 \in \mathbb{S}_1$ and $t_1, t_2 \in \mathbb{S}_2$. Then (s_1, s_2) and (t_1, t_2) are in $\mathbb{S}_1 \times \mathbb{S}_2$. Now

$$\begin{aligned} \mathcal{T}_{L\times M}[(s_1,t_1) *_1 (s_2,t_2)] &= \mathcal{T}_{L\times M}(s_1 *_1 s_2, t_1 *_1 t_2) \\ &= \min\{\mathcal{T}_L(s_1 *_1 s_2), \mathcal{T}_M(t_1 *_1 t_2)\} \\ &\geqslant \min\{\min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}, \min\{\mathcal{T}_M(t_1), \mathcal{T}_M(t_2)\}\} \\ &= \min\{\min\{\mathcal{T}_L(s_1), \mathcal{T}_M(t_1)\}, \min\{\mathcal{T}_L(s_2), \mathcal{T}_M(t_2)\}\} \\ &= \min\{\mathcal{T}_{L\times M}(s_1,t_1), \mathcal{T}_{L\times M}(s_2,t_2)\} \end{aligned}$$

$$\begin{aligned} &1-\mathcal{F}_{L\times M}[(s_1,t_1) *_1 (s_2,t_2)] \\ &\quad = 1-\mathcal{F}_{L\times M}(s_1 *_1 s_2, t_1 *_1 t_2) \\ &\quad \geqslant \min\{1-\mathcal{F}_L(s_1 *_1 s_2), 1-\mathcal{F}_M(t_1 *_1 t_2)\} \\ &\quad \geqslant \min\{\min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\}, \min\{1-\mathcal{F}_M(t_1), 1-\mathcal{F}_M(t_2)\}\} \\ &\quad = \min\{\min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_M(t_1)\}, \min\{1-\mathcal{F}_L(s_2), 1-\mathcal{F}_M(t_2)\}\} \\ &\quad = \min\{1-\mathcal{F}_{L\times M}(s_1,t_1), 1-\mathcal{F}_{L\times M}(s_2,t_2)\} \end{aligned}$$

Thus, $V_{L\times M}[(s_1,t_1) *_1 (s_2,t_2)] \geqslant \min\{V_{L\times M}(s_1,t_1), V_{L\times M}(s_2,t_2)\}$. Similarly,

$$V_{L\times M}[(s_1,t_1) *_2 (s_2,t_2)] \geqslant \min\{V_{L\times M}(s_1,t_1), V_{L\times M}(s_2,t_2)\},$$

$$V_{L\times M}[(s_1,t_1) *_3 (s_2,t_2)] \geqslant \min\{V_{L\times M}(s_1,t_1), V_{L\times M}(s_2,t_2)\}.$$

Hence $L \times M$ is a VSBS. □

THEOREM 3.3. *Let L be fuzzy subset of $\mathbb{S}$. Then $L = (\mathcal{T}_L, \mathcal{F}_L)$ is a VSBS if and only if all non empty level sets $L_{(\alpha,\beta)}(\alpha, \beta \in [0,1])$ is a SBS.*

PROOF. Assume that $L = (\mathcal{T}_L, \mathcal{F}_L)$ is a VSBS. For each $\alpha, \beta \in [0,1]$ and $s_1, s_2 \in L_{(\alpha,\beta)}$. We have $\mathcal{T}_L(s_1) \geqslant \alpha$, $\mathcal{T}_L(s_2) \geqslant \alpha$, $1-\mathcal{F}_L(s_1) \geqslant \beta$ and $1-\mathcal{F}_L(s_2) \geqslant \beta$. Since L is a VSBS, $\mathcal{T}_L(s_1 *_1 s_2) \geqslant \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} \geqslant \alpha$. Similarly, $1 - \mathcal{F}_L(s_1 *_1 s_2) \geqslant \min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\} \geqslant \beta$, This implies that $s_1 *_1 s_2 \in L_{(\alpha,\beta)}$. Similarly, $s_1 *_2 s_2 \in L_{(\alpha,\beta)}$ and $s_1 *_3 s_2 \in L_{(\alpha,\beta)}$. Therefore $L_{(\alpha,\beta)}$ is a SBS for each $\alpha, \beta \in [0,1]$.

