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SHEFFER STROKE BG-ALGEBRAS BY FILTERS AND HOMOMORPHISMS

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ABSTRACT. In this paper, a Sheffer stroke BG-algebra, a subalgebra and a normal subset of a Sheffer stroke BG-algebra are given. After determining a filter of Sheffer stroke BG-algebra, a congruence relation on a Sheffer stroke BG-algebra is defined by means of its filter, and quotient of a Sheffer stroke BG-algebra by a congruence relation on this algebra is built. Besides, a homomorphism between Sheffer stroke BG-algebras is introduced and it is presented that their features are preserved under this homomorphism.

1. Introduction

In 1996, Y. Imai and K. Iséki introduced a new algebraic structure called BCK-algebra. K. Iséki introduced the new idea be called BCI-algebra [**5**]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 2002, J. Neggers and H. S. Kim [**10**] constructed a new algebraic structure. They took some properties from BCI and BCK-algebras be called B-algebra. In 2005, C. B. Kim and H. S. Kim [**6**] introduced the notion of a BG-algebra which is a generalization of B-algebras. A non-empty set X with a binary operation $*$ and a constant 0 satisfying some axioms will construct an algebraic structure be called BG-algebra. With these ideas, fuzzy subalgebras of BG-algebra were developed by Ahn and Lee [**2**]. Muthuraj et al. [**8**] presented fuzzy ideals in BG-algebra in 2010. Muthuraj and Devi [**9**] introduced the concept of multi-fuzzy subalgebra of BG-algebra in 2016.

On the other hand, H. M. Sheffer defined originally Sheffer stroke (or Sheffer operation) in 1913 [**17**]. Since any Boolean function or operation can be stated by

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only this operation [**7**], it attracts many researchers' attention. It also causes to reduction of axiom systems of many structures. So, many researchers want to use this operation on their studies. Also, some applications of this operation has been appeared in algebraic structures such as Sheffer stroke non-associative MV-algebras [**3**] and filters [**14**], (fuzzy) filters of Sheffer stroke BL-algebras [**15**], Sheffer stroke Hilbert algebras [**12**] and filters [**13**], Sheffer stroke UP-algebras [**16**], Sheffer stroke BG-algebras [**11**] and Sheffer operation in ortholattices [**4**].

After giving basic definitions and notions about a Sheffer stroke BG-algebra, a subalgebra and a normal subset of a Sheffer stroke BG-algebra, their features are presented. A filter on a Sheffer stroke BG-algebra is defined. It is shown that the family of all filters of a Sheffer stroke BG-algebra forms a complete lattice, and for a subset of a Sheffer stroke BG-algebra, there exists the minimal filter containing this subset. It is proved that a subalgebra of a Sheffer stroke BG-algebra is a filter. Then a medial Sheffer stroke BG-algebra is determined and it is indicated that every non-empty subset of a medial Sheffer stroke BG-algebra is a filter. Besides, it is proved that the intersection of arbitrary number of filters of a Sheffer stroke BG-algebra is a filter while the union of arbitrary number of filters of a Sheffer stroke BG-algebra is generally not a filter. Morever, defining a homomorphism between Sheffer stroke BG-algebras, it is demonstrated that mentioned notions such as filter is preserved under this homomorphism. A kernel of a homomorphism is built and it is proved that the kernel is a filter. Finally, a congruence relation on a Sheffer stroke BG-algebra is determined by its filter and related concepts are given. It is shown that a quotient of a Sheffer stroke BG-algebra defined by a congruence relation is a Sheffer stroke BG-algebra. It is proved that the kernel is a normal subalgebra.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke, a BG-algebra and a Sheffer stroke BG-algebra.

DEFINITION 2.1. ([**4**]) Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be *Sheffer stroke* if it satisfies the following conditions:

$(S1)$ $a_1|a_2 = a_2|a_1$,

$(S2)$ $(a_1|a_1)|(a_1|a_2) = a_1$,

$(S3)$ $a_1|((a_2|a_3)|(a_2|a_3)) = ((a_1|a_2)|(a_1|a_2))|a_3$,

$(S4)$ $(a_1|((a_1|a_1)|(a_2|a_2)))|(a_1|((a_1|a_1)|(a_2|a_2))) = a_1$.

DEFINITION 2.2. [**6**] A BG-algebra is a non-empty set A with a constant 0 and a binary operation $*$ satisfying the following axioms:

$(BG.1)$ $a_1 * a_1 = 0$,

$(BG.2)$ $a_1 * 0 = a_1$,

$(BG.3)$ $(a_1 * a_2) * (0 * a_2) = a_1$, for all $a_1, a_2 \in A$.

