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GENERALIZED c-ALMOST PERIODIC FUNCTIONS AND APPLICATIONS

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Abstract. In this paper, we introduce the class of (equi-)Weyl-(p, c)-almost periodic functions as well as the classes of quasi-asymptotically c-almost periodic functions, S-asymptotically (ω, c)-periodic functions and their Stepanov generalizations, where $c \in \mathbb{C} \smallsetminus \{0\}$. Composition principles for quasi-asymptotically c-almost periodic functions and related composition principles are established, as well.

1. Introduction and preliminaries

The notion of almost periodicity was studied by Bohr around 1925 and later generalized by many others. We refer the interested reader to the research monographs by Diagana [**1**], Fink [**3**], Guérékata [**4**], Kostić [**7**] and Zaidman [**10**] for the basic introduction to the theory of almost periodic functions.

The main aim of this paper is to continue our recent research study [**6**], where we have introduced and analyzed the class of c-almost periodic functions. We basically aim to report how the ideas followed in [**6**] can be used to define many other function spaces of almost periodic functions. In this paper, we concretely introduce and analyze the class of (equi-)Weyl-(p, c)-almost periodic functions as well as the classes of quasi-asymptotically c-almost periodic functions, S-asymptotically

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(ω, c)-periodic functions and their Stepanov generalizations, where $c \in \mathbb{C} \setminus \{0\}$. Concerning the corresponding classes of functions depending on two variables, we will only consider quasi-asymptotically c-almost periodic functions and formulate some composition principles in this direction.

The organization of paper can be briefly described as follows. After collecting some basic definitions and results about almost periodic type functions, in Section 2 we introduce and analyze the class of (equi-)Weyl-(p, c)-almost periodic functions. Quasi-asymptotically c-almost periodic functions and S-asymptotically (ω, c)-periodic functions are introduced and analyzed in Subsection 3; in Subsection 3.1, we consider composition principles for quasi-asymptotically c-almost periodic functions. Special attention is paid to the analysis of invariance of introduced types of c-almost periodicity under the actions of convolution products and giving some illustrative applications.

We use the standard notation. By $(E, \|\cdot\|)$ we denote a complex Banach space. By $L^p_{loc}(I : E)$, $C(I : E)$ and $C_0(I : E)$ we denote the vector spaces consisting of all p-locally integrable functions $f : I \to E$, all continuous functions $f : I \to E$ and all continuous functions $f : I \to E$ satisfying that $\lim_{|t|\to+\infty} \|f(t)\| = 0$, respectively $(1 \leqslant p < \infty)$. If $f : \mathbb{R} \to E$, then we define $\check{f} : \mathbb{R} \to E$ by $\check{f}(x) := f(-x)$, $x \in \mathbb{R}$; $S_1 := \{z \in \mathbb{C}\,;\, |z| = 1\}$.

In this paper, we will always assume that $c \in \mathbb{C} \setminus \{0\}$. Let $f : I \to E$ be a continuous function and let a number $\epsilon > 0$ be given. We call a number $\tau > 0$ an (ϵ, c)-period for $f(\cdot)$ if $\|f(t+\tau) - cf(t)\| \leqslant \epsilon$ for all $t \in I$. By $\vartheta_c(f, \epsilon)$ we denote the set consisting of all (ϵ, c)-periods for $f(\cdot)$. It is said that $f(\cdot)$ is c-almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta_c(f, \epsilon)$ is relatively dense in $[0, \infty)$, which means that for each $\epsilon > 0$ there exists $l > 0$ such that any subinterval I' of $[0, \infty)$ of length l contains an element of set $\vartheta_c(f, \epsilon)$. The space consisting of all c-almost periodic functions from the interval I into E will be denoted by $AP_c(I : E)$. If $c = 1$, resp. $c = -1$, then we also say that the function $f(\cdot)$ is almost periodic, resp. almost anti-periodic.

