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POSITIVE SOLUTION FOR THIRD-ORDER SINGULAR SEMIPOSITIVE BVPs ON THE HALF LINE WITH FIRST-ORDER DERIVATIVE DEPENDANCE

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Abstract. In this paper, we investigate the existence of a positive solution to the singular third-order boundary value problem

$$\begin{cases} -u'''(t) + k^2 u'(t) = f(t, u(t), u'(t)), \text{ a.e. } t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases}$$

where k is a positive constant and the function $f : (0, +\infty)^3 \to \mathbb{R}$ is continuous and may be singular at $t = 0$, $u = 0$ and at $u' = 0$.

The main existence result is proved by means of Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

1. Introduction

This article deals with existence of positive solutions to the third-order boundary value problem (bvp for short),

$$(1.1) \qquad \begin{cases} -u'''(t) + k^2 u'(t) = f(t, u(t), u'(t)), \text{ a.e. } t \in I \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases}$$

where k is a positive constant, $I = (0, +\infty)$ and $f : I^3 \to \mathbb{R}$ is a Carathéodory function, that is

- $f(\cdot, u, v)$ is a measurable function for all $u, v \in I$, and
- $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in I$.

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Throughout, we assume that

$$(1.2)\qquad \left\{\begin{array}{l}\text{There exists a measurable function } q: I\to\mathbb{R}^+ \text{ such that}\\ \int_0^{+\infty} e^{ks}q(s)ds<\infty \text{ and } f(t,u,v)+q(t)\geqslant 0 \text{ for all } t,u,v>0,\end{array}\right.$$

$$(1.3)\quad \left\{\begin{array}{l}\text{for all } \rho>0 \text{ there exist two functions } \omega_\rho:(0,+\infty)\to\mathbb{R}^+\\ \text{and } \Psi_\rho:(0,+\infty)\times(0,+\infty)\to(0,+\infty) \text{ such that}\\ \Psi_\rho \text{ is nonincreasing following its two variables,}\\ \left|f(t,e^{kt}w,e^{kt}z)\right|\leqslant\omega_\rho(t)\,\Psi_\rho(w,z) \text{ for all } t,w,z\geqslant 0 \text{ with } |(w,z)|\leqslant\rho,\\ \text{for all } r\in(0,\rho]\,,\ \lim_{s\to+\infty}\omega_\rho(s)\,\Psi_\rho\left(re^{-ks}\gamma(s),re^{-ks}\widetilde{\gamma}(s)\right)=0 \text{ and}\\ \int_0^{+\infty}\omega_\rho(s)\,\Psi_\rho\left(re^{-ks}\gamma(s),re^{-ks}\widetilde{\gamma}(s)\right)ds<\infty,\end{array}\right.$$

where

$$\gamma_1(t)=(e^{2kt}-1)e^{-4kt},$$
$$\widetilde{\gamma}(t)=k^*e^{kt}\gamma_1(t)=k^*\left(1-e^{-kt}\right)(1+e^{-kt})e^{-kt},$$
$$\gamma(t)=\int_0^t\widetilde{\gamma}(s)ds=\frac{k^*}{3k}\left(2-3e^{-kt}+e^{-3kt}\right)=\frac{k^*}{3k}\left(1-e^{-ks}\right)^2(2+e^{-ks}),$$

and $k^*=\min(1,k)/2$.

By positive solution to the bvp (1.1), we mean a function $u\in C^2(\mathbb{R}^+)\cap W^{3,1}(I)$ such that $u>0$ in I and $u(0)=u'(0)=\lim_{t\to+\infty}u'(t)=0$, satisfying the differential equation in (1.1).

Notice that the nonlinearity f may exhibit singular at the solution and at its derivative. It is well known that the bvp (1.1) is called positone if $q(t)=0$ a.e. $t\in I$, and semipositone if $q(t)>0$ a.e. t in some interval of I.

BVPs for third-order differential equation originate from many applications in physics and engineering. For example, the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves, gravity driven flows produce third-order boundary-value problems. During the last two decades, there has been many works dealing with several aspects of such BVPs, see, [**1, 4, 12, 14, 19, 23**] and the references therein. Often, for physical considerations, the positivity of the solution is required and many authors established existence and multiplicity results for positive solutions to such bvps posed on bounded intervals, where the nonlinear term is positive and satisfies either superlinear or sublinear growth conditions, see [**5, 11, 13, 20, 22, 24, 25, 26, 27**] and the references therein.