A BG-algebra is called bounded if it has the greatest element.

Definition 2.3. ([**6**]) A nonempty subset S of a BG-algebra A is called a BG-subalgebra if $a_1 * a_2 \in S$, for all $a_1, a_2 \in S$.

Definition 2.4. ([**10**]) Let A be a BG-algebra. A nonempty subset N of A is said to be normal if $(a_1 * x) * (a_2 * y) \in N$, for any $a_1 * a_2$, $x * y \in N$.

Definition 2.5. ([**1**]) A BG-algebra A is called medial if $a_1 * (a_1 * a_2) = a_2$, for all $a_1, a_2 \in A$.

Definition 2.6. ([**11**]) A Sheffer stroke BG-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A and the following axioms are satisfied:

$(sBG.1)$ $(a_1|(a_1|a_1)|(a_1|(a_1|a_1) = 0$,

$(sBG.2)$ $(0|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) = a_1|a_1$,

for all $a_1, a_2 \in A$.

Lemma 2.1 ([**11**]). *Let A be a Sheffer stroke BG-algebra. Then the following features hold for all $a_1, a_2, a_3 \in A$:*

(1) $(0|0)|(a_1|a_1) = a_1$,

(2) $(a_1|(0|0))|(a_1|(0|0)) = a_1$,

(3) $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = (a_3|(a_2|a_2))|(a_3|(a_2|a_2))$ *implies* $a_1 = a_3$,

(4) $(0|(0|(a_1|a_1))) = a_1|a_1$,

(5) *If* $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$ *then* $a_1 = a_2$,

(6) *If* $(0|(a_1|a_1)) = (0|(a_2|a_2))$ *then* $a_1 = a_2$,

(7) $(((a_1|(0|(a_1|a_1)))|(a_1|(0|(a_1|a_1))))|(a_1|a_1)) = a_1|a_1$,

(8) $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$.

Definition 2.7. ([**11**]) A non-empty subset S of a Sheffer stroke BG-algebra A is called a Sheffer stroke BG-subalgebra of A if $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in S$, for all $a_1, a_2 \in S$.

Definition 2.8. ([**11**]) Let A be a Sheffer stroke BG-algebra. A non-empty subset N of A is said to be normal subset of A if

$$(((a_1|(x|x))|(a_1|(x|x)))|(a_2|(y|y)))|(((a_1|(x|x))|(a_1|(x|x)))|(a_2|(y|y))) \in N,$$

for any $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)), (x|(y|y))|(x|(y|y)) \in N$.

Theorem 2.1 ([**11**]). *Every normal subset N of a Sheffer stroke BG-algebra A is a Sheffer stroke BG-subalgebra of A.*

Lemma 2.2 ([**11**]). *Let N be a Sheffer stroke normal subalgebra of a Sheffer stroke BG-algebra A and let $a_1, a_2 \in N$. If $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in N$, then*

$$(a_2|(a_1|a_1))|(a_2|(a_1|a_1)) \in N.$$

3. On filters of Sheffer stroke BG-algebras

We introduce the notion of a filter in a Sheffer stroke BG-algebra in this section.

Definition 3.1. A filter of a Sheffer stroke BG-algebra A is a nonempty subset $F \subseteq A$ which satisfies the following:

$(SF.1)$ If $a_1, a_2 \in F$, then

$$(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))) \in F \text{ and } (a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) \in F.$$

$(SF.2)$ If $a_1 \in F$ and $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$, then $a_2 \in F$.

Theorem 3.1. *Let A be a Sheffer stroke BG-algebra. Then the family K_A of all filters of A forms a complete lattice.*

Proof. Let $\{F_i\}_{i\in I}$ be a family of filters of A. If $a_1, a_2 \in \bigcap_{i\in I} F_i$, then $a_1, a_2 \in F_i$, for all $i \in I$. Since F_i is a filter of A, then we have

$$(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))), (a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) \in F_i.$$

Then

$$(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))), (a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) \in \bigcap_{i\in I} F_i.$$

(i) Suppose that $a_1 \in \bigcap_{i\in I} F_i$ and $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in \bigcap_{i\in I} F_i$ holds for $a_1, a_2 \in A$, that is $a_1 \in F_i$ and $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$ hold for all $i \in I$. Then it is obtained from (SF.2) that $a_2 \in F_i$ for all $i \in I$. Then $a_2 \in \bigcap_{i\in I} F_i$.