Let $1 \leqslant p < \infty$. We continue by recalling that a function $f \in L^p_{loc}(I : E)$ is said to be Stepanov p-bounded if and only if

$$\|f\|_{S^p} := \sup_{t \in I} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

Equipped with the above norm, the space $L^p_S(I : E)$ consisted of all Stepanov p-bounded functions is a Banach space. A function $f \in L^p_S(I : E)$ is said to be Stepanov (p, c)-almost periodic, resp. Stepanov p-almost periodic, if and only if the function $\hat{f} : I \to L^p([0, 1] : E)$, defined by

$$\hat{f}(t)(s) := f(t + s), \quad t \in I,\ s \in [0, 1],$$

is c-almost periodic, resp. almost periodic. Furthermore, we say that a function $f \in L^p_S(I : E)$ is asymptotically Stepanov (p, c)-almost periodic if and only if there exist a Stepanov (p, c)-almost periodic (Stepanov p-almost periodic) function

$g \in L_S^p(I : E)$ and a function $q \in L_S^p(I : E)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : E))$.

2. Weyl-(p, c)-almost periodic type functions

We introduce the notion of an (equi-)Weyl-(p, c)-almost periodic function as follows:

Definition 2.1. Let $1 \leqslant p < \infty$ and $f \in L_{loc}^p(I : E)$.

(i) We say that the function $f(\cdot)$ is equi-Weyl-(p, c)-almost periodic, $f \in e - W_{ap;c}^p(I : E)$ for short, if and only if for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \left\| f(t + \tau) - cf(t) \right\|^p dt \right]^{1/p} \leqslant \epsilon. \tag{2.1}$$

(ii) We say that the function $f(\cdot)$ is Weyl-(p, c)-almost periodic, $f \in W_{ap;c}^p(I : E)$ for short, if and only if for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l \to +\infty} \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \left\| f(t + \tau) - cf(t) \right\|^p dt \right]^{1/p} \leqslant \epsilon.$$

If $c = 1$, resp. $c = -1$, then we also say that $f(\cdot)$ is (equi-)Weyl-p-almost periodic, resp. (equi-)Weyl-p-almost anti-periodic.

It is clear that any equi-Weyl-(p, c)-almost periodic function is Weyl-(p, c)-almost periodic. The proofs of following results are trivial and therefore omitted:

Proposition 2.1. *Suppose that $f : I \to E$ is (equi-)Weyl-(p, c)-almost periodic. Then $\|f\| : I \to [0, \infty)$ is (equi-)Weyl-p-almost periodic.*

Proposition 2.2. *Let $1 \leqslant p < \infty$ and $f \in L_{loc}^p(I : E)$. If the function $f(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic and $I = \mathbb{R}$, then the function $\check{f} : \mathbb{R} \to E$ is (equi-)Weyl-$(p, 1/c)$-almost periodic.*

We will include the proof of following proposition for the sake of completeness:

Proposition 2.3. *Let $1 \leqslant p < \infty$ and $f \in L_{loc}^p(I : E)$. If the function $f(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic and $m \in \mathbb{N}$, then the function $f(\cdot)$ is (equi-)Weyl-(p, c^m)-almost periodic.*

Proof. We will give the proof for the class of equi-Weyl-(p, c)-almost periodic functions. Let $\epsilon > 0$ be fixed; then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (2.1) holds true. Clearly, we have ($x \in I$, $\tau > 0$):

$$f\big(x + m\tau\big) - c^m f(x) = \sum_{j=0}^{m-1} c^j \Big[f\big(x + (m - j)\tau\big) - cf\big(x + (m - j - 1)\tau\big) \Big]. \tag{2.2}$$

Integrating this estimate over the segment $[x, x+l]$, where $x \in I$, we obtain the existence of a finite constant $c_p > 0$ such that

$$\begin{aligned}
&\left[\frac{1}{l}\int_x^{x+l}\big\|f(t+m\tau)-c^m f(t)\big\|^p\,dt\right]^{1/p} \\
&\quad\leqslant c_p\left[\sum_{j=0}^{m-1}\frac{|c|^{jp}}{l}\int_x^{x+l}\Big\|f\big(t+(m-j)\tau\big)-cf\big(t+(m-j-1)\tau\big)\Big\|^p\,dt\right]^{1/p} \\
&\quad\leqslant c_p\left[\sum_{j=0}^{m-1}\frac{|c|^{jp}}{l}\int_{x+(m-j-1)\tau}^{x+l+(m-j-1)\tau}\Big\|f\big(t+\tau\big)-cf\big(t\big)\Big\|^p\,dt\right]^{1/p} \\
&\quad\leqslant c_p\epsilon\left[\sum_{j=0}^{m-1}|c|^{jp}\right]^{1/p}.
\end{aligned}$$

