

## CONVEXITY AND AFPP IN THE DIGITAL PLANE

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ABSTRACT. We examine the relationship between convexity and the approximate fixed point property (AFPP) for digital images in  $\mathbb{Z}^2$ .

### 1. Introduction

The study of fixed points is prominent in many branches of mathematics. In digital topology, it has become worthwhile to broaden the study to “approximate fixed points.” The Approximate Fixed Point Property (AFPP), a generalization of the classical fixed point property (FPP), was introduced in [7]. In this paper, we show that for digital images  $X \subset \mathbb{Z}^2$ , convexity can help us show whether  $(X, c_2)$  has the AFPP.

### 2. Preliminaries

Much of this section is quoted or paraphrased from papers that are listed in the references, especially [4, 5, 6, 7].

We use  $\mathbb{Z}$  to indicate the set of integers;  $\mathbb{R}$  for the set of real numbers.

For  $(x, y) \in \mathbb{Z}^2$ , the projection functions  $p_1, p_2 : \mathbb{Z}^2 \rightarrow \mathbb{Z}$  are

$$p_1(x, y) = x, \quad p_2(x, y) = y.$$

**2.1. Adjacencies.** A digital image is a graph  $(X, \kappa)$ , where  $X$  is a subset of  $\mathbb{Z}^n$  for some positive integer  $n$ , and  $\kappa$  is an adjacency relation for the points of  $X$ . The  $c_u$ -adjacencies are commonly used. Let  $x, y \in \mathbb{Z}^n$ ,  $x \neq y$ , where we consider these points as  $n$ -tuples of integers:

$$x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n).$$

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Let  $u \in \mathbb{Z}$ ,  $1 \leq u \leq n$ . We say distinct points  $x$  and  $y$  are  $c_u$ -adjacent if

- There are at most  $u$  indices  $i$  for which  $|x_i - y_i| = 1$ .
- For all indices  $j$  such that  $|x_j - y_j| \neq 1$  we have  $x_j = y_j$ .

Often, a  $c_u$ -adjacency is denoted by the number of points adjacent to a given point in  $\mathbb{Z}^n$  using this adjacency. E.g.,

- In  $\mathbb{Z}^1$ ,  $c_1$ -adjacency is 2-adjacency.
- In  $\mathbb{Z}^2$ ,  $c_1$ -adjacency is 4-adjacency and  $c_2$ -adjacency is 8-adjacency.
- In  $\mathbb{Z}^3$ ,  $c_1$ -adjacency is 6-adjacency,  $c_2$ -adjacency is 18-adjacency, and  $c_3$ -adjacency is 26-adjacency.

We write  $x \leftrightarrow_{\kappa} x'$ , or  $x \leftrightarrow x'$  when  $\kappa$  is understood, to indicate that  $x$  and  $x'$  are  $\kappa$ -adjacent. Similarly, we write  $x \rightleftharpoons_{\kappa} x'$ , or  $x \rightleftharpoons x'$  when  $\kappa$  is understood, to indicate that  $x$  and  $x'$  are  $\kappa$ -adjacent or equal.

A subset  $Y$  of a digital image  $(X, \kappa)$  is  $\kappa$ -connected [12], or *connected* when  $\kappa$  is understood, if for every pair of points  $a, b \in Y$  there exists a sequence  $\{y_i\}_{i=0}^m \subset Y$  such that  $a = y_0$ ,  $b = y_m$ , and  $y_i \leftrightarrow_{\kappa} y_{i+1}$  for  $0 \leq i < m$ .

Given a digital image  $(X, \kappa)$  and  $x \in X$ , we denote by  $N^*(X, \kappa, x)$  the set  $\{y \in X \mid y \rightleftharpoons_{\kappa} x\}$ .

**2.2. Digitally continuous functions.** The following generalizes a definition of [12].

DEFINITION 2.1. ([2]) Let  $(X, \kappa)$  and  $(Y, \lambda)$  be digital images. A single-valued function  $f : X \rightarrow Y$  is  $(\kappa, \lambda)$ -continuous if for every  $\kappa$ -connected  $A \subset X$  we have that  $f(A)$  is a  $\lambda$ -connected subset of  $Y$ .  $\square$

When the adjacency relations are understood, we will simply say that  $f$  is *continuous*. Continuity can be expressed in terms of adjacency of points:

THEOREM 2.2 ([12, 2]). A function  $f : X \rightarrow Y$  is continuous if and only if  $x \leftrightarrow x'$  in  $X$  implies  $f(x) \rightleftharpoons f(x')$  in  $Y$ .  $\square$

See also [8, 9], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

Let  $Y \subset X$  and let  $r : X \rightarrow Y$  be  $(\kappa, \kappa)$ -continuous such that  $r(y) = y$  for all  $y \in Y$ . Then  $r$  is a  $\kappa$ -retraction, and  $Y$  is a  $\kappa$ -retract of  $X$ .