(ii) Let η be the family of all filters of A containing the union $\bigcup_{i\in I} F_i$. Then $\bigcap \eta$ is a filter of A from (i). If we define $\bigwedge_{i\in I} F_i = \bigcap_{i\in I} F_i$ and $\bigvee_{i\in I} F_i = \bigcap \eta$, then $(K_A, \bigwedge, \bigvee)$ is a complete lattice. □

Lemma 3.1. *Let A be a Sheffer stroke BG-algebra and B be a subset of A. Then there is the minimal filter $\langle B\rangle$ containing the subset B.*

Proof. Let $\varepsilon = \{F : F \text{ } is \text{ } a \text{ } filter \text{ } of \text{ } A \text{ } containing \text{ } B\}$. Then

$$\langle B\rangle = \{x \in A : x \in \bigcap_{F\in\varepsilon} F\}$$

is the minimal filter of A containing B. □

Theorem 3.2. *Let A be a Sheffer stroke BG-algebra and S be a subalgebra of A. Then S is a filter of A.*

Proof. $(SF.1)$ Let S be a subalgebra of A and $a_1, a_2 \in S$. Then

$$(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in S \text{ and } (a_2|(a_1|a_1))|(a_2|(a_1|a_1)) \in S.$$

So

$$(a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) \in S$$

and

$$(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))) \in S.$$

$(SF.2)$ Let $a_1 \in S$ and $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0 = (a_2|(a_2|a_2))|(a_2|(a_2|a_2))$. Then $a_1 = a_2$ from (sBG.1) and Lemma 2.1 (3). Then $a_2 \in S$. □

Corollary 3.1. *Let S be a normal subset of A. Then S is a filter of A.*

Definition 3.2. A Sheffer stroke BG-algebra A is called a medial Sheffer stroke BG-algebra if

$$(a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) = a_2,$$

for all $a_1, a_2 \in A$.

Theorem 3.3. *Let A be a medial Sheffer stroke BG-algebra. Then every nonempty subset S of A is a filter of A.*

Proof. Let A be a non-empty subset of A.

$(SF.1)$ Let $a_1, a_2 \in S$. Since A is a medial Sheffer stroke BG-algebra,

$$(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))) = a_1 \in S,$$

and

$$(a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) = a_2 \in S.$$

$(SF.2)$ Let $a_1 \in S$ and $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$. Then

$$a_2 = (a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) = (a_1|(0|0))|(a_1|(0|0)) = a_1$$

from Lemma 2.1 (2). Then $a_2 \in S$. □

Proposition 3.1. *Let A be a Sheffer stroke BG-algebra and let $\{Fi, i \in \lambda\}$ be a family of filters of A. Then $\bigcap_{i\in\lambda} F_i$ is a filter of A.*

Proof. Let $\{F_i, i \in \lambda\}$ be a family of filters of A.

$(SF.1)$ If $a_1, a_2 \in \bigcap_{i\in\lambda} F_i$, then $a_1, a_2 \in F_i$ for all $i \in \lambda$. Since F_i is a filter of A, then

$$(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))), (a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) \in F_i.$$

Then

$$(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))), (a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) \in \bigcap_{i\in\lambda} F_i.$$

$(SF.2)$ Let $a_1 \in \bigcap_{i\in\lambda} F_i$ such that $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$. Then $a_1 \in F_i$ for all $i \in \lambda$. Since F_i is a filter of A, we obtain $a_2 \in F_i$ for all $i \in \lambda$. Then $a_2 \in \bigcap_{i\in\lambda} F_i$.

Therefore, we get $\bigcap_{i\in\lambda} F_i$ is a filter of A. □

Remark 3.1. The union of filters of a Sheffer stroke BG-algebra may be not a Sheffer stroke BG-filter.

Proposition 3.2. *Let A be a Sheffer stroke BG-algebra and let $\{F_i, i \in \lambda\}$ be a chain of filters of A. Then $\bigcup_{i\in\lambda} F_i$ is a filter of A.*

Proof. Let $\{F_i, i \in \lambda\}$ be a chain of filters of A.

$(SF.1)$ If $a_1, a_2 \in \bigcup_{i\in\lambda} F_i$, for all $i \in \lambda$, then there exist $F_j, F_k \in \{F_i\}_{i\in\lambda}$ such that $a_1 \in F_j$ and $a_2 \in F_k$. So, either $F_j \subseteq F_k$ or $F_k \subseteq F_j$. If $F_j \subseteq F_k$, then $a_1 \in F_k$

and $a_2 \in F_k$. Since F_k is a filter of A, we have $(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))$ and $(a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))) \in F_k$. Similarly, if $F_k \subseteq F_j$. Then

$$(a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))), (a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))) \in \bigcup_{i\in\lambda} F_i.$$

$(SF.2)$ Let $a_1 \in \bigcup_{i\in\lambda} F_i$ such that $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$. There exists $j \in \lambda$ such that $a_1 \in F_j$. Since F_i is a filter of A, we obtain $a_2 \in F_j$. Thus $a_2 \in \bigcup_{i\in\lambda} F_i$. Therefore, $\bigcup_{i\in\lambda} F_i$ is a filter of A. □

4. Homomorphisms on Sheffer stroke BG-algebras

In this section, we present some definitions and concepts about homomorphism between Sheffer stroke BG-algebras.

