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ON THE ROMAN DOMINATION POLYNOMIAL OF GRAPHS

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Abstract. A Roman dominating function on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value

$$W(f(V)) = \sum_{u\in V(G)} f(u).$$

The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G and is denoted by $\gamma_R(G)$. In this paper, we introduce the Roman domination polynomial of a graph G as

$$R(G, x) = \sum_{j=\gamma_R(G)}^{2n} r(G, j)x^j,$$

where $r(G, j)$ is the number of Roman dominating functions of G of weight j. We establish new and interested results on this domination polynomial by studying some of its properties and obtain its formulas for some specific graphs and some graph operations.

1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of all vertices and edges of G, respectively. The open neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are defined by

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$$N(v) = \{u \in V(G) : uv \in E\} \text{ and } N[v] = N(v) \cup \{v\},$$

respectively. The cardinality of $N(v)$ is called the degree of the vertex v and denoted by $deg(v)$ in G. For more terminology and notations about graph, we refer the reader to [**6, 8**].

A subset D of $V(G)$ is called dominating set if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that v is adjacent to u. The minimum cardinality of a dominating set in G is called the domination number of G and denoted by $\gamma(G)$. For more details about domination of graphs, we refer to [**9**].

The domination polynomial $D(G, x)$ of a graph G is defined by

$$D(G, x) = \sum_{j=\gamma(G)}^{n} d(G, j)x^j,$$

where $d(G, j)$ is the number of all the dominating sets of G of size j [**5**]. The dominating sets and the domination polynomial of graphs have been studied extensively in [**5, 3, 4, 2**]. Recently, the injective domination polynomial of graphs has been studied in [**1**].

The Roman domination of graph has been suggested in [**11**]. A Roman dominating function of a graph $G = (V, E)$ (or in brief RDF of G) is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value

$$W(f(V)) = \sum_{u \in V(G)} f(u).$$

The minimum weight of a Roman dominating function of a graph G is called the Roman domination number of G which is denoted by $\gamma_R(G)$. For more details about Roman domination and its properties, we refer to [**7**].

The join $G_1 \vee G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $|V(G_1)| = n_1$, $|V(G_2)| = n_2$ is the graph obtained by connecting each vertex of G_1 to each vertex of G_2, while keeping all edges of both graphs.

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2, where $|V(G_1)| = n_1$, $|V(G_2)| = n_2$ is the graph obtained by taking n_1 copies of G_2 and joining the ith vertex of G_1 to each vertex of the i-th copy of G_2.

There are many graph polynomials have been introduced and studied extensively like Characteristic polynomial, Chromatic polynomial, Matching polynomial, Tutte polynomial...etc. The graph polynomial is one of the ways for algebraic graph representation. By the analysis of graph polynomial and studying its properties we get some information about the graph, which motivated us to introduce this new type of graph polynomial. In this paper, we introduce the Roman domination polynomial of graphs. Some properties of $R(G, x)$ are obtained and exact formulas for some specific graphs and graph operations are computed.

2. Definition and Properties

In this section, we define the Roman domination polynomial of a graph G and study some of its properties.

Definition 2.1. Let G be a graph on n vertices. The Roman domination polynomial of G is denoted by $R(G, x)$ and defined as

$$R(G,x) = \sum_{j=\gamma_R(G)}^{2n} r(G,j)x^j,$$

where $\gamma_R(G)$ is the Roman domination number of G and $r(G, j)$ is the number of Roman dominating functions of G of weight j. The roots of $R(G, x)$ are called the Roman dominating roots of the graph G.

To illustrate Definition 2.1, consider the cycle graph C_4. The cycle C_4 has four Roman dominating functions of weight three ($\gamma_R(G) = 3$), fifteen Roman dominating functions of weight four, sixteen Roman dominating functions of weight five, ten Roman dominating functions of weight six, four Roman dominating functions of weight seven and one Roman dominating function of weight eight, then the Roman domination polynomial of C_4 is

$$R(C_4, x) = x^8 + 4x^7 + 10x^6 + 16x^5 + 15x^4 + 4x^3.$$

It is easy to see that, if a graph G consists of m components $G_1, \dots, G_m$, then $R(G, x) = R(G_1, x) \dots R(G_m, x)$.