The notation  $C(X, \kappa)$  denotes  $\{f : X \rightarrow X \mid f \text{ is } (\kappa, \kappa)\text{-continuous}\}$ .

**2.3. Approximate fixed points and the AFPP.** Let  $f \in C(X, \kappa)$  and let  $x \in X$ . We say

- $x$  is a *fixed point* of  $f$  if  $f(x) = x$ ;
- If  $f(x) \rightleftharpoons_{\kappa} x$ , then  $x$  is an *almost fixed point* [12, 14] or *approximate fixed point* [7] of  $(f, \kappa)$ .
- A digital image  $(X, \kappa)$  has the *approximate fixed point property* (AFPP) [7] if for every  $f \in C(X, \kappa)$  there is an approximate fixed point of  $f$ . This generalizes the *fixed point property* (FPP): a digital image  $(X, \kappa)$  has the FPP if every  $f \in C(X, \kappa)$  has a fixed point.

The AFPP gathered attention in part because only a digital image with a single point has the FPP [7].

A. Rosenfeld’s paper [12] states the following as its Theorem 4.1 (quoted verbatim).

Let  $I$  be a digital picture, and let  $f$  be a continuous function from  $I$  into  $I$ ; then there exists a point  $P \in I$  such that  $f(P) = P$  or is a neighbor or diagonal neighbor of  $P$ .

We quote from [4]:

Several subsequent papers have incorrectly concluded that this [Rosenfeld’s] result implies that  $I$  with some  $c_u$  adjacency has the AFPP $_S$ . By *digital picture* Rosenfeld means a digital cube,  $I = [0, n]_{\mathbb{Z}}^v$ . By a “continuous function” he means a  $(c_1, c_1)$ -continuous function; by “a neighbor or diagonal neighbor of  $P$ ” he means a  $c_v$ -adjacent point.

Thus, Rosenfeld’s result was important but different from Theorem 2.3(6), below.

**THEOREM 2.3.** *The following digital images have the AFPP.*

- (1) Any digital interval  $([a, b]_{\mathbb{Z}}, c_1)$  [12, 7].
- (2) Any digital image  $(Y, \lambda)$  that is isomorphic to  $(X, \kappa)$  such that  $(X, \kappa)$  has the AFPP [7]
- (3) Any digital image  $(Y, \kappa)$  that is a retract of  $(X, \kappa)$  such that  $(X, \kappa)$  has the AFPP [7].
- (4) Any digital image  $(T, \kappa)$  that is a tree [5].
- (5) Any digital image  $(X, c_{m+n})$  such that  $X = X' \times \prod_{i=1}^n [a_i, b_i]_{\mathbb{Z}}$ ,  $X' \subset \mathbb{Z}^m$ , and  $(X', c_m)$  has the AFPP [5].
- (6) Any digital cube  $(\prod_{i=1}^n [a_i, b_i]_{\mathbb{Z}}, c_n)$  [5].

The next result suggests that “most” digital images  $(X, c_u) \subset \mathbb{Z}^v$  that have the AFPP have  $u = v$ .

**THEOREM 2.4.** [4] *Let  $X \subset \mathbb{Z}^v$  be such that  $X$  has a subset  $Y = \prod_{i=1}^v [a_i, b_i]_{\mathbb{Z}}$ , where  $v > 1$ ; for all indices  $i$ ,  $b_i \in \{a_i, a_i + 1\}$ ; and, for at least 2 indices  $i$ ,  $b_i = a_i + 1$ . Then  $(X, c_u)$  fails to have the AFPP for  $1 \leq u < v$ .*

**EXAMPLE 2.5.** ([7]) A digital simple closed curve of at least 4 points does not have the AFPP.

**2.4. Digital convexity, disks.** Material in this section is quoted or paraphrased from [6].

Let  $\kappa \in \{c_1, c_2\}$ ,  $n > 1$ . We say a  $\kappa$ -connected set  $S = \{x_i\}_{i=1}^n \subset \mathbb{Z}^2$  is a (*digital*) *line segment* if the members of  $S$  are collinear.