Therefore, for this number $\epsilon > 0$, we can take the numbers $l > 0$ and $mL > 0$ in definition of equi-Weyl-(p, c)-almost periodicity. This completes the proof. $\square$

Consider now the following condition:

$$m \in \mathbb{Z} \smallsetminus \{0\},\ n \in \mathbb{N},\ (m, n) = 1,\ |c| = 1 \text{ and } \arg(c) = \pi m/n. \tag{2.3}$$

The next corollary of Proposition 2.3 follows immediately:

Corollary 2.1. *Let $1 \leqslant p < \infty$, $f \in L^p_{loc}(I : E)$, and let (2.3) hold.*

- (i) *If m is even and $f(\cdot)$ is an (equi-)Weyl-(p, c)-almost periodic function, then $f(\cdot)$ is (equi-)Weyl-p-almost periodic.*
- (ii) *If m is odd and $f(\cdot)$ is an (equi-)Weyl-(p, c)-almost periodic function, then $f(\cdot)$ is (equi-)Weyl-p-almost anti-periodic.*

Proposition 2.4. *Let $1 \leqslant p < \infty$, $f \in L^p_{loc}(I : E)$, and let $|c| = 1$, $\arg(c)/\pi \notin \mathbb{Q}$. If $f(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic and Stepanov p-bounded, then $f(\cdot)$ is (equi-)Weyl-(p, c')-almost periodic for all $c' \in S_1$.*

Proof. It suffices to consider case in which the function $f(\cdot)$ is not almost everywhere equal to zero. Let the numbers $c' \in S_1$ and $\epsilon > 0$ be fixed; then the set $\{c^l : l \in \mathbb{N}\}$ is dense in S_1 and therefore there exists an increasing sequence (l_k) of positive integers such that $\lim_{k\to+\infty} c^{l_k} = c'$. Let $k \in \mathbb{N}$ be such that $|c^{l_k} - c'| < \epsilon/(2\|f\|_{S^p})$, and let $\epsilon > 0$ be given. Then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (2.1) holds. Then we have

$$\big\|f(x+\tau)-c'f(x)\big\| \leqslant \big\|f(x+\tau)-c^{l_k}f(x)\big\| + \big|c^{l_k}-c'\big|\cdot\|f(x)\|,$$

for any $x \in I$. Then the conclusion follows from Proposition 2.3, after integrating the above estimate over the segment $[x, x+l]$ and using the estimate

$$\frac{1}{l}\int_x^{x+l}\|f(t)\|^p\,dt \leqslant \frac{1}{l}(1+\lfloor l\rfloor)\|f\|^p_{S^p}.$$

$\square$

The main structural properties of (equi-)Weyl-(p, c)-almost periodic functions are collected in the following theorem (see also [**7**, Proposition 2.3.5]):

THEOREM 2.1. *Let $f : I \to E$ be (equi-)Weyl-(p, c)-almost periodic, and let $\alpha \in \mathbb{C}$. Then we have:*

(i) *$\alpha f(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic.*
(ii) *If $E = \mathbb{C}$ and $ess \inf_{x\in\mathbb{R}} |f(x)| = m > 0$, then $1/f(\cdot)$ is (equi-)Weyl-$(p, 1/c)$-almost periodic).*
(iii) *If $(g_n : I \to E)_{n\in\mathbb{N}}$ is a sequence of bounded, continuous, (equi-)Weyl-(p, c)-almost periodic functions and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g : I \to E$, then $g(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic.*
(iv) *If $a \in I$ and $b \in I \setminus \{0\}$, then the functions $f(\cdot + a)$ and $f(b\,\cdot)$ are likewise (equi-)Weyl-(p, c)-almost periodic.*

Now we will provide two illustrative examples:

EXAMPLE 2.1. Set $f(t) := \chi_{[0,1/2]}(t)$, $t \in \mathbb{R}$. Then for each number $l > 0$ we have

$$\frac{1}{l}\int_x^{x+l} |f(t+\tau) - cf(t)|^p\, dt \leqslant \frac{1}{2l}(1 + |c|)^p, \quad x \in \mathbb{R}.$$

This implies that $f(\cdot)$ is equi-Weyl-(p, c)-almost periodic for each complex number $c \in \mathbb{C} \setminus \{0\}$ and for each finite exponent $p \geqslant 1$.