Conversely, assume that $L_{(\alpha,\beta)}$ is a SBS for each $\alpha, \beta \in [0,1]$. Let $s_1, s_2 \in \mathbb{S}$. Then $s_1, s_2 \in L_{(\alpha,\beta)}$,where $\alpha = \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}$ and $\beta = \min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\}$. Thus, $s_1 *_1 s_2 \in L_{(\alpha,\beta)}$ imply $\mathcal{T}_L(s_1 *_1 s_2) \geqslant \alpha = \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}$ and $1 - \mathcal{F}_L(s_1 *_1 s_2) \geqslant \beta = \min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\}$. Similarly, $\mathcal{T}_L(s_1 *_2 s_2) \geqslant \alpha = \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}$ and $1 - \mathcal{F}_L(s_1 *_2 s_2) \geqslant \beta = \min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\}$ and $\mathcal{T}_L(s_1 *_3 s_2) \geqslant \alpha = \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}$ and $1 - \mathcal{F}_L(s_1 *_3 s_2) \geqslant \beta = \min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\}$. Hence $L = (\mathcal{T}_L, \mathcal{F}_L)$ is a VSBS. □

THEOREM 3.4. *Let L be the VSBS and W be the strongest vague relation of $\mathbb{S}$. Then L is a VSBS if and only if W is a VSBS of $\mathbb{S} \times \mathbb{S}$.*

PROOF. Let L is a VSBS and W be the strongest vague relation of $\mathbb{S}$. For $s = (s_1, s_2)$ and $t = (t_1, t_2)$ are in $\mathbb{S} \times \mathbb{S}$. We have

$$\begin{aligned}
\mathcal{T}_W(s *_1 t) &= \mathcal{T}_W[((s_1, s_2) *_1 (t_1, t_2)] \\
&= \mathcal{T}_W(s_1 *_1 t_1, s_2 *_1 t_2) \\
&= \min\{\mathcal{T}_L(s_1 *_1 t_1), \mathcal{T}_L(s_2 *_1 t_2)\} \\
&\geqslant \min\{\min\{\mathcal{T}_L(s_1), \mathcal{T}_L(t_1)\}, \min\{\mathcal{T}_L(s_2), \mathcal{T}_L(t_2)\}\} \\
&= \min\{\min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}, \min\{\mathcal{T}_L(t_1), \mathcal{T}_L(t_2)\}\} \\
&= \min\{\mathcal{T}_W(s_1, s_2), \mathcal{T}_W(t_1, t_2)\} \\
&= \min\{\mathcal{T}_W(s), \mathcal{T}_W(t)\}
\end{aligned}$$

$$\begin{aligned}
1 - \mathcal{F}_W(s *_1 t) &= 1 - \mathcal{F}_W[((s_1, s_2) *_1 (t_1, t_2)] \\
&= 1 - \mathcal{F}_W(s_1 *_1 t_1, s_2 *_1 t_2) \\
&= \min\{1 - \mathcal{F}_L(s_1 *_1 t_1), 1 - \mathcal{F}_L(s_2 *_1 t_2)\} \\
&\geqslant \min\{\min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(t_1)\}, \min\{1 - \mathcal{F}_L(s_2), 1 - \mathcal{F}_L(t_2)\}\} \\
&= \min\{\min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\}, \min\{1 - \mathcal{F}_L(t_1), 1 - \mathcal{F}_L(t_2)\}\} \\
&= \min\{1 - \mathcal{F}_W(s_1, s_2), 1 - \mathcal{F}_W(t_1, t_2)\} \\
&= \min\{1 - \mathcal{F}_W(s), 1 - \mathcal{F}_W(t)\}
\end{aligned}$$

Thus, $V_W(s *_1 t) \geqslant \min\{V_W(s), V_W(t)\}$. Similarly, $V_W(s *_2 t) \geqslant \min\{V_W(s), V_W(t)\}$ and $V_W(s *_3 t) \geqslant \min\{V_W(s), V_W(t)\}$. Hence W is a VSBS of $\mathbb{S} \times \mathbb{S}$.