**REMARK 2.6.** ([6]) A digital line segment must be vertical, horizontal, or have slope of  $\pm 1$ . We say a segment with slope of  $\pm 1$  is *slanted*.

A (*digital*)  $\kappa$ -closed curve is a path  $S = \{s_i\}_{i=0}^{m-1}$  such that  $i \neq j$  implies  $s_i \neq s_j$ , and  $s_i \leftrightarrow_{\kappa} s_{(i+1) \bmod m}$  for  $0 \leq i \leq m - 1$ . If  $s_i \leftrightarrow_{\kappa} s_j$  implies  $j = (i \pm 1) \bmod m$ ,

$S$  is a (*digital*)  $\kappa$ -simple closed curve. For a simple closed curve  $S \subset \mathbb{Z}^2$  we generally assume

- $m \geq 8$  if  $\kappa = c_1$ , and
- $m \geq 4$  if  $\kappa = c_2$ .

These requirements are necessary for the Jordan Curve Theorem of digital topology, below, as a  $c_1$ -simple closed curve in  $\mathbb{Z}^2$  needs at least 8 points to have a nonempty finite complementary  $c_2$ -component, and a  $c_2$ -simple closed curve in  $\mathbb{Z}^2$  needs at least 4 points to have a nonempty finite complementary  $c_1$ -component. Examples in [11] show why it is desirable to consider  $S$  and  $\mathbb{Z}^2 \setminus S$  with different adjacencies.

**THEOREM 2.7 ([11]).** (Jordan Curve Theorem for digital topology) *Let  $\{\kappa, \kappa'\} = \{c_1, c_2\}$ . Let  $S \subset \mathbb{Z}^2$  be a simple closed  $\kappa$ -curve such that  $S$  has at least 8 points if  $\kappa = c_1$  and such that  $S$  has at least 4 points if  $\kappa = c_2$ . Then  $\mathbb{Z}^2 \setminus S$  has exactly 2  $\kappa'$ -connected components.*

One of the  $\kappa'$ -components of  $\mathbb{Z}^2 \setminus S$  is finite and the other is infinite. This suggests the following.

**DEFINITION 2.8. ([6])** Let  $S \subset \mathbb{Z}^2$  be a  $c_2$ -closed curve such that  $\mathbb{Z}^2 \setminus S$  has two  $c_1$ -components, one finite and the other infinite. The union  $D$  of  $S$  and the finite  $c_1$ -component of  $\mathbb{Z}^2 \setminus S$  is a (*digital*) *disk*.  $S$  is a *bounding curve* of  $D$ . The finite  $c_1$ -component of  $\mathbb{Z}^2 \setminus S$  is the *interior of  $S$* , denoted  $Int(S)$ , and the infinite  $c_1$ -component of  $\mathbb{Z}^2 \setminus S$  is the *exterior of  $S$* , denoted  $Ext(S)$ .

Note a disk may have multiple distinct bounding curves [6].

A set  $X$  in a Euclidean space  $\mathbb{R}^n$  is *convex* if for every pair of distinct points  $x, y \in X$ , the line segment  $\overline{xy}$  from  $x$  to  $y$  is contained in  $X$ . The *convex hull* of  $Y \subset \mathbb{R}^n$ , denoted  $hull(Y)$ , is the smallest convex subset of  $\mathbb{R}^n$  that contains  $Y$ . If  $Y \subset \mathbb{R}^2$  is a finite set, then  $hull(Y)$  is a single point if  $Y$  is a singleton; a line segment if  $Y$  has at least 2 members and all are collinear; otherwise,  $hull(Y)$  is a polygonal disk, and the endpoints of the edges of  $hull(Y)$  are its *vertices*.

A digital version of convexity can be stated for subsets of the digital plane  $\mathbb{Z}^2$  as follows. A finite set  $Y \subset \mathbb{Z}^2$  is (*digitally*) *convex* [6] if either

- $Y$  is a single point, or
- $Y$  is a digital line segment, or
- $Y$  is a digital disk with a bounding curve  $S$  such that the endpoints of the maximal digital line segments of  $S$  are the vertices of  $hull(Y) \subset \mathbb{R}^2$ .

### 3. Retractions, convexity, and the AFPP

Due to assertions (3) and (6) of Theorem 2.3, the following theorem can be useful in determining whether  $(X, c_2)$  has the AFPP, for  $X \subset \mathbb{Z}^2$ .