Proposition 2.1. *Let G be the empty graph $\overline{K_n}$ on n vertices. Then*

$$R(G,x) = \sum_{j=0}^{n} \binom{n}{j} x^{n+j} = x^n\big(1+x\big)^n.$$

The following proposition is an easy consequence from the definition of the Roman domination polynomial of graphs.

Proposition 2.2. *Let G be a non trivial graph on n vertices. Then*

(1) *$R(G, x)$ has no constant term.*
(2) *$R(G, x)$ has no term of degree one.*
(3) *Zero is a root of $R(G, x)$, with multiplicity $\gamma_R(G)$.*
(4) *$R(G, x)$ never equal x^p for any $2 \leqslant p \leqslant 2n$.*
(5) *For any graph G, $r(G, 2n) = 1$ and $r(G, 2n - 1) = n$.*
(6) *$r(G, j) = 0$ if and only if $j < \gamma_R(G)$ or $j > 2n$.*
(7) *$R(G, x)$ is a strictly increasing function in $[0, \infty)$.*
(8) *The only polynomial of degree two can $R(G, x)$ be equal is $x^2 + x$ if and only if $G \cong K_1$.*
(9) *Let H be any induced subgraph of G. Then*

$$deg\big(R(G,x)\big) \geqslant deg\big(R(H,x)\big).$$

In the next theorem, we show that using a Roman domination polynomial of a graph G, we can obtain the number of isolated vertices, the number of vertices of degree one, the number of vertices of degree two and the number of K_2-components in the graph G.

THEOREM 2.1. *Let G be a graph on n vertices with i isolated vertices, t vertices of degree one and l vertices of degree two. Suppose $R(G,x) = \sum_{j=\gamma_R(G)}^{2n} r(G,j)x^j$ is the Roman domination polynomial of G. Then the following hold*

(1) $r(G, 2n-1) = n$.
(2) $i = \dfrac{n(n+1)}{2} - r(G, 2n-2)$.
(3) $r(G, 2n-3) = 2\dbinom{n}{2} + \dbinom{n}{3} - i(n-1) - t$.
(4) *If G has s K_2-components, then*

$$r(G, 2n-4) = \binom{n}{2} + 3\binom{n}{3} + \binom{n}{4} - i(n-1) + \binom{i}{2} - t(n-1) + s - l.$$

(5) *If $G \neq K_2$, then $r(G,2) = |\{v \in V(G) : |deg(v) = n-1\}|$.*

PROOF.
(1) It is clear that, for any vertex $v \in V(G)$ the function $f : V(G) \to \{0,1,2\}$ with $f(v) = 1$ and weight $W(f(V)) = 2n-1$ is a Roman dominating function of G. Hence, $r(G, 2n-1) = |V(G)| = n$.

(2) Suppose $I \subseteq V(G)$ is the set of all isolated vertices in G. Then $|I| = i$ (by assumption). If G is connected, it is easy to see that

$$r(G, 2n-2) = \binom{n}{1} + \binom{n}{2} = \frac{n(n+1)}{2}.$$

But since G has i isolated vertices, then each function $f : V(G) \to \{0,1,2\}$ with weight $2n-2$ in which $f(v) = 0$, where $v \in I$ is not Roman dominating function of G. Hence, $r(G, 2n-2) = \dfrac{n(n+1)}{2} - i$.

(3) The number of all functions $f : V(G) \to \{0,1,2\}$ with $W(f(V)) = 2n-3$ is $2\dbinom{n}{2} + \dbinom{n}{3}$, but some of them are not Roman dominating functions of G. To determine the number of non Roman dominating functions of G in this situation, we have two cases:

Case 1. Suppose $v \in I \subseteq V(G)$ is an isolated vertex of G. Then each function $f : V(G) \to \{0,1,2\}$ with $W(f(V)) = 2n-3$ having $f(v) = 0$ and $f(w) = 1$ for any vertex $w \in V(G) \setminus \{v\}$ is not RDF of G. Therefore, we have $i(n-1)$ non RDFs of G.

Case 2. Suppose $u, w \in V(G)$ are any two adjacent vertices such that $deg(u) = 1$. Then any function $f : V(G) \to \{0,1,2\}$ with $W(f(V)) = 2n-3$ having $f(u) = 0$

and $f(w) = 1$ is not RDF of G. Thus, we have t non RDFs of G. This complete the proof.