Conversely, assume that W is a VSBS of $\mathbb{S} \times \mathbb{S}$, $s = (s_1, s_2)$ and $t = (t_1, t_2)$ are in $\mathbb{S} \times \mathbb{S}$.

$$\begin{aligned}
\min\{\mathcal{T}_L(s_1 *_1 t_1), \mathcal{T}_L(s_2 *_1 t_2)\} &= \mathcal{T}_W(s_1 *_1 t_1, s_2 *_1 t_2) \\
&= \mathcal{T}_W[(s_1, s_2) *_1 (t_1, t_2)] \\
&= \mathcal{T}_W(s *_1 t) \\
&\geqslant \min\{\mathcal{T}_W(s), \mathcal{T}_W(t)\} \\
&= \min\{\mathcal{T}_W(s_1, s_2)\}, \mathcal{T}_W(t_1, t_2)\} \\
&= \min\{\min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}, \min\{\mathcal{T}_L(t_1), \mathcal{T}_L(t_2)\}\}
\end{aligned}$$

If $\mathcal{T}_L(s_1 *_1 t_1) \leqslant \mathcal{T}_L(s_2 *_1 t_2)$, then $\mathcal{T}_L(s_1) \leqslant \mathcal{T}_L(s_2)$ and $\mathcal{T}_L(t_1) \leqslant \mathcal{T}_L(t_2)$.
We get $\mathcal{T}_L(s_1 *_1 t_1) \geqslant \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(t_1)\}$ for all $s_1, t_1 \in \mathbb{S}$.

And $\min\{1-\mathcal{F}_L(s_1 *_1 t_1), 1-\mathcal{F}_L(s_2 *_1 t_2)\}$

$$\begin{aligned}
&= 1-\mathcal{F}_W(s_1 *_1 t_1, s_2 *_1 t_2)\\
&= 1-\mathcal{F}_W[(s_1, s_2) *_1 (t_1, t_2)]\\
&= 1-\mathcal{F}_W(s *_1 t)\\
&\geqslant \min\{1-\mathcal{F}_W(s), 1-\mathcal{F}_W(t)\}\\
&= \min\{1-\mathcal{F}_W(s_1, s_2)\}, 1-\mathcal{F}_W(t_1, t_2)\}\\
&= \min\{\min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\}, \min\{1-\mathcal{F}_L(t_1), 1-\mathcal{F}_L(t_2)\}\}
\end{aligned}$$

If $1-\mathcal{F}_L(s_1 *_1 t_1) \leqslant 1-\mathcal{F}_L(s_2 *_1 t_2)$, then $1-\mathcal{F}_L(s_1) \leqslant 1-\mathcal{F}_L(s_2)$ and $1-\mathcal{F}_L(t_1) \leqslant 1-\mathcal{F}_L(t_2)$. We get $1-\mathcal{F}_L(s_1 *_1 t_1) \geqslant \min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(t_1)\}$ for all $s_1, t_1 \in \mathbb{S}$. Thus, $V_L(s_1 *_1 t_1) \geqslant \min\{V_L(s_1), V_L(t_1)\}$. Similarly we can prove that, $V_L(s_1 *_2 t_1) \geqslant \min\{V_L(s_1), V_L(t_1)\}$ and $V_L(s_1 *_3 t_1) \geqslant \min\{V_L(s_1), V_L(t_1)\}$. Hence L is a VSBS. □

THEOREM 3.5. (i) *If L is a VSBS, then $H_1 = \{s | s \in \mathbb{S} : \mathcal{T}_L(s) = 1, \mathcal{F}_L(s) = 0\}$ is either empty or is a FSBS.*

(ii) *If L is a VSBS, then $H_2 = \{\langle s, \mathcal{T}_L(s)\rangle : 0 < \mathcal{T}_L(s) \leqslant 1, \mathcal{F}_L(s) = 0\}$ is either empty or FSBS.*

(iii) *If L is a VSBS, then $H_3 = \{\langle s, \mathcal{T}_L(s)\rangle : 0 < \mathcal{T}_L(s) \leqslant 1\}$ is either empty or FSBS.*

(iv) *If L is a VSBS, then $H_4 = \{\langle s, \mathcal{F}_L(s)\rangle : 0 < \mathcal{F}_L(s) \leqslant 1\}$ is either empty or anti FSBS.*

THEOREM 3.6. *If L is a VSBS of $(\mathbb{S}, *_1, *_2, *_3)$, then $\Box L$ is a VSBS.*