**THEOREM 3.1.** *Let  $X \subset Y = [a, b]_{\mathbb{Z}} \times [c, d]_{\mathbb{Z}} \subset \mathbb{Z}^2$ , such that  $X$  is a digitally convex disk. Let  $S$  be a bounding curve for  $X$ . Then there is a  $c_2$ -retraction  $r : Y \rightarrow X$  such that  $r(Y \setminus Int(S)) = S$ .*

PROOF. We define a function  $r : Y \rightarrow X$  as follows. For  $y \in X$ ,  $r(y) = y$ . For  $y \notin X$  we proceed as follows. Let

$$m = \min\{p_1(x) \mid x \in X\}, \quad L = \{x \in S \mid p_1(x) = m\},$$

$$M = \max\{p_1(x) \mid x \in X\}, \quad R = \{x \in S \mid p_1(x) = M\}.$$

- Suppose  $y = (a, b)$  for  $m \leq a \leq M$ .
  - If  $x = (a, c) \in S$  such that  $b > c = \max\{n \mid (a, n) \in X\}$  then  $r(y) = x$ . (See  $y_1, y_2, y_3$  in Figure 1.)
  - If  $x = (a, d) \in S$  such that  $b < d = \min\{n \mid (a, n) \in X\}$  then  $r(y) = x$ . (See  $y_4$  in Figure 1.)

- Suppose  $y = (a, b)$  for  $a < m$ ; then there is a unique nearest (in the Euclidean metric)  $y' \in L$  to  $y$ , determined as follows. Let

$$s_0 = \min\{p_2(x) \mid x \in L\}, \quad s_1 = \max\{p_2(x) \mid x \in L\}.$$

- If  $b > s_1$  then  $y' = (m, s_1)$ . (See  $y_5$  in Figure 1.)
- If  $s_0 \leq b \leq s_1$  then  $y' = (m, b)$ . (See  $y_6$  in Figure 1.)
- If  $b < s_0$  then  $y' = (m, s_0)$ . (See  $y_7$  in Figure 1.)

Let  $r(y) = y'$ .

- Suppose  $y = (a, b)$  for  $a > M$ ; then there is a unique nearest (in the Euclidean metric)  $y' \in R$  to  $y$ , determined as follows. Let

$$s_2 = \min\{p_2(x) \mid x \in R\}, \quad s_3 = \max\{p_2(x) \mid x \in R\}.$$

- If  $b > s_3$  then  $y' = (M, s_3)$ . (See  $y_8$  in Figure 1.)
- If  $s_2 \leq b \leq s_3$  then  $y' = (M, b)$ . (See  $y_9$  in Figure 1.)
- If  $b < s_2$  then  $y' = (M, s_2)$ . (See  $y_{10}$  in Figure 1.)

Let  $r(y) = y'$ .

In order to show  $r$  is a  $c_2$ -retraction, we must show  $r \in C(X, c_2)$ . Let  $y \leftrightarrow_{c_2} y'$  in  $X$ .

- Suppose  $y \in X$ .
  - If  $y' \in X$ , then we have  $r(y') = y' \leftrightarrow_{c_2} y = r(y)$ .
  - If  $y' \notin X$  then we must have  $y \in S$ . Then either  $r(y') = r(y)$ , or, since  $X$  is convex, it follows from Remark 2.6 that  $r(y') \leftrightarrow_{c_2} y = r(y)$ .
- Suppose  $y$  is vertically above or below a point  $x \in S$ , so  $r(y) = x$ . Since  $X$  is convex, it follows from Remark 2.6 that  $r(y') \leftrightarrow_{c_2} x = r(y)$ .
- Suppose  $p_1(y) < m$ . Since  $X$  is convex, it follows from Remark 2.6 that  $r(y') \leftrightarrow_{c_2} r(y)$ .
- Suppose  $p_1(y) > M$ . Since  $X$  is convex, it follows from Remark 2.6 that  $r(y') \leftrightarrow_{c_2} r(y)$ .