EXAMPLE 2.2. Set $f(t) := \chi_{[0,\infty)}(t)$, $t \in \mathbb{R}$. Then for each number $l > 0$ we have

$$\sup_{x\in\mathbb{R}} \frac{1}{l}\int_x^{x+l} |f(t+\tau) - cf(t)|^p\, dt \geqslant |1 - c|^p,$$

so that $f(\cdot)$ cannot be Weyl-(p, c)-almost periodic for $c \neq 1$. On the other hand, it is well known that $f(\cdot)$ is Weyl-$(p, 1)$-almost periodic for any finite exponent $p \geqslant 1$.

Concerning the invariance of (equi-)Weyl-(p, c)-almost periodicity under the actions of convolution products, we will only note that the statements of [**7**, Proposition 2.11.1, Theorem 2.11.4, Proposition 2.11.6] can be simply reformulated in our framework.

3. S-asymptotically (ω, c)-periodic functions

We start this section by introducing the following notion:

DEFINITION 3.1. Let $\omega \in I$. Then we say that a continuous function $f : I \to E$ is S-asymptotically (ω, c)-periodic if and only if $\lim_{|t|\to\infty} \|f(t + \omega) - cf(t)\| = 0$; a continuous function $f : I \to E$ is said to be S_c-asymptotically periodic if and only if there exists $\omega > 0$ such that $f(\cdot)$ is S-asymptotically (ω, c)-periodic. By $SAP_{\omega;c}(I : E)$ and $SAP_c(I : E)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is S-asymptotically ω-anti-periodic, resp. S-asymptotically anti-periodic.

This definition extends the well known definition of an S-asymptotically ω-periodic function, introduced by H. Henríquez et al. [**5**] for case $I = \mathbb{R}$ and M. Kostić [**8**] for case $I = [0, \infty)$. For the Stepanov classes, we will use the following notion:

DEFINITION 3.2. Let $p \in [1, \infty)$. A p-locally integrable function $f(\cdot)$ is said to be Stepanov p-asymptotically (ω, c)-periodic if and only if

$$\lim_{|t|\to\infty} \int_t^{t+1} \big\| f(s+\omega) - cf(s) \big\|^p \, ds = 0;$$

a p-locally integrable function $f : I \to E$ is said to be Stepanov p_c-asymptotically periodic if and only if there exists $\omega > 0$ such that $f(\cdot)$ is Stepanov p-asymptotically (ω, c)-periodic. By $S^pSAP_{\omega;c}(I : E)$ and $S^pSAP_c(I : E)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov p-asymptotically ω-anti-periodic, resp. Stepanov p-asymptotically anti-periodic.

Now we will introduce the class of quasi-asymptotically c-almost periodic functions:

DEFINITION 3.3. It is said that a continuous function $f : I \to E$ is quasi-asymptotically c-almost periodic if and only if for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\epsilon, \tau) > 0$ such that

$$\|f(t+\tau) - cf(t)\| \leqslant \epsilon, \text{ provided } t \in I \text{ and } |t| \geqslant M(\epsilon, \tau).$$

Denote by $Q - AAP_c(I : E)$ the set consisting of all quasi-asymptotically c-almost periodic functions from I into E; if $c = -1$, then we also say that the function $f(\cdot)$ is quasi-asymptotically almost anti-periodic.

The various notions of Stepanov quasi-asymptotically almost periodic functions in Lebesgue spaces with variable exponent, introduced and analyzed in [**2**], can be slightly generalized by the use of the difference $f(\cdot + \tau) - cf(\cdot)$. For simplicity, we will consider the following notion of Stepanov (p, c)-quasi-asymptotically almost periodicity, only:

DEFINITION 3.4. A p-locally integrable function $f(\cdot)$ is said to be Stepanov (p, c)-quasi-asymptotically almost periodic if and only if for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\epsilon, \tau) > 0$ such that

$$\int_t^{t+1} \|f(s+\tau) - cf(s)\|^p \, ds \leqslant \epsilon^p, \text{ provided } t \in I \text{ and } |t| \geqslant M(\epsilon, \tau).$$

By $S^pQ - AAP_c(I : E)$ we denote the set consisting of all Stepanov p-quasi-asymptotically c-almost periodic functions from I into E; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov p-quasi-asymptotically almost anti-periodic.