(4) Clearly, the number of all functions $f : V(G) \to \{0, 1, 2\}$ with weight $W(f(V)) = 2n - 4$ is $\binom{n}{2} + 3\binom{n}{3} + \binom{n}{4}$. We have the graph G with i isolated vertices, t vertices of degree one, l vertices of degree two and here also G has s K_2-components. To determine the number of non RDFs of G in this situation, we have three cases:

Case 1. Suppose $v \in I \subseteq V(G)$. If $f(v) = 0$ and any other vertex $w \in V(G) \setminus \{v\}$ has the value $f(w) = 0$ under a function $f : V(G) \to \{0, 1, 2\}$ of weight $W(f(V)) = 2n - 4$, then f is not RDF of G. Therefore, we have $i(n-1)$ non RDFs of G, but if $w \in I$, we need to reduce $\binom{i}{2}$ because v and w counted twice. Hence, we have $i(n-1) - \binom{i}{2}$ non RDFs of G.

Case 2. Suppose $u, w \in V(G)$ are any two adjacent vertices such that $deg(u) = 1$. Then for any function $f : V(G) \to \{0, 1, 2\}$ with $W(f(V)) = 2n - 4$ having $f(u) = 0$, we characterize the following:

- If $f(w) = 0$, then f is not RDF of G, and since G has s K_2-components, then we have $t - s$ non RDFs of G in this case.
- If $f(w) = 1$, again f is not RDF of G and therefore, we have $t(n-2)$ non RDFs of G.

Case 3. Suppose that $v \in V(G)$ is a vertex of G with $deg(v) = 2$ and $u, w \in V(G)$ are adjacent to v. Then any function $f : V(G) \to \{0, 1, 2\}$ of weight $W(f(V)) = 2n - 4$ with $f(v) = 0$ and $f(u) = f(w) = 1$ is not RDF of G. Hence, in this case we have l non RDFs of G and the proof is complete.

(5) From the definition of domination in a graph G on n vertices, any vertex $v \in V(G)$ of degree $n - 1$ form a dominating set of G. Therefore, any function $f : V(G) \to \{0, 1, 2\}$ with $f(v) = 2$ and $f(w) = 0$ for all $w \in V(G) \setminus \{v\}$ is a RDF of G of weight $W(f(V)) = 2$. In case of K_2 graph, we can get one more RDF of K_2 of weight 2 by labeling the both vertices by 1. $\square$

3. Roman domination polynomial for some specific graphs

In this section, we obtain the Roman domination polynomial for some families of graphs and graph operations. We will start by the complete graph K_n.

Theorem 3.1. *For any complete graph K_n, with $n \geqslant 2$,*

$$R(K_n, x) = x^n + \sum_{j=2}^{2n} \left[\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n}{r} \binom{n-r}{j-2r} \right] x^j.$$

PROOF. It is easy to see that, $\gamma_R(K_n) = 2$. Therefore, we have

$$\sum_{r=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n}{r}\binom{n-r}{j-2r}$$

RDFs of K_n with weight j, where $j = 2, 3, \ldots, 2n$ (note that r denotes to the number of vertices that have chosen from n whose taking the value 2 under a RDF $f : V(G) \to \{0, 1, 2\}$ of K_n with $W(f(V)) = j$), and also, we have one more function with weight n when all the vertices of K_n taking the value 1. □

In the next theorem, we obtain the Roman domination polynomial of the star graph S_n.

THEOREM 3.2. *For any star graph S_n, with $n \geqslant 3$,*

$$\begin{aligned} R(S_n, x) =& x^n + \sum_{j=2}^{2n} \Bigg[\sum_{r=1}^{\lfloor \frac{j-2}{2} \rfloor} \binom{n-1}{r}\binom{n-r-1}{j-2(r+1)} + \binom{n-1}{j-2} \Bigg] x^j \\ &+ \sum_{r=1}^{n-1} \binom{n-1}{r} (x^{n+r} + x^{n+r-1}). \end{aligned}$$

PROOF. We have here $\gamma_R(S_n) = 2$. Thus, we can characterize the RDFs of S_n into three types (note that as in the proof of Theorem 3.1, r denotes to the number of vertices that have chosen from n whose taking the value 2 under a RDF $f : V(G) \to \{0, 1, 2\}$ of S_n with $W(f(V)) = j$):

Type 1. RDFs of S_n with weight j in which the center vertex takes value 2, and they are $\sum_{r=1}^{\lfloor \frac{j-2}{2} \rfloor} \binom{n-1}{r}\binom{n-r-1}{j-2(r+1)} + \binom{n-1}{j-2}$ functions.