PROOF. Let L be the VSBS of a bisemiring $\mathbb{S}$, $L = \{\langle s_1, \mathcal{T}_L(s_1), \mathcal{F}_L(s_1)\rangle\}$ for all $s_1 \in \mathbb{S}$. Take $\Box L = M = \{\langle s_1, \mathcal{T}_M(s_1), \mathcal{F}_M(s_1)\rangle\}$,where $\mathcal{T}_M(s_1) = \mathcal{T}_L(s_1)$, $\mathcal{F}_M(s_1) = 1-\mathcal{T}_L(s_1)$. Clearly $\mathcal{T}_M(s_1 *_1 s_2) \geqslant \min\{\mathcal{T}_M(s_1), \mathcal{T}_M(s_2)\}$, for all $s_1, s_2 \in \mathbb{S}$. Since L is an VSBS. Then $\mathcal{T}_L(s_1 *_1 s_2) \geqslant \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}$ implies that $1-\mathcal{F}_M(s_1 *_1 s_2) \geqslant \min\{(1-\mathcal{F}_M(s_1)), (1-\mathcal{F}_M(s_2))\}$. Hence $\Box L$ is VSBS. □

REMARK 3.1. The reverse of the Theorem 3.6 fails by the Example 3.2.

$$\Box L = \begin{cases} \langle z_1, 0.45, 0.55\rangle \\ \langle z_2, 0.35, 0.65\rangle \\ \langle z_3, 0.25, 0.75\rangle \\ \langle z_4, 0.15, 0.85\rangle \end{cases} \quad L = \begin{cases} \langle z_1, 0.45, 0.55\rangle \\ \langle z_2, 0.35, 0.58\rangle \\ \langle z_3, 0.25, 0.50\rangle \\ \langle z_4, 0.15, 0.85\rangle \end{cases}$$

Clearly $\Box L$ is the VSBS, but L is not a VSBS.

THEOREM 3.7. *If L is a VSBS of $(\mathbb{S}, *_1, *_2, *_3)$ then $\Diamond L$ is a VSBS.*

PROOF. Let L be a VSBS. Consider $L = \{\langle s_1, \mathcal{T}_L(s_1), \mathcal{F}_L(s_1)\rangle\}$, $\forall s_1 \in \mathbb{S}$. Take $\Diamond L = M = \{\langle s_1, \mathcal{T}_M(s_1), \mathcal{F}_M(s_1)\rangle\}$ where $\mathcal{T}_M(s_1) = 1-\mathcal{F}_L(s_1)$, $\mathcal{F}_M(s_1) = \mathcal{F}_L(s_1)$ and the applying Theorem 3.6. Hence $\Diamond L$ is an VSBS. □

Remark 3.2. The inversion of Theorem 3.7 fails by the Example 3.2,

$$\diamond L = \begin{cases} \langle z_1, 0.70, 0.30\rangle \\ \langle z_2, 0.60, 0.40\rangle \\ \langle z_3, 0.50, 0.50\rangle \\ \langle z_4, 0.40, 0.60\rangle \end{cases} \quad L = \begin{cases} \langle z_1, 0.70, 0.30\rangle \\ \langle z_2, 0.45, 0.40\rangle \\ \langle z_3, 0.48, 0.50\rangle \\ \langle z_4, 0.40, 0.60\rangle \end{cases}$$

Clearly $\diamond L$ is the VSBS, but L is not a VSBS.

Definition 3.2. Let $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ and $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. Let $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ be any function and L be an VSBS in $\mathbb{S}_1$, W be an VSBS in $\phi(\mathbb{S}_1) = \mathbb{S}_2$, the image of vague set can be defined by $V_{\phi(W)}(s_2) = [\mathcal{T}_{\phi(W)}(s_2), \mathcal{F}_{\phi(W)}(s_2)]$, where $\mathcal{T}_{\phi(W)}(s_2) = \mathcal{T}_W\phi(s_2)$ and $\mathcal{F}_{\phi(W)}(s_2) = \mathcal{F}_W\phi(s_2)$.