REMARK 3.1. A p-locally integrable function $f(\cdot)$ is Stepanov (p, c)-quasi-asymptotically almost periodic if and only if the function $f : I \to L^p([0,1] : E)$ is quasi-asymptotically c-almost periodic. Similar statements hold for the classes of Stepanov p-asymptotically (ω, c)-periodic functions and Stepanov p_c-asymptotically periodic functions. This observation enables one to see that many results clarified below, like Proposition 3.1, Corollary 3.1 and Theorem 3.1, continue to hold for the corresponding Stepanov classes of functions under our consideration.

It is very simple to prove that any asymptotically c-almost periodic function is quasi-asymptotically c-almost periodic. Furthermore, (2.2) easily implies:

PROPOSITION 3.1. *Let $\omega > 0$, $f : I \to E$ be an S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), and let $m \in \mathbb{N}$. Then $f(\cdot)$ is S-asymptotically $(m\omega, c^m)$-periodic (S_{c^m}-asymptotically periodic, quasi-asymptotically c^m-almost periodic).*

The next corollary of Proposition 3.1 follows immediately:

COROLLARY 3.1. *Let $f : I \to E$ be a continuous function, and let (2.3) hold.*

(i) *If m is even and $f(\cdot)$ is S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), then $f(\cdot)$ is S-asymptotically ω-anti-periodic (S-asymptotically anti-periodic, quasi-asymptotically almost anti-periodic).*

(ii) *If m is odd and $f(\cdot)$ is S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), then $f(\cdot)$ is S-asymptotically ω-periodic (S-asymptotically periodic, quasi-asymptotically almost periodic).*

Therefore, if $\arg(c)/\pi \in \mathbb{Q}$, then the class of S-asymptotically (ω, c)-periodic functions (S_c-asymptotically periodic functions, quasi-asymptotically c-almost periodic functions) is always contained in the class of S-asymptotically ω-periodic functions (S-asymptotically periodic functions, quasi-asymptotically almost periodic functions).

Arguing as in the proof of [**6**, Proposition 2.11], we may conclude that the following result holds true:

COROLLARY 3.2. *Let $|c| = 1$ and $\arg(c)/\pi \notin \mathbb{Q}$. If $f(\cdot)$ is bounded S-asymptotically (ω, c)-periodic (bounded S_c-asymptotically periodic, bounded quasi-asymptotically c-almost periodic), then $f(\cdot)$ is S-asymptotically ω-periodic (S-asymptotically periodic, quasi-asymptotically almost periodic).*

Further on, a slight modification of the proof of [**8**, Theorem 2.5] shows that the following statement holds:

THEOREM 3.1. *Let $F(I : E)$ be any space consisting of continuous functions $h : I \to E$ such that $\sup_{t\in I} \|h(t+\tau) - ch(t)\| = \sup_{t\geqslant a} \|h(t+\tau) - ch(t)\|$, $a \in I$. Then the following holds:*

(i) $AAA_c(I : E) \cap Q - AAP_c(I : E) = AAP_c(I : E)$.

(ii) $AA_c(\mathbb{R} : E) \cap Q - AAP_c(\mathbb{R} : E) = AP_c(\mathbb{R} : E)$.

We will include the proof of the following proposition for the sake of completeness (see also the proof of [**8**, Proposition 2.7]):

PROPOSITION 3.2. *Let* $|c| \leqslant 1$. *Then* $SAP_{\omega;c}(I : E) \subseteq Q - AAP_c(I : E)$.