Because of a lack of compactness, the case where such bvps are posed on unbounded intervals is somewhat complicated and they has not been so extensively investigated. This case have been considered in [**2, 3, 4, 7, 8, 9, 10, 15, 16, 17, 18, 21**] and, to the authors' knowledge, there are no papers in the literature consedering the singular semipositone version of such bvps. Thus, the purpose of this paper is to fill in the gap in this area.

Our approach in this work is based on a fixed point formulation of the bvp (1.1) and the main existence result in this work is then proved by the Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space. Let us recall some basic facts related to the use of this principal.

Let $(E, ||.||)$ be a real Banach space. A nonempty closed convex subset C in E is said to be a cone in E, if $C \cap (-C) = \{0_E\}$ and $tC \subset C$ for all $t \geqslant 0$.

Let Ω be a nonempty subset in E, a mapping $A : \Omega \to E$ is said to be compact if it is continuous and $A(\Omega)$ is relatively compact in E.

THEOREM 1.1. *Let P be a cone in E and let Ω_1, Ω_2 be open bounded subsets of E such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. If $T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is a compact operator such that, either*

(1) *$\|Tu\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_2$, or*
(2) *$\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_2$.*

Then T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

2. Fixed point formulation

In all this paper, we let

$$E = \{u \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} e^{-kt}u(t) = 0,\ \lim_{t\to+\infty} e^{-kt}u'(t) = 0\}.$$

Endowed with the norm $\|u\| = \|u\|_k + \|u'\|_k$ where $\|u\|_k = \sup_{t\geqslant 0} e^{-kt}|u(t)|$, E becomes a Banach space.

The following lemma is an adapted version to the case of the space E of Corduneanu's compactness criterion ([**6**], p. 62). It will be used in this work to prove that some operator is compact.

LEMMA 2.1. *A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) *M is bounded in E,*
(b) *the sets $\{u : u(t) = e^{-kt}x(t),\ x \in M\}$ and $\{u : u(t) = e^{-kt}x'(t),\ x \in M\}$ are locally equicontinuous on $[0, +\infty)$, and*
(c) *the sets $\{u : u(t) = e^{-kt}x(t),\ x \in M\}$ and $\{u : u(t) = e^{-kt}x'(t),\ x \in M\}$ are equiconvergent at $+\infty$.*

Throughout, P denotes the cone in E defined by

$$P = \{u \in E :\ u'(t) \geqslant \widetilde{\gamma}(t)||u|| \text{ and } u(t) \geqslant \gamma(t)||u|| \text{ for all } t > 0\}$$

Let $G, \widetilde{G} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be the functions defined by

$$G(t,s) = \frac{1}{k^2}\begin{cases} e^{-ks}\left(\cosh(kt) - 1\right) & \text{if } t \leqslant s, \\ -e^{-kt}\sinh(ks) + \left(1 - e^{-ks}\right) & \text{if } s \leqslant t, \end{cases}$$
$$\widetilde{G}(t,s) = \frac{\partial G}{\partial t}(t,s) = \frac{1}{k}\begin{cases} e^{-ks}\sinh(kt) & \text{if } t \leqslant s, \\ e^{-kt}\sinh(ks) & \text{if } s \leqslant t. \end{cases}$$

LEMMA 2.2. *The functions G and $\widetilde{G}$ satisfy:*

(a): *For all $t, s \in \mathbb{R}^+$ we have $G(t,s) \geqslant 0$ and $\widetilde{G}(t,s) \geqslant 0$.*
(b): *The functions G and $\widetilde{G}$ are continuous and for all $s \geqslant 0$, we have*

$$G(0,s) = \widetilde{G}(0,s) = 0. \tag{2.1}$$

(c): *For all* $t, s \geqslant 0$, *we have*

$$G(t,s) \leqslant \frac{1}{k^2}(1-e^{-ks}) \leqslant \frac{1}{k^2}, \quad \widetilde{G}(t,s) \leqslant \widetilde{G}(s,s) \leqslant \frac{1}{2k}.$$

(d): *For all* $s, t, \tau \geqslant 0$, *we have*

$$e^{-ks}\widetilde{G}(s,s) \geqslant \widetilde{G}(t,s)e^{-kt} \geqslant \widetilde{\gamma}(t)\widetilde{G}(\tau,s)e^{-k\tau}.$$

(e): *For all* $t_2, t_1 \geqslant 0$, *we have*

$$\left|e^{-kt_2}G(t_2,s) - e^{-kt_1}G(t_1,s)\right| \leqslant \frac{3}{2k}\left|t_2 - t_1\right| \tag{2.2}$$

$$\left|e^{-kt_2}\widetilde{G}(t_2,s) - e^{-kt_1}\widetilde{G}(t_1,s)\right| \leqslant \left|t_2 - t_1\right| \tag{2.3}$$