Definition 3.3. Let $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ and $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. Let $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ be any function. Let W be a vague set in $\phi(\mathbb{S}_1) = \mathbb{S}_2$. Then the inverse image of W, ϕ^{-1} is the vague set in $\mathbb{S}_1$ by $V_{\phi^{-1}(W)}(s_1) = [\mathcal{T}_{\phi^{-1}(W)}(s_1), \mathcal{F}_{\phi^{-1}(W)}(s_1)]$, where $\mathcal{T}_{\phi^{-1}(W)}(s_1) = \mathcal{T}_W(\phi^{-1}(s_1))$ and $\mathcal{F}_{\phi^{-1}(W)}(s_1) = \mathcal{F}_W(\phi^{-1}(s_1))$.

Theorem 3.8. *Let $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ and $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic image of the VSBS of $\mathbb{S}_1$ is also VSBS of $\mathbb{S}_2$.*

Proof. Let $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2), \phi(s_1 \oplus_2 s_2) = \phi(s_1) \odot_2 \phi(s_2)$ and $\phi(s_1 \oplus_3 s_2) = \phi(s_1) \odot_3 \phi(s_2)$ for all $s_1, s_2 \in \mathbb{S}_1$. Let $W = \phi(L)$, L is a VSBS of $\mathbb{S}_1$. Let $\phi(s_1), \phi(s_2) \in \mathbb{S}_2$, $\mathcal{T}_W(\phi(s_1) \odot_1 \phi(s_2)) \geqslant \mathcal{T}_L(s_1 \oplus_1 s_2) \geqslant \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\} = \min\{\mathcal{T}_W\phi(s_1), \mathcal{T}_W\phi(s_2)\}$. And $1-\mathcal{F}_W(\phi(s_1)\odot_1\phi(s_2)) \geqslant 1-\mathcal{F}_L(s_1\oplus_1 s_2) \geqslant \min\{1-\mathcal{F}_L(s_1), 1-\mathcal{F}_L(s_2)\} = \min\{1-\mathcal{F}_W\phi(s_1), 1-\mathcal{F}_W\phi(s_2)\}$. Thus, $V_W(\phi(s_1) \odot_1 \phi(s_2)) \geqslant \min\{V_W\phi(s_1), V_W\phi(s_2)\}$. Similarly, $V_W(\phi(s_1)\odot_2\phi(s_2)) \geqslant \min\{V_W\phi(s_1), V_W\phi(s_2)\}$ and $V_W(\phi(s_1)\odot_3\phi(s_2)) \geqslant \min\{V_W\phi(s_1), V_W\phi(s_2)\}$. Hence W is a VSBS of $\mathbb{S}_2$. □

Theorem 3.9. *Let $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ and $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic preimage of a VSBS of $\mathbb{S}_2$ is a VSBS of $\mathbb{S}_1$.*

Proof. Let $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2), \phi(s_1\oplus_2 s_2) = \phi(s_1)\odot_2\phi(s_2)$ and $\phi(s_1\oplus_3 s_2) = \phi(s_1)\odot_3\phi(s_2)\ \forall\ s_1, s_2 \in \mathbb{S}_1$. Let $W = \phi(L)$, where W is an VSBS of $\mathbb{S}_2$. Now, $\mathcal{T}_L(s_1 \oplus_1 s_2) = \mathcal{T}_W(\phi(s_1) \odot_1 \phi(s_2)) \geqslant \min\{\mathcal{T}_W(\phi(s_1)), \mathcal{T}_W(\phi(s_2))\} = \min\{\mathcal{T}_L(s_1), \mathcal{T}_L(s_2)\}$. And $1 - \mathcal{F}_L(s_1 \oplus_1 s_2) = 1 - \mathcal{F}_W(\phi(s_1) \odot_1 \phi(s_2)) \geqslant \min\{1 - (\mathcal{F}_W\phi(s_1)), 1 - (\mathcal{F}_W\phi(s_2))\} = \min\{1 - \mathcal{F}_L(s_1), 1 - \mathcal{F}_L(s_2)\}$. Thus, $V_L(s_1 \oplus_1 s_2) \geqslant \min\{V_L(s_1), V_L(s_2)\}$. Similarly, $V_L(s_1 \oplus_2 s_2) \geqslant \min\{V_L(s_1), V_L(s_2)\}$ and $V_L(s_1 \oplus_3 s_2) \geqslant \min\{V_L(s_1), V_L(s_2)\}$. Hence L is a VSBS of $\mathbb{S}_1$. □