PROOF. Let $\epsilon > 0$ be given. Then we can take $L(\epsilon) = 2\omega$ in definition of space $Q - AAP_c(I : E)$. Then any interval $I' \subseteq I$ of length $L(\epsilon)$ contains a number $\tau = n\omega$ for some $n \in \mathbb{N}$. For this n and ϵ, there exists a finite number $M(\epsilon, n) > 0$ such that $\|f(t + \omega) - cf(t)\| \leqslant \epsilon/n\omega$ for $|t| \geqslant M(\epsilon, n)$. Then we have

$$\|f(t + n\omega) - cf(t)\| \leqslant \sum_{k=0}^{n-1} |c|^{n-k-1} \|f(t + (k + 1)\omega) - cf(t + k\omega)\|$$

$$\leqslant \sum_{k=0}^{n-1} \|f(t + (k + 1)\omega) - cf(t + k\omega)\| \leqslant \sum_{k=0}^{n-1} \frac{\epsilon}{n\omega} = \epsilon/\omega,$$

provided $|t| \geqslant M(\epsilon, n) + n\omega$. This completes the proof. □

The following proposition can be deduced from the argumentation contained in the proof of [**8**, Proposition 2.12]:

PROPOSITION 3.3. *We have* $S^pQ - AAP_c(I : E) \subseteq W^p_{ap;c}(I : E)$.

The structural properties of quasi-asymptotically almost periodic functions clarifed in [**8**, Theorem 2.13] can be slightly generalized in the following manner:

THEOREM 3.2. *Let* $f : I \to E$ *be a quasi-asymptotically c-almost periodic function (Stepanov* (p, c)*-quasi-asymptotically almost periodic function). Then we have:*

(i) $\alpha f(\cdot)$ *is quasi-asymptotically c-almost periodic (Stepanov* (p, c)*-quasi-asymptotically almost periodic) for any* $\alpha \in \mathbb{C}$.
(ii) *If* $E = \mathbb{C}$ *and* $\inf_{x\in I} |f(x)| = m > 0$ *(ess*$\inf_{x\in I} |f(x)| = m > 0$*), then* $1/f(\cdot)$ *is quasi-asymptotically* $1/c$*-almost periodic (Stepanov* $(p, 1/c)$*-quasi-asymptotically almost periodic).*
(iii) *If* $(g_n : I \to E)_{n\in\mathbb{N}}$ *is a sequence of quasi-asymptotically c-almost periodic functions and* $(g_n)_{n\in\mathbb{N}}$ *converges uniformly to a function* $g : I \to E$*, then* $g(\cdot)$ *is quasi-asymptotically c-almost periodic.*
(iv) *If* $(g_n : I \to E)_{n\in\mathbb{N}}$ *is a sequence of Stepanov* (p, c)*-quasi-asymptotically almost periodic functions and* $(g_n)_{n\in\mathbb{N}}$ *converges to a function* $g : I \to E$ *in the space* $L^p_S(I : E)$*, then* $g(\cdot)$ *is Stepanov* (p, c)*-quasi-asymptotically almost periodic.*
(v) *The functions* $f(\cdot + a)$ *and* $f(b\,\cdot)$ *are likewise quasi-asymptotically c-almost periodic (Stepanov* (p, c)*-quasi-asymptotically almost periodic), where* $a \in I$ *and* $b \in I \smallsetminus \{0\}$.

The space of quasi-asymptotically c-almost periodic functions is not closed under pointwise addition and multiplication, as easily approved (see also [**8**, Proposition 2.15, Example 2.16-Example 2.18]).

Concerning the invariance of quasi-asymptotical c-almost periodicity under the actions of convolution products, the structural results clarified in [**8**, Section 3] continue to hold for (Stepanov p-) bounded forcing terms $f(\cdot)$:

PROPOSITION 3.4. (i) *Suppose that* $(R(t))_{t>0} \subseteq L(X,E)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\|\,ds < \infty$. *If the function* $f \in Q-AAP_c([0,\infty) : X)$ *is bounded, then the function* $F(\cdot)$, *defined through*

$$F(t) := \int_0^t R(t-s)f(s)\,ds,\ t \geqslant 0, \tag{3.1}$$

belongs to the class $Q-AAP_c([0,\infty) : E)$.

(ii) *Suppose that* $(R(t))_{t>0} \subseteq L(X,E)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\|\,ds < \infty$. *If* $f \in Q-AAP_c(\mathbb{R} : X)$ *is bounded, then the function* $\mathbf{F}(t)$, *defined through*

$$\mathbf{F}(t) := \int_{-\infty}^t R(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{3.2}$$

belongs to the class $Q-AAP_c(\mathbb{R} : E)$.