Thus  $r \in C(X, c_2)$ . Therefore,  $r$  is a retraction. Clearly,  $r(Y \setminus \text{Int}(S)) = S$ . This completes the proof.  $\square$

**THEOREM 3.2.** *Let  $X \subset Y = [a, b]_{\mathbb{Z}} \times [c, d]_{\mathbb{Z}} \subset \mathbb{Z}^2$ , such that  $X$  is digitally convex. Then  $(X, c_2)$  has the AFPP.*

PROOF. By Theorem 2.3(5),  $(Y, c_2)$  has the AFPP. By Theorem 3.1,  $(X, c_2)$  is a retract of  $(Y, c_2)$ . By Theorem 2.3(3),  $(X, c_2)$  has the AFPP.  $\square$

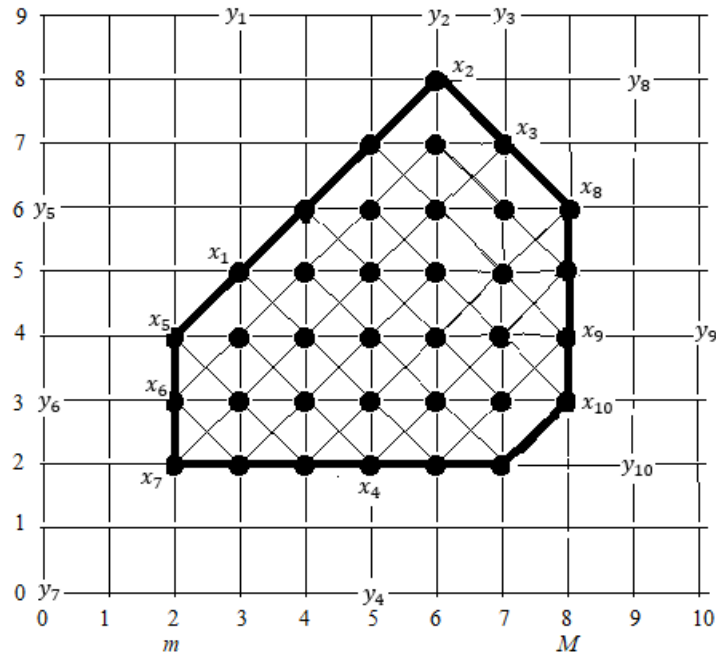


FIGURE 1. Retraction  $r$  of a digital image  $Y$  to a subset  $X$  that is a convex disk as in Theorem 3.1. Here,  $s_0 = 2$ ,  $s_1 = 4$ ,  $s_2 = 3$ ,  $s_3 = 6$ .

- a) Each point vertically above or below the disk is mapped to its nearest vertical neighbor in  $X$ , e.g.,  $r(y_i) = x_i$ ,  $i \in \{1, 2, 3, 4\}$ .
- b) Each point to the left (not necessarily horizontally) of  $X$  is mapped to the nearest member of  $X$  with minimal first coordinate, e.g.,  $r(y_i) = x_i$ ,  $i \in \{5, 6, 7\}$ .
- c) Each point to the right (not necessarily horizontally) of  $X$  is mapped to the nearest member of  $X$  with maximal first coordinate, e.g.,  $r(y_i) = x_i$ ,  $i \in \{8, 9, 10\}$ .

COROLLARY 3.3. Let  $X = X_1 \times X_2$ , where  $X_1 \subset \mathbb{Z}^n$ ,  $(X_1, c_n)$  has the AFPP,  $X_2 \subset \mathbb{Z}^2$ , and  $X_2$  is a digitally convex disk. Then  $(X, c_{n+2})$  has the AFPP.

PROOF. Clearly there exists  $Y = [a, b]_{\mathbb{Z}} \times [c, d]_{\mathbb{Z}}$  such that  $X_2 \subset Y$ . By Theorem 2.3(4),  $(X_1 \times Y, c_{n+2})$  has the AFPP. By Theorem 3.1, there is a  $c_2$ -retraction  $r : Y \rightarrow X_2$ . Then  $\text{id}_{X_1} \times r : X_1 \times Y \rightarrow X$  is a  $c_{n+2}$ -retraction. The assertion follows from Theorem 2.3(3).  $\square$

PROPOSITION 3.4. Let  $X \subset \mathbb{Z}^2$  be finite. Suppose  $X' \subset \mathbb{Z}^2$  is a convex disk with bounding curve  $S$  such that  $X' \setminus S$  is a  $c_1$ -component of  $\mathbb{Z}^2 \setminus X$ . Then there is a  $c_2$ -retraction of  $X$  onto  $S$ .