Type 2. RDFs of S_n with weight $n + r$ in which the center vertex takes the value 1, and they are $\binom{n-1}{r}$ functions, where r starting from zero (note that in this type no vertex can take the value 0 under any RDFs of S_n).

Type 3. RDFs of S_n with weight $n + r - 1$ in which the center vertex takes the value 0, and they are $\binom{n-1}{r}$ functions, where r starting from one (note that in this type also no vertex can take the value 0 under any RDFs of S_n except the center vertex). This complete the proof. □

The next theorem give us the formula of the Roman domination polynomial of the wheel graph W_n.

THEOREM 3.3. *Let $G \cong W_n$ on $n \geqslant 4$ vertices. Then*

$$\begin{aligned} R(G, x) =& \sum_{j=2}^{2n} \Bigg[\sum_{r=1}^{\lfloor \frac{j-2}{2} \rfloor} \binom{n-1}{r}\binom{n-r-1}{j-2(r+1)} + \binom{n-1}{j-2} \Bigg] x^j \\ &+ (x+1)R(C_{n-1}, x) - x^{n-1}. \end{aligned}$$

Proof. For the wheel graph W_n, we know that $\gamma_R(W_n) = 2$. Since $W_n = K_1 \vee C_{n-1}$, we have two cases:

Case 1. RDFs of W_n with weight j in which the center vertex takes the value 2, and they are $\sum_{r=1}^{\lfloor \frac{j-2}{2} \rfloor} \binom{n-1}{r}\binom{n-r-1}{j-2(r+1)} + \binom{n-1}{j-2}$ functions.

Case 2. In this case, we have two types of RDFs of W_n. In the first types, the center vertex takes the value 1, and in the second types, the center vertex takes the value 0 and in both of them any RDF of C_{n-1} is also a RDF of W_n except one function in the second types, when all the vertices of C_{n-1} taking the value 1. Hence, we will obtain the term $(x+1)R(C_{n-1}, x) - x^{n-1}$. $\square$

In the next theorem, we get the Roman domination polynomial of the complete bipartite graph $K_{n,m}$.

Theorem 3.4. *Let $G \cong K_{n,m}$ with $n, m \geqslant 2$. Then*

$$\begin{aligned}R(G,x) =&x^{n+m} + \sum_{r=1}^{n}\binom{n}{r}\Big[\sum_{t=0}^{n-r}\binom{n-r}{t}\Big(\sum_{s=1}^{m}\binom{m}{s}\sum_{l=0}^{m-s}\binom{m-s}{l}\Big)\Big]x^{2r+t+2s+l}\\ &+\sum_{t=0}^{n}\binom{n}{t}\Big[\sum_{s=1}^{m}\binom{m}{s}\Big]x^{m+s+t} + \sum_{l=0}^{m}\binom{m}{l}\Big[\sum_{r=1}^{n}\binom{n}{r}\Big]x^{n+r+l}.\end{aligned}$$

Proof. Let $G \cong K_{n,m}$ with $n, m \geqslant 2$. Then we have three cases:

Case 1. RDFs of $K_{n,m}$ with weight $2r + t + 2s + l$, where r, s denote to the number of vertices that have chosen from n, m, respectively, whose taking the value 2, and they are $\binom{n}{r}\Big[\sum_{t=0}^{n-r}\binom{n-r}{t}\Big(\sum_{s=1}^{m}\binom{m}{s}\sum_{l=0}^{m-s}\binom{m-s}{l}\Big)\Big]$ functions (note that, in this case at least one vertex of n and one vertex of m taking the value 2).

Case 2. RDFs of $K_{n,m}$ with weight $m + s + t$, where t denotes to the number of vertices that have chosen from n whose taking the value 1, and they are $\binom{n}{t}\Big[\sum_{s=1}^{m}\binom{m}{s}\Big]$ functions, where t starting from zero (note that, in this case no vertex of m takes the value 0).

Case 3. RDFs of $K_{n,m}$ with weight $n + r + l$, where l denotes to the number of vertices that have chosen from m whose taking the value 1, and they are $\binom{m}{l}\Big[\sum_{r=1}^{n}\binom{n}{r}\Big]$ functions, where l starting from zero (note that, in this case no vertex of n takes the value 0).