PROPOSITION 3.5. (i) *Suppose that* $1/p+1/q = 1$, $(R(t))_{t>0} \subseteq L(X,E)$ *is a strongly continuous operator family and* $\sum_{k=0}^\infty \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. *If* $f \in S^pQ-AAP_c([0,\infty) : X)$ *is Stepanov* p*-bounded, then the function* $F(\cdot)$, *defined by* (3.1), *belongs to the class* $Q-AAP_c([0,\infty) : E)$.

(ii) *Suppose that* $1/p+1/q = 1$, $(R(t))_{t>0} \subseteq L(X,E)$ *is a strongly continuous operator family and* $\sum_{k=0}^\infty \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. *If* $f \in S^pQ-AAP_c(\mathbb{R} : X)$ *is Stepanov* p*-bounded, then the function* $\mathbf{F}(\cdot)$, *defined by* (3.2), *belongs to the class* $Q-AAP_c(\mathbb{R} : E)$.

3.1. Composition principles for quasi-asymptotically c-almost periodic functions. The main aim of this subsection is to introduce the class of quasi-asymptotically c-almost periodic functions depending on two parameters, its Stepanov generalization and to formulate several composition principles for quasi-asymptotically c-almost periodic functions. The following notion has recently been introduced in [**2**]:

DEFINITION 3.5. Let $\mathbf{B} \subseteq P(E)$, where $P(E)$ denotes the power set of E. A continuous function $F : I \times X \to E$ is said to be uniformly continuous on $\mathbf{B}$, uniformly for $t \in I$ if and only if for every $\epsilon > 0$ and for every $B \in \mathbf{B}$ there exists a number $\delta_{\epsilon,B} > 0$ such that $\|F(t,x)-F(t,y)\| \leqslant \epsilon$ for all $t \in I$ and all $x,\ y \in B$ satisfying that $\|x-y\| \leqslant \delta_{\epsilon,B}$.

We continue by introducing the folowing definition:

DEFINITION 3.6. Suppose that $F : I \times X \to E$ is a continuous function and $\mathcal{F}$ is a non-empty collection of subsets of X. Then we say that $F(\cdot,\cdot)$ is quasi-asymptotically c-almost periodic, uniformly on $\mathcal{F}$ if and only if for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number holds with a number $M(\epsilon, \tau) > 0$ such that for each subset $B \in \mathcal{F}$ we have:

$$\left\|F(t+\tau, x) - cF(t, x)\right\|_Y \leqslant \epsilon, \text{ provided } t \in I,\ x \in B \text{ and } |t| \geqslant M(\epsilon, \tau).$$

Denote by $Q - AAP_{c;\mathcal{F}}(I \times X : E)$ the set consisting of all quasi-asymptotically c-almost periodic functions $F : I \times X \to E$ on $\mathcal{F}$.

Suppose that $F : I \times X \to E$ is a continuous function and there exists a finite constant $L > 0$ such that

$$\|F(t,x) - F(t,y)\| \leqslant L\|x-y\|_X, \quad t \in I,\ x,\ y \in X. \tag{3.3}$$

Define $\mathcal{F}(t) := F(t, f(t))$, $t \in I$. We need the following estimates ($\tau \geqslant 0$, $t \in I$):

$$\begin{aligned} &\Big\|F(t+\tau, f(t+\tau)) - cF(t, f(t))\Big\| \\ &\leqslant \Big\|F(t+\tau, f(t+\tau)) - F\Big(t+\tau, cf(t)\Big)\Big\| + \Big\|F\Big(t+\tau, cf(t)\Big) - cF(t, f(t))\Big\| \\ &\leqslant L\Big\|f(t+\tau) - cf(t)\Big\| + \Big\|F\Big(t+\tau, cf(t)\Big) - cF(t, f(t))\Big\|. \end{aligned} \tag{3.4}$$

Using (3.4) and the proofs of [**1**, Theorem 3.30, Theorem 3.31], we may deduce the following composition principles:

THEOREM 3.3. *Suppose that $F \in Q - AAP_c(I \times X : E)$ and $f \in Q - AAP_c(I : X)$. If there exists a finite number $L > 0$ such that*

$$\left\|F(t,x) - F(t,y)\right\|_Y \leqslant L\|x-y\|,\ x,\ y \in X,\ t \in I$$

and for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that

$$\Big\|F\big(t+\tau, cf(t)\big) - cF(t, f(t))\Big\| \leqslant \epsilon, \quad t \in I, \tag{3.5}$$

then the function $t \mapsto F(t, f(t))$, $t \in I$ belongs to the class $Q - AAP_c(I : E)$.