Definition 4.1. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BG-algebras. A mapping $f : A \longrightarrow B$ is called homomorphism if

$$f(a_1|_A a_2) = f(a_1)|_B f(a_2),$$

for all $a_1, a_2 \in A$.

A Sheffer stroke BG-homomorphism f is called a Sheffer stroke BG-monomorphism if it is injective.

Lemma 4.1. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BG-algebras and $f : A \longrightarrow B$ be a monomorphism. Then if F is a filter of A, then $f(F)$ is a filter of B.*

Proof. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BG-algebras and $f : A \longrightarrow B$ be a monomorphism.

• Let F be a filter of A and $a_1, a_2 \in f(F)$. Then there exist $x, y \in F$ such that $a_1 = f(x)$ and $a_2 = f(y)$. Since F is a filter, we obtain

$$\begin{aligned}&(a_2|_B(a_2|_B(a_1|_Ba_1)))|_B(a_2|_B(a_2|_B(a_1|_Ba_1)))\\&= (f(y)|_B(f(y)|_B(f(x)|_Bf(x))))|_B(f(y)|_B(f(y)|_B(f(x)|_Bf(x))))\\&= f((y|_A(y|_A(x|_Ax)))|_A(y|_A(y|_A(x|_Ax)))) \in f(F).\end{aligned}$$

Hence we obtain

$$(a_2|_B(a_2|_B(a_1|_Ba_1)))|_B(a_2|_B(a_2|_B(a_1|_Ba_1))) \in f(F),$$

and similarly,

$$(a_1|_B(a_1|_B(a_2|_Ba_2)))|_B(a_1|_B(a_1|_B(a_2|_Ba_2))) \in f(F).$$

• Let $a_1 \in f(F)$ such that $(a_1|_B(a_2|_Ba_2))|_B(a_1|_B(a_2|_Ba_2)) = 0_B$. Then there exist $x, y \in F$ such that $a_1 = f(x), a_2 = f(y)$. Hence

$$\begin{aligned}&(a_1|_B(a_2|_Ba_2))|_B(a_1|_B(a_2|_Ba_2))\\&= (f(x)|_B(f(y)|_Bf(y)))|_B(f(x)|_B(f(y)|_Bf(y)))\\&= f((x|_A(y|_Ay))|_A(x|_A(y|_Ay))) = 0_B = f(0_A).\end{aligned}$$

Since f is injective, we get $(x|_A(y|_Ay))|_A(x|_A(y|_Ay)) = 0_A$. Thus $y \in F$. So,

$a_2 = f(y) \in f(F)$.

Therefore $f(F)$ is a filter of B. □

THEOREM 4.1. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BG-algebras and $f : A \longrightarrow B$ be a homomorphism. If F is a filter of B, then $f^{-1}(F)$ is a filter of A.*

PROOF. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BG-algebras and $f : A \longrightarrow B$ be a homomorphism. Suppose that F is a filter of A.

• Let $a_1, a_2 \in f^{-1}(F)$. Then $f(a_1), f(a_2) \in F$. Since F is a filter,
$(f(a_2)|_B((f(a_2)|_B(f(a_1)|_Bf(a_1)))))|_B(f(a_2)|_B((f(a_2)|_B(f(a_1)|_Bf(a_1)))$
$= f((a_2|_A(a_2|_A(a_1|_Aa_1)))|_A(a_2|_A(a_2|_A(a_1|_Aa_1)))) \in F$.
Therefore, we obtain

$$(a_2|_A(a_2|_A(a_1|_Aa_1)))|_A(a_2|_A(a_2|_A(a_1|_Aa_1))) \in f^{-1}(F)$$

and

$$(a_1|_A(a_1|_A(a_2|_Aa_2)))|_A(a_1|_A(a_1|_A(a_2|_Aa_2))) \in f^{-1}(F).$$

• Let $a_1 \in f^{-1}(F)$ such that $(a_1|_A(a_2|_Aa_2))|_A(a_1|_A(a_2|_Aa_2)) = 0_A$. Then $f(a_1) \in F$ and

$$\begin{aligned} f((a_1|_A(a_2|_Aa_2))|_A(a_1|_A(a_2|_Aa_2))) &= (f(a_1)|_B(f(a_2)|_Bf(a_2)))|_B \\ &\quad\ (f(a_1)|_B(f(a_2)|_Bf(a_2))) \\ &= f(0_A) \\ &= 0_B. \end{aligned}$$

Hence $f(a_2) \in F$. Thus $a_2 \in f^{-1}(F)$.