Finally, there is one more RDF of $K_{n,m}$ in which all the vertices taking the value 1, which has the weight $n + m$. The proof is complete. $\square$

A firefly graph $F_{s,t,l}$ on n vertices, where $n = 2s + 2t + l + 1$ such that $s \geqslant 0$, $t \geqslant 0$ and $l \geqslant 0$, is a graph consists of s triangles, t pendent paths of length 2 and l pendent edges, sharing a common vertex.

Let $\mathfrak{F}_n$ be the set of all firefly graphs $F_{s,t,l}$. Note that $\mathfrak{F}_n$ contains the stars S_n ($\cong F_{0,0,n-1}$), stretched stars ($\cong F_{0,t,l}$), friendship graphs ($\cong F_{\frac{n-1}{2},0,0}$) and butterfly graphs ($\cong F_{s,0,l}$), [**10**].

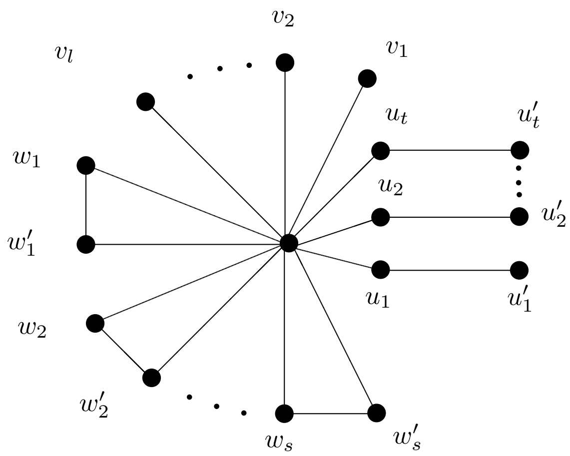

FIGURE 1. Firefly graph $F_{s,t,l}$

In the following, we obtain the Roman domination polynomial of the firefly graph in case $t = 0$. We start with the following lemma.

LEMMA 3.1. *Let $G \cong F_{s,0,l}$ be a firefly graph, where $t = 0$, $s > 0$ and $n > 2s+1$. Then $\gamma_R(F_{s,0,l}) = 2$.*

THEOREM 3.5. *For the firefly graph $F_{s,0,l}$, where $t = 0$, $s > 0$ and $n > 2s + 1$,*

$$R(F_{s,0,l}, x) = R(S_n, x) + \sum_{r=0}^{l} \binom{l}{r} \sum_{i=0}^{s} \binom{s}{i} \left[x^{n+r} + x^{n+r-1}\right].$$

PROOF. It is easy to see that, S_n is a spanning subgraph of $F_{s,0,l}$. Thus all types of RDFs of S_n in the proof of Theorem 3.2 are also RDFs of $F_{s,0,l}$, it just remains the RDFs of Types 2 and 3, when the vertices of s taking the values 0 or 2 and they are $\binom{l}{r} \sum_{i=0}^{s} \binom{s}{i}$ functions, where r starting from zero. $\square$

In the next theorems, we obtain the Roman domination polynomial of the join $G_1 \vee G_2$ of two connected graphs G_1 and G_2 and the corona product $K_n \circ K_1$.

THEOREM 3.6. *Let G_1, G_2 be any two connected graphs with $|G_1| = n_1$, $|G_2| = n_2$, respectively. Then*

$$\begin{aligned}R(G_1 \vee G_2, x) &= R(G_1, x)\sum_{l=0}^{n_2}\binom{n_2}{l}x^l - x^{n_1}\sum_{l=0}^{n_2-1}\binom{n_2}{l}x^l \\ &+ \sum_{r=1}^{n_1}\binom{n_1}{r}\Big[\sum_{t=0}^{n_1-r}\binom{n_1-r}{t}\Big(\sum_{s=1}^{n_2}\binom{n_2}{s}\sum_{l=0}^{n_2-s}\binom{n_2-s}{l}\Big)\Big]x^{2r+t+2s+l} \\ &+ R(G_2, x)\sum_{t=0}^{n_1}\binom{n_1}{t}x^t - x^{n_2}\sum_{t=0}^{n_1-1}\binom{n_1}{t}x^t - x^{n_1+n_2}.\end{aligned}$$

PROOF. Let G_1, G_2 be any two connected graphs with $|G_1| = n_1$, $|G_2| = n_2$, respectively. Then we have the following cases:

Case 1. RDFs of $G_1 \vee G_2$ with weight $2r + t + 2s + l$, where r, s denote to the number of vertices that have chosen from n_1, n_2, respectively, whose taking the value 2, and they are

$$\binom{n_1}{r}\Big[\sum_{t=0}^{n_1-r}\binom{n_1-r}{t}\Big(\sum_{s=1}^{n_2}\binom{n_2}{s}\sum_{l=0}^{n_2-s}\binom{n_2-s}{l}\Big)\Big]$$

functions (note that, in this case at least one vertex of n_1 and one vertex of n_2 taking the value 2).