THEOREM 3.4. *Suppose that $F \in Q - AAP_c(I \times X : E)$ and $f \in Q - AAP_c(I : X)$. If the function $x \mapsto F(t,x)$, $t \in I$ is uniformly continuous on $R(f)$ uniformly for $t \in I$ and for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that* (3.5) *holds, then the function $t \mapsto F(t, f(t))$, $t \in I$ belongs to the class $Q - AAP_c(I : E)$.*

The notion of a Stepanov (p,c)-quasi-asymptotically almost periodic function depending on two parameters can be also introduced, and [**8**, Theorem 2.23, Theorem 2.24] can be slightly generalized in this framework.

It is clear that Proposition 3.4 and Proposition 3.5 can be applied at any place where the variation of parameteters formula takes effect. In [**8**, Section 4], we have analyzed the qualitative solutions of the abstract nonautonomous differential equations $u'(t) = A(t)u(t) + f(t)$, $t \in \mathbb{R}$,

$$u'(t) = A(t)u(t) + f(t), \quad t > 0;\ u(0) = x \tag{3.6}$$

and their semilinear analogues. We close the paper with the observation that the structural results established in [**8**, Theorem 4.1, Theorem 4.3] can be simply reformulated in our context; for example, in the formulation of [**8**, Theorem 4.1], we can assume that

$$\sum_{k=0}^{\infty}\big\|\Gamma(t+\tau,t+\tau-\cdot)-c\Gamma(t,t-\cdot)\big\|_{L^q[k,k+1]}\leqslant\epsilon,\quad\text{provided } t\geqslant M(\epsilon,\tau),$$

in place of condition [**8**, (4.1)]. Then the unique mild solution $u(\cdot)$ of the abstract Cauchy problem (3.6) will belong to the class $Q-AAP_c([0,\infty):E)+\mathcal{F}$; see [**8**] for the notation. The structural results established for the abstract nonautonomous semilinear differential equations [**8**, Theorem 4.6, Theorem 4.7] can be slightly generalized in our framework, as well.

References

[1] T. Diagana. *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces.* Springer-Verlag, New York, 2013.

[2] W.-S. Du and M. Kostić. Generalized almost periodicity in Lebesgue spaces with variable exponents. Part II. in: *Fixed Point Theory and Dynamical Systems with Applications*, special issue of Mathematics: *Mathematics* **8**(7)(2020), 1052; https://doi.org/10.3390/math8071052.

[3] A. M. Fink. *Almost Periodic Differential Equations.* Springer-Verlag, Berlin, 1974.

[4] G. M. N'Guérékata. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces.* Kluwer Acad. Publ, Dordrecht, 2001.

[5] H. R. Henríquez, M. Pierri and P. Táboas, *On* S-*asymptotically ω-periodic functions on Banach spaces and applications.* J. Math. Appl. Anal. **343** (2008), 1119–1130.

[6] M. T. Khalladi, M. Kostić, M. Pinto, A. Rahmani and D. Velinov. c-Almost periodic functions and applications. preprint. *Nonauton. Dyn. Syst.*, **7** (2020), 176–193.

[7] M. Kostić. *Almost Periodic and Almost Automorphic Type Solutions to Integro-Differential Equations.* W. de Gruyter, Berlin, 2019.

[8] M. Kostić. Quasi-asymptotically almost periodic functions and applications. *Bull. Brazilan Math. Soc.*, in press. https://link.springer.com/article/10.1007/s00574-020-00197-7.

[9] M. Kostić. Almost periodic type functions and densities. *Evol. Equ.Control. Theory.*, (submitted). https://hal.archives-ouvertes.fr/hal-02523952.

[10] S. Zaidman. *Almost-Periodic Functions in Abstract Spaces.* Pitman Research Notes in Math, Vol. **126**, Pitman, Boston, 1985.

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