Therefore, $f^{-1}(F)$ is a filter of A. □

PROPOSITION 4.1. *Let A and B be Sheffer stroke BG-algebras and $f : (A, |_A, 0_A) \to (B, |_B, 0_B)$ be a Sheffer stroke BG-homomorphism. Then $Ker(f)$ is a filter of A.*

PROOF. $(SF.1)$: Let $a_1, a_2 \in Ker(f)$. Then $f(a_1) = 0_B$, $f(a_2) = 0_B$. From Lemma 2.1 (4) we obtain
$f((a_2|_A(a_2|_A(a_1|_Aa_1)))|_A(a_2|_A(a_2|_A(a_1|_Aa_1))))$
$= (f(a_2)|_B(f(a_2)|_B(f(a_1)|_Bf(a_1))))|_B(f(a_2)|_B(f(a_2)|_B(f(a_1)|_Bf(a_1))))$
$= 0_B$.
Hence

$$(a_2|_A(a_2|_A(a_1|_Aa_1)))|_A(a_2|_A(a_2|_A(a_1|_Aa_1))) \in Ker(f)$$

and

$$(a_1|_A(a_1|_A(a_2|_Aa_2)))|_A(a_1|_A(a_1|_A(a_2|_Aa_2))) \in Ker(f).$$

$(SF.2)$: Let $a_1 \in Ker(f)$ such that $(a_1|_A(a_2|_Aa_2))|_A(a_1|_A(a_2|_Aa_2)) = 0_A$. Then $f(a_1) = 0_B$. Hence,

$$\begin{aligned} f((a_1|_A(a_2|_Aa_2))|_A(a_1|_A(a_2|_Aa_2))) &= (f(a_1)|_B(f(a_2)|_Bf(a_2)))|_B \\ &\quad (f(a_1)|_B(f(a_2)|_Bf(a_2))) \\ &= f(0_A) \\ &= 0_B. \end{aligned}$$

Thereby, $(0_B|_B(f(a_2)|_Bf(a_2)))|_B(0_B|_B(f(a_2)|_Bf(a_2))) = (f(a_2)|_B(f(a_2)|_Bf(a_2)))$ $|_B(f(a_2)|_B(f(a_2)|_Bf(a_2)))$. We obtain $f(a_2) = 0_B$ from Lemma 2.1 (3). Thus $a_2 \in Ker(f)$.

Therefore, $Ker(f)$ is a filter of A. □

Lemma 4.2. *Let A be a Sheffer stroke BG-algebra and N be a normal subalgebra of A. Define a relation $\sim_N$ on A by*

$$a_1 \sim_N a_2 \text{ if and only if } (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in N,$$

for $a_1, a_2 \in A$. Then $\sim_N$ is an equivalence relation on A.

Proof. $\cdot \sim_N$ is reflexive since $0 \in A$, we have $(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0 \in A$, i.e., $a_1 \sim_N a_1$ for all $a_1 \in A$.

$\cdot$ Symmetric: Let $a_1 \sim_N a_2$, where $a_1, a_2 \in A$. Then $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in N$ and

$$(a_2|(a_1|a_1))|(a_2|(a_1|a_1)) \in N$$

by Lemma 2.2. We obtain $a_2 \sim_N a_1$.

$\cdot$ Transitive: Let $a_1 \sim_N a_2$ and $a_2 \sim_N a_3$. Then

$$(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in N$$

and

$$(a_2|(a_3|a_3))|(a_2|(a_3|a_3)) \in N.$$

By Lemma 2.2, we have $(a_3|(a_2|a_2))|(a_3|(a_2|a_2)) \in N$. Since N is normal subalgebra, then

$(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_2|a_2)))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_2|a_2)))$
$= (((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(0|0))|((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(0|0))$
$= ((a_1|(a_3|a_3))|(a_1|(a_3|a_3)) \in N$.