Case 2. In this case, we will take all the RDFs of G_1 such that no vertex in G_2 take the value 2 [note that, any RDF of G_1 (G_2) is also a RDF of $G_1 \vee G_2$ except when all the vertices of G_1 (G_2) taking the value 1 and there is at least one vertex in the other side takes value 0]. Therefore, we obtain the terms

$$R(G_1, x)\sum_{l=0}^{n_2}\binom{n_2}{l}x^l - x^{n_1}\sum_{l=0}^{n_2-1}\binom{n_2}{l}x^l,$$

where l denotes to the number of vertices that have chosen from n_2 whose taking the value 1.

Case 3. Here, we will do the same for G_2 as in Case 1, thus we get the terms

$$R(G_2, x)\sum_{t=0}^{n_1}\binom{n_1}{t}x^t - x^{n_2}\sum_{t=0}^{n_1-1}\binom{n_1}{t}x^t,$$

where t denotes to the number of vertices that have chosen from n_1 whose taking the value 1.

Note that, the RDF of the term $x^{n_1+n_2}$ has been counted twice (once in Case 1 and again in Case 2), thus we have to subtract this term from the formula once. $\square$

LEMMA 3.2. *Let $G \cong K_n \circ K_1$ with $n \geqslant 3$. Then $\gamma_R(G) = n + 1$.*

Now, we get our result for $K_n \circ K_1$.

THEOREM 3.7. *Let $G \cong K_n \circ K_1$ with $n \geqslant 3$. Then*

$$R(K_n \circ K_1, x) = \sum_{s=1}^{n} \binom{n}{s} x^{2n+s} + \sum_{t=1}^{n-1} \binom{n-1}{t} x^{2n+t} + 2^{n+1} x^{2n}$$
$$+ \sum_{r=1}^{n} \binom{n}{r} \sum_{t=0}^{n-r} \binom{n-r}{t} \sum_{s=0}^{n-r} \binom{n-r}{s} \Big[x^{n+r+t+s} + x^{n+2r+t+s} + x^{n+3r+t+s} \Big].$$

PROOF. We have $G \cong K_n \circ K_1$ with $n \geqslant 3$. To compute the Roman dominating polynomial of $K_n \circ K_1$ we have the following cases:

Case 1. RDFs of $K_n \circ K_1$ with weights $(n+r+t+s)$, $(n+2r+t+s)$ and $(n+3r+t+s)$, where r, t denote to the number of vertices that have chosen from K_n whose taking the value 2, 1, respectively, and s denotes to the number of vertices that have chosen from the join vertices to K_n whose taking the value 2, and they are $\binom{n}{r} \sum_{t=0}^{n-r} \binom{n-r}{t} \sum_{s=0}^{n-r} \binom{n-r}{s}$ functions (note that in this case at least one vertex of K_n takes the value 2).

Case 2. RDFs of $K_n \circ K_1$ with weight $2n+s$, where s denotes to the number of vertices that have chosen from the join vertices to K_n which is taking the value 2, and they are $\binom{n}{s}$ functions (note that in this case all the vertices of K_n taking the value 1 and at least one vertex from the join vertices to K_n takes the value 2).

Case 3. RDFs of $K_n \circ K_1$ with weight $2n+t$, where t denotes to the number of vertices that have chosen from K_n which is taking the value 1, and they are $\binom{n-1}{t}$ functions, where t starting from one (note that in this case no vertex of K_n take the value 2 and all the join vertices to K_n taking the value 2).

Case 4. Finally, the RDFs of $K_n \circ K_1$ with weight $2n$ and they are (in this case) $\binom{n}{l}$ functions (l starting from zero), where l denotes to the number of vertices of K_n which taken the value 1 (note that, in this case when $l = 0$ all the vertices of K_n taking the value 0 and all the join vertices to K_n taking the value 2 and in case $l = 1$ one vertex of K_n will take the value 1 and its join vertex will take 1 also). □

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