Theorem 3.10. *Let $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ and $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. If $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ is a homomorphism, then $\phi(L_{(\alpha,\beta)})$ is a level SBS of an VSBS W of $\mathbb{S}_2$.*

PROOF. Let $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2), \phi(s_1 \oplus_2 s_2) = \phi(s_1) \odot_2 \phi(s_2)$ and $\phi(s_1 \oplus_3 s_2) = \phi(s_1) \odot_3 \phi(s_2)$ for all $s_1, s_2 \in \mathbb{S}_1$. Let $W = \phi(L)$, L is an VSBS of $\mathbb{S}_1$. By Theorem 3.8, W is a VSBS of $\mathbb{S}_2$. Let $L_{(\alpha,\beta)}$ be a level SBS of L. Suppose $s_1, s_2 \in L_{(\alpha,\beta)}$. Then $\phi(s_1 \oplus_1 s_2), \phi(s_1 \oplus_2 s_2)$ and $\phi(s_1 \oplus_3 s_2) \in L_{(\alpha,\beta)}$. Now, $\mathcal{T}_W(\phi(s_1)) \geqslant \mathcal{T}_L(s_1) \geqslant \alpha, \mathcal{T}_W(\phi(s_2)) \geqslant \mathcal{T}_L(s_2) \geqslant \alpha$. Then $\mathcal{T}_W(\phi(s_1) \odot_1 \phi(s_2)) \geqslant \mathcal{T}_L(s_1 \oplus_1 s_2) \geqslant \alpha$, for all $\phi(s_1), \phi(s_2) \in \mathbb{S}_2$. And $1 - \mathcal{F}_W(\phi(s_1)) \geqslant 1 - \mathcal{F}_L(s_1) \geqslant \beta, 1 - \mathcal{F}_W(\phi(s_2)) \geqslant 1 - \mathcal{F}_L(s_2) \geqslant \beta$. Then $1 - \mathcal{F}_W(\phi(s_1) \odot_1 \phi(s_2)) \geqslant 1 - \mathcal{F}_L(s_1 \oplus_1 s_2) \geqslant \beta$, for all $\phi(s_1), \phi(s_2) \in \mathbb{S}_2$. Thus, $V_W(\phi(s_1) \odot_1 \phi(s_2)) \geqslant [\alpha, \beta]$. Similarly, $V_W(\phi(s_1) \odot_2 \phi(s_2)) \geqslant [\alpha, \beta]$ and $V_W(\phi(s_1) \odot_3 \phi(s_2)) \geqslant [\alpha, \beta]$. Hence $\phi(L_{(\alpha,\beta)})$ is a level SBS of an VSBS W of $\mathbb{S}_2$. □

THEOREM 3.11. *Let* $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ *and* $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ *be any two bisemirings. If* $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ *is a homomorphism, then* $L_{(\alpha,\beta)}$ *is a level SBS of an VSBS* L *of* $\mathbb{S}_1$.

PROOF. Let $\phi : \mathbb{S}_1 \to \mathbb{S}_2$ be a homomorphism. Then $\phi(s_1 \oplus_1 s_2) = \phi(s_1) \odot_1 \phi(s_2), \phi(s_1 \oplus_2 s_2) = \phi(s_1) \odot_2 \phi(s_2)$ and $\phi(s_1 \oplus_3 s_2) = \phi(s_1) \odot_3 \phi(s_2)$ for all $s_1, s_2 \in \mathbb{S}_1$. Let $W = \phi(L)$, W is an VSBS of $\mathbb{S}_2$. By Theorem 3.9, L is a VSBS of $\mathbb{S}_1$. Let $\phi(L_{(\alpha,\beta)})$ be a level SBS of W. Suppose $\phi(s_1), \phi(s_2) \in \phi(L_{(\alpha,\beta)})$. Then $\phi(s_1 \oplus_1 s_2), \phi(s_1 \oplus_2 s_2)$ and $\phi(s_1 \oplus_3 s_2) \in \phi(L_{(\alpha,\beta)})$. Now, $\mathcal{T}_L(s_1) = \mathcal{T}_W(\phi(s_1)) \geqslant \alpha, \mathcal{T}_L(s_2) = \mathcal{T}_W(\phi(s_2)) \geqslant \alpha$. Then $\mathcal{T}_L(s_1 \oplus_1 s_2) \geqslant \alpha$. And $1 - \mathcal{F}_L(s_1) = 1 - \mathcal{F}_W(\phi(s_1)) \geqslant \beta$, $1 - \mathcal{F}_L(s_2) = 1 - \mathcal{F}_W(\phi(s_2)) \geqslant \beta$. Then $1 - \mathcal{F}_L(s_1 \oplus_1 s_2) \geqslant \beta$. Thus, $V_L(s_1 \oplus_1 s_2) \geqslant [\alpha, \beta]$. Similarly, $V_L(s_1 \oplus_2 s_2) \geqslant [\alpha, \beta]$ and $V_L(s_1 \oplus_3 s_2) \geqslant [\alpha, \beta]$. Hence $L_{(\alpha,\beta)}$ is a level SBS of an VSBS L of $\mathbb{S}_1$. □