We obtain $a_1 \sim_N a_3$. Hence, $\sim_N$ is transitive. □

Lemma 4.3. *An equivalence relation $\sim_*$ is a congruence relation if and only if $a_1 \sim_* a_2$ and $x \sim_* y$ imply $(a_1|(x|x))|(a_1|(x|x)) \sim_* (a_2|(y|y))|(a_2|(y|y))$.*

Lemma 4.4. *Let A be a Sheffer stroke BG-algebra and N be a normal subalgebra of A. Then $\sim_N$ is a congruence relation on A.*

Proof. Let x, y, a_1, a_2 be any elements in A such that $a_1 \sim_N a_2$ and $x \sim_N y$,i.e., $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in N$ and $(x|(y|y))|(x|(y|y)) \in N$. Since N is a normal subalgebra, we obtain

$$(((a_1|(x|x))|(a_1|(x|x)))|(a_2|(y|y)))|(((a_1|(x|x))|(a_1|(x|x)))|(a_2|(y|y))) \in N.$$

Then $(a_1|(x|x))|(a_1|(x|x)) \sim_* (a_2|(y|y))|(a_2|(y|y))$. Therefore, $\sim_N$ is a congruence relation on A. □

Denote the equivalence class containing a_1 by $[a_1]_N$, i.e., $[a_1]_N = \{a_2 \in N \mid a_1 \sim_N a_2\}$ and $A/N = \{[a_1]_N \mid a_1 \in A\}$.

THEOREM 4.2. *Let A be a Sheffer stroke BG-algebra and N be a normal subalgebra of a Sheffer stroke BG-algebra A. Then A/N is a Sheffer stroke BG-algebra.*

PROOF. If we define

$$[a_1]_N|[a_2]_N := [a_1|a_2]_N,$$

then the operation $|$ is well defined, since if $a_1 \sim_N p$ and $a_2 \sim_N q$, then

$$(a_1|(p|p))|(a_1|(p|p)) \in N$$

and

$$(a_2|(q|q))|(a_2|(q|q)) \in N$$

imply

$$(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(p|(q|q)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(p|(q|q))) \in N$$

by normality of N. Hence, $(((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \sim_N (((p|(q|q))|(p|(q|q)))$ and so $((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))_N = (((p|(q|q))|(p|(q|q))))_N$. Note that $[0]_N = \{a_1 \in A \mid a_1 \sim_N 0\} = \{a_1 \in A \mid (a_1|(0|0))|(a_1|(0|0)) \in N\} = \{a_1 \in A \mid a_1 \in N\} = N$.

$(sBG.1)$ $([a_1]_N|([a_1]_N|[a_1]_N)|([a_1]_N|([a_1]_N|[a_1]_N) = [0]_N$,

$(sBG.2)$ $(0|([a_2]_N|[a_2]_N))|(([a_1]_N|([a_2]_N|[a_2]_N))|([a_1]_N|(([a_2]_N|[a_2]_N))))$
$= [a_1]_N|\ [a_1]_N$. □

The Sheffer stroke BG-algebra A/N discussed in Theorem 4.2 is called the quotient Sheffer stroke BG-algebra of A by N.

THEOREM 4.3. *Let A be a Sheffer stroke BG-algebra and N be a normal subalgebra of A. If F is a filter of A, then F/N is a filter of Sheffer stroke BG-algebra A/N.*

PROOF. N be a normal subalgebra and F be a filter of A. Then $(A/N, |^{'}, [0]_N)$ is a Sheffer stroke BG-algebra from Theorem 4.2.

• Let $[a_1]_N, [a_2]_N \in F/N$. So $([a_2]_N|^{'}([a_2]_N|^{'}([a_1]_N\ |^{'}[a_1]_N)))|^{'}([a_2]_N|^{'}([a_2]_N|^{'}([a_1]_N|^{'}\ [a_1]_N))) = [(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))]_N$. Since F is a filter of A and $(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))) \in F$, we obtain

$$([a_2]_N|^{'}([a_2]_N|^{'}([a_1]_N|^{'}[a_1]_N)))|^{'}([a_2]_N|^{'}([a_2]_N|^{'}([a_1]_N|^{'}[a_1]_N))) \in F/N$$

and

$$([a_1]_N|^{'}([a_1]_N|^{'}([a_2]_N|^{'}[a_2]_N)))|^{'}([a_1]_N|^{'}([a_1]_N|^{'}([a_2]_N|^{'}[a_2]_N))) \in F/N.$$

• Let $a_1 \in F/N$ such that $([a_1]_N|^{'}([a_2]_N|^{'}[a_2]_N))|^{'}([a_1]_N|^{'}([a_2]_N|^{'}[a_2]_N)) = [0]_N$. Then $[(a_1|(a_2|a_2))|(a_1|(a_2|a_2))]_N = [0]_N$. Hence

$$((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|0)|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|0) \in N.$$

Thereby, $((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \in N$, then $a_2 \in [a_1]_N$. We obtain $[a_2]_N = [a_1]_N$, then $[a_2]_N \in F/N$. Therefore, F/N is a filter of A/N. □

THEOREM 4.4. *Let N be a normal subalgebra of a Sheffer stroke BG-algebra A. Then the mapping $\gamma : A \to A/N$ by $\gamma(a_1) := [a_1]_N$ is a surjective Sheffer stroke BG-homomorphism and $Ker\gamma = N$.*

This mapping γ is called the naturalhomomorphism of A onto A/N.