PROOF. By Theorem 3.1, there is a  $c_2$ -retraction  $r : X \cup X' \rightarrow X'$  such that  $r(X) = S$ . Then  $r|_X : X \rightarrow S$  is a retraction.  $\square$

THEOREM 3.5. *Let  $X \subset \mathbb{Z}^2$  be finite. Suppose  $X' \subset X$  is a convex disk with bounding curve  $S$  such that  $X' \setminus S$  is a  $c_1$ -component of  $\mathbb{Z}^2 \setminus X$ . Suppose there is a continuous  $F : S \rightarrow S$  such that*

$$(3.1) \quad \text{for each } x \in S, N^*(X, \kappa, x) \cap N^*(X, \kappa, F(x)) = \emptyset.$$

*Then  $(X, c_2)$  does not have the AFPP.*

PROOF. By Proposition 3.4, there is a  $c_2$ -retraction  $r : X \rightarrow S$ . Let  $F$  be as described above and let  $f : X \rightarrow X$  be the function  $f(x) = F \circ r(x)$ . Since composition preserves continuity, we have  $f \in C(X, c_2)$ .

For  $y \in X$ , consider the following cases.

- Suppose  $y \not\approx_{c_2} x$  for all  $x \in S$ . Then in particular,  $y \not\approx_{c_2} f(y)$ , so  $y$  is not an approximate fixed point for  $f$ .
- Suppose  $y \approx_{c_2} x$  for some  $x \in S$ . Then the continuity of  $f$  implies  $f(y) \approx_{c_2} f(x)$ . It follows from (3.1) that  $y$  is not an approximate fixed point for  $f$ .

Thus  $f$  does not have an approximate fixed point. The assertion follows.  $\square$

EXAMPLE 3.6. Let  $X = [-3, 3]_{\mathbb{Z}}^2 \setminus A$ , where  $A = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| \leq 1\}$ . See Figure 2. As a bounding curve for  $A$ , we can take  $S = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| = 2\}$ . Then  $S$  is a  $c_2$ -simple closed curve. Let  $F : S \rightarrow S$  be the map  $F(x, y) = (-x, -y)$ . Then we may apply Theorem 3.5 to conclude that  $(X, c_2)$  does not have the AFPP.

#### 4. Further remarks

We have explored relationships between the convexity of digital images in  $\mathbb{Z}^2$  and the AFPP.

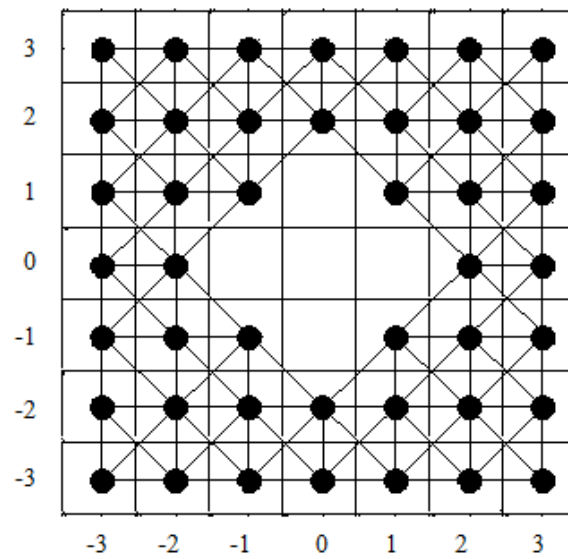
In classical topology, every *absolute retract* (a contractible compactum with certain “nice” local properties for which we need not consider analogs in digital topology) has the FPP [1]. Since under the definition of digital homotopy in [2], a digital simple closed curve of 4 points is digitally contractible [3], Example 2.5 shows that contractibility based on [2] is not sufficient for the AFPP. Recent papers of Staecker [13] and Lupton, Oprea, and Scoville [10] have developed a different notion of homotopy under which a digital simple closed curve of 4 points is not digitally contractible; Staecker calls this *strong homotopy*. This suggests the following questions concerning possible extensions of Theorem 3.2.

QUESTION 4.1. Let  $X \subset \mathbb{Z}^2$  be finite and  $c_2$ -strongly contractible, i.e., contractible with respect to strong homotopy. Does  $(X, c_2)$  have the AFPP?

If Question 4.1 and the following Question 4.2 both have affirmative answers, the latter result would be contained in the former.

QUESTION 4.2. Let  $X \subset \mathbb{Z}^2$  be a digital disk. Does  $(X, c_2)$  have the AFPP?

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FIGURE 2. The image  $X$  of Example 3.6

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