4. Vague Normal Subbisemirings

DEFINITION 4.1. A fuzzy subset L of $\mathbb{S}$ is said to be a VNSBS if

$$\begin{cases} V_L(s_1 *_1 s_2) = V_L(s_2 *_1 s_1) \\ V_L(s_1 *_2 s_2) = V_L(s_2 *_2 s_1) \\ V_L(s_1 *_3 s_2) = V_L(s_2 *_3 s_1) \end{cases}$$

$$i.e. \begin{cases} \mathcal{T}_L(s_1 *_1 s_2) = \mathcal{T}_L(s_2 *_1 s_1) \\ \mathcal{T}_L(s_1 *_2 s_2) = \mathcal{T}_L(s_2 *_2 s_1) \\ \mathcal{T}_L(s_1 *_3 s_2) = \mathcal{T}_L(s_2 *_3 s_1) \end{cases} \quad \begin{cases} 1 - \mathcal{F}_L(s_1 *_1 s_2) = 1 - \mathcal{F}_L(s_2 *_1 s_1) \\ 1 - \mathcal{F}_L(s_1 *_2 s_2) = 1 - \mathcal{F}_L(s_2 *_2 s_1) \\ 1 - \mathcal{F}_L(s_1 *_3 s_2) = 1 - \mathcal{F}_L(s_2 *_3 s_1) \end{cases}$$

$\forall\ s_1, s_2 \in \mathbb{S}$.

THEOREM 4.1. *The intersection of a family of VNSBS's is a VNSBS.*

THEOREM 4.2. *If* L *and* M *are any two VNSBS's of* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *respectively, then* $L \times M$ *is a VNSBS.*

THEOREM 4.3. *Let* L *be a VNSBS of* $\mathbb{S}$ *and* W *be the strongest intuitionistic fuzzy relation of* $\mathbb{S}$. *Then* L *is a VNSBS of* $\mathbb{S}$ *if and only if* W *is a VNSBS of* $\mathbb{S} \times \mathbb{S}$.

THEOREM 4.4. *Let* $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ *and* $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ *be any two bisemirings. The homomorphic image of a VNSBS of* $\mathbb{S}_1$ *is a VNSBS of* $\mathbb{S}_2$.

THEOREM 4.5. *Let* $(\mathbb{S}_1, \oplus_1, \oplus_2, \oplus_3)$ *and* $(\mathbb{S}_2, \odot_1, \odot_2, \odot_3)$ *be any two bisemirings. The homomorphic preimage of a VNSBS of* $\mathbb{S}_2$ *is a VNSBS of* $\mathbb{S}_1$.

5. Conclusion

The main goal of this work is to present a subsemirings of semirings to vague subbisemirings of bisemirings. So in future, we should consider the cubic subbisemirings of bisemirings and cubic soft subbisemirings of bisemirings.

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M. PALANIKUMAR: DEPARTMENT OF MATHEMATICS, ANNAMALAI UNIVERSITY, INDIA
E-mail address: palanimaths86@gmail.com

K.ARULMOZHI: DEPARTMENT OF MATHEMATICS, ANNAMALAI UNIVERSITY, INDIA.
E-mail address: arulmozhiems@gmail.com