THEOREM 4.5. *Let A and B be Sheffer stroke BG-algebras and $\varphi : A \to B$ be a Sheffer stroke BG-homomorphism. Then φ is injective if and only if $Ker\varphi = \{0_A\}$.*

PROOF. Let $a_1, a_2 \in A$ with $\varphi(a_1) = \varphi(a_2)$. Then from (sBG.1), we obtain $(\varphi(a_1)|(\varphi(a_2)|\varphi(a_2)))|(\varphi(a_1)|(\varphi(a_2)|\varphi(a_2))) = 0_B$. So $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in Ker\varphi$. Since $Ker\varphi = \{0_A\}$, $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0_A$. Then $a_1 = a_2$ by Lemma 2.1 (5). Hence φ is injective. The converse is trivial. □

THEOREM 4.6. *Let A and B be Sheffer stroke BG-algebras and $\varphi : A \to B$ be a Sheffer stroke BG-homomorphism. Then $Ker\varphi$ is a normal subalgebra of A.*

PROOF. Since $0_A \in Ker\varphi$, Ker $\varphi \neq \emptyset$. Let $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)), (x|(y|y))|(x|(y|y)) \in Ker\varphi$. Then $(\varphi(a_1)|(\varphi(a_2)|\varphi(a_2)))|(\varphi(a_1)|(\varphi(a_2)|\varphi(a_2))) = 0 = (\varphi(x)|(\varphi(y)|\varphi(y)))|(\varphi(x)|(\varphi(y)|\varphi(y)))$. Then from Lemma 2.1 (5), $\varphi(a_1) = \varphi(a_2)$ and $\varphi(x) = \varphi(y)$. In this case we obtain

$$\begin{aligned}
&\varphi((((a_1)|(x|x))|(a_1|(x|x)))|(a_2|(y|y)))|((((a_1)|(x|x))|(a_1|(x|x)))|(a_2|(y|y)))\\
&= (((\varphi(a_1)|\varphi(x)|\varphi(x))|(\varphi(a_1)|\varphi(x)|\varphi(x)))|(\varphi(a_2)|\varphi(y)|\varphi(y)))|\\
&\quad (((\varphi(a_1)|\varphi(x)|\varphi(x))|(\varphi(a_1)|\varphi(x)|\varphi(x)))|(\varphi(a_2)|\varphi(y)|\varphi(y)))\\
&= (((\varphi(a_1)|\varphi(x)|\varphi(x))|(\varphi(a_1)|\varphi(x)|\varphi(x)))|(\varphi(a_1)|\varphi(x)|\varphi(x)))|\\
&\quad (((\varphi(a_1)|\varphi(x)|\varphi(x))|(\varphi(a_1)|\varphi(x)|\varphi(x)))|(\varphi(a_1)|\varphi(x)|\varphi(x)))\\
&= 0.
\end{aligned}$$

Consequently,

$$((((a_1)|(x|x))|(a_1|(x|x)))|((a_2|(y|y))|(a_2|(y|y))|(a_2|(y|y))|(a_2|(y|y)))) \in Ker\varphi.$$

Hence $Ker\varphi$ is a normal subalgebra of A. □

By Theorem 4.4 and Theorem 4.6, if $\varphi : A \to B$ is a Sheffer stroke BG-homomorphism, then $A/Ker\varphi$ is a Sheffer stroke BG-algebra.

THEOREM 4.7. *Let A and B be Sheffer stroke BG-algebras and $\varphi : A \to B$ be a Sheffer stroke BG-homomorphism. Then*

$$A/Ker\varphi \cong Im\varphi.$$

In particular, if φ is surjective, then

$$A/Ker \cong B.$$

THEOREM 4.8. *Let A be a Sheffer stroke BG-algebra. Let N and K be two normal subalgebras of A and let $K \subseteq N$. Then*

$$A/N \cong \frac{A/K}{N/K}$$

.

THEOREM 4.9. *Let A, B and C be Sheffer stroke BG-algebras, $h : A \to B$ be a Sheffer stroke BG-epimorphism and $g : A \to C$ be a Sheffer stroke BG-homomorphism. If $Ker(h) \subseteq Ker(g)$, then there exists a unique Sheffer stroke BG-homomorphism $f : A \to B$ satisfying $f \circ h = g$.*

THEOREM 4.10. *Let A, B and C be Sheffer stroke BG-algebras, $h : B \to C$ be a Sheffer stroke BG-homomorphism and $g : A \to C$ be a Sheffer stroke BG-monomorphism. If $Im(g) \subseteq Im(h)$, then there exists a unique Sheffer stroke BG-homomorphism $f : A \to B$ satisfying $h \circ f = g$.*

PROOF. For each $a_1 \in A$, we have $g(a_1) \in Im(g) \subseteq Im(h)$. Since h is a Sheffer stroke BG-monomorphism, there exists a unique $a_2 \in B$ such that $h(a_2) = g(a_1)$. Define a map $f : A \to B$ by $f(a_1) = a_2$. Then $h \circ f = g$. Let $a_3, a_4 \in A$. Then $g((a_3|(a_4|a_4))|(a_3|(a_4|a_4))) = h(f((a_3|(a_4|a_4))|(a_3|(a_4|a_4))))$. Since h is injective,

$$\begin{aligned} f((a_3|(a_4|a_4))|(a_3|(a_4|a_4))) &= f(a_3|(a_4|a_4))|f((a_3|(a_4|a_4))) \\ &= f(a_3|f(a_4)|f(a_4))|f((a_3)|f(a_4)|f(a_4)). \end{aligned}$$

As a consequence, f is a Sheffer stroke BG-homomorphism. The uniqueness of f follows from the fact that h is a Sheffer stroke BG-monomorphism. □

THEOREM 4.11. *Let A and B be Sheffer stroke BG-algebras and let $f : A \to B$ be a Sheffer stroke BG-homomorphism. If N is a normal subalgebra of A such that $N \subseteq Ker(f)$, then $\bar{f} : A/N \to B$ defined by $\bar{f}([a_1]_N) := f(a_1)$ for all $a_1 \in A$ is a unique Sheffer stroke BG-homomorphism such that $\bar{f} \circ \gamma = f$, where $\gamma : A \to A/N$ is a natural Sheffer stroke BG-homomorphism.*

THEOREM 4.12. *Let A and B be Sheffer stroke BG-algebras and let $f : A \to B$ be a Sheffer stroke BG-homomorphism. If N is a normal subalgebra of A such that $N \subseteq Ker(f)$, then the following are equivalent:*
(i) There exists a unique Sheffer stroke BG-homomorphism $\bar{f} : A/N \to B$ such that $\bar{f} \circ \gamma = f$, where $\gamma : A \to A/N$ is the natural Sheffer stroke BG-homomorphism,
(ii) $N \subseteq Ker(f)$.
Furthermore, $\bar{f}$ is a Sheffer stroke BG-monomorphism if and only if $N = Ker(f)$.

PROOF. $(ii) \Rightarrow (i)$: It is obtained from Theorem 4.9.
$(i) \Rightarrow (ii)$: If $a_1 \in N$, then

$$f(a_1) = (\bar{f} \circ \gamma)(a_1) = \bar{f}([a_1]_N) = \bar{f}([0]_N) = f(0) = 0.$$

Thus, $a_1 \in Ker(f)$. Furthermore, $\bar{f}$ is a monomorphism if and only if $Ker\bar{f} = \{N\}$ if and only if $f(a_1) = 0$ implies $[a_1]_N = [0]_N = N$ if and only if $Ker(f) \subseteq N$. □

5. Conclusions

In this study, a Sheffer stroke BG-algebra, a filter, a congruence relation, a homomorphism, a quotient structure, kernel and their features are studied. Basic definitions and notions about a Sheffer stroke BG-algebra, a subalgebra and a normal subset are given. Besides, by defining a filter on Sheffer stroke BG-algebra, it is proved that the family of all filters of a Sheffer stroke BG-algebra forms a complete lattice, and that for a subset of a Sheffer stroke BG-algebra, there exists the minimal filter containing this subset. It is shown that a subalgebra of a Sheffer stroke BG-algebra is a filter. A medial Sheffer stroke BG-algebra is determined and it is proved that every non-empty subset of a medial Sheffer stroke BG-algebra is a filter. Morever, a homomorphism between two Sheffer stroke BG-algebras is described and it is stated that mentioned notions are preserved under this homomorphism. Finally, a congruence relation on a Sheffer stroke BG-algebra is determined and it is shown that a quotient of a Sheffer stroke BG-algebra is a Sheffer stroke BG-algebra.

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