Proof. Assertions (a), (b) and (c) are easy to prove, Assertion (d) is proved in [**9**]. Assertion (e) is obtained by the mean value theorem. □

Lemma 2.3. *Assume that Hypothesis (1.2) holds, then the function* ϕ *where for* $t \in I$, $\phi(t) = \int_0^{+\infty} G(t,s)q(s)ds$, *satisfies the following upper bound:*

$$\phi(t) \leqslant \phi^*\gamma(t), \quad \phi'(t) \leqslant \phi^*\widetilde{\gamma}(t) \ \text{ for all } t \in I$$

where

$$\phi^* = \max\left(\sup_{t>0}\frac{\phi(t)}{\gamma(t)}, \sup_{t>0}\frac{\phi'(t)}{\widetilde{\gamma}(t)}\right).$$

Proof. For all $t > 0$, we have

$$\begin{aligned}
\frac{\phi'(t)}{\widetilde{\gamma}(t)} &= \frac{3k}{k^*}\frac{\int_0^{+\infty}\widetilde{G}(t,s)q(s)ds}{(1-e^{-kt})(1+e^{-kt})e^{-kt}} \leqslant \frac{3k}{k^*}\frac{\int_0^{+\infty}\widetilde{G}(t,s)q(s)ds}{(1-e^{-kt})e^{-kt}} \\
&= \frac{3}{2k^*}\left(\frac{\int_0^t \sinh(ks)q(s)ds}{(1-e^{-kt})} + \frac{\sinh(kt)\int_t^{+\infty}e^{-ks}q(s)ds}{(1-e^{-kt})e^{-kt}}\right) \\
&= \frac{3}{2k^*}\left(\frac{\int_0^t \sinh(ks)e^{-ks}e^{ks}q(s)ds}{(1-e^{-kt})} + \frac{\sinh(kt)\int_t^{+\infty}e^{-2ks}e^{ks}q(s)ds}{(1-e^{-kt})e^{-kt}}\right) \\
&\leqslant \frac{3}{2k^*}\frac{\sinh(kt)e^{-kt}}{(1-e^{-kt})}\int_0^{+\infty}e^{ks}q(s)ds = \frac{3}{2k^*}\left(1+e^{-kt}\right)\int_0^{+\infty}e^{ks}q(s)ds \\
&\leqslant \frac{3}{k^*}\int_0^{+\infty}e^{ks}q(s)ds := \overline{\phi}.
\end{aligned}$$

This proves that $\sup_{t>0}(\phi'(t)/\widetilde{\gamma}(t)) < \infty$.

Therefore, we have

$$\frac{\phi(t)}{\gamma(t)} = \frac{\int_0^t \phi'(s)ds}{\gamma(t)} \leqslant \frac{\int_0^t \overline{\phi}\widetilde{\gamma}(s)ds}{\gamma(t)} = \overline{\phi},$$

leading to $\sup_{t>0}(\phi(t)/\gamma(t)) < \infty$. The proof is complete. □

LEMMA 2.4. *Assume that Hypoheses (1.2) and (1.3) hold. Then for all* $r, R \in \mathbb{R}$ *with* $R > r > \phi^*$ *there exists a compact operator* $T_{r,R} : P \cap (\overline{B}(0,R) \setminus B(0,r)) \to P$ *such that if* v *is a fixed point of* $T_{r,R}$ *then* $u = v - \phi$ *is a positive solution to the bvp (1.1).*

PROOF. Let r, R be two real numbers such that $R > r > \phi^*$ and set $\Omega = P \cap (\overline{B}(0,R) \setminus B(0,r))$. In all this proof, we let by Φ the function defined by

$$\Phi(s) = \omega_R(s)\, \Psi_R\left((r-\phi^*)\, e^{-ks}\gamma(s), (r-\phi^*)\, e^{-ks}\widetilde{\gamma}(s)\right),$$

where ω_R and Ψ_R are the functions given by Hypothesis (1.3) for $\rho = R$ and ϕ^* is the constant given by Lemma 2.3. The proof is divided into four steps.

Step 1. In this step we prove the existence of the operator $T_{r,R}$. We have from the definition of the cone P and Lemma 2.3 that, for all $v \in \Omega$ and all $t > 0$,

$$v(t) - \phi(t) \geqslant (\|v\| - \phi^*)\, \gamma(t) \geqslant (r-\phi^*)\, \gamma(t) > 0.$$
$$v'(t) - \phi'(t) \geqslant (\|v\| - \phi^*)\, \widetilde{\gamma}(t) \geqslant (r-\phi^*)\, \widetilde{\gamma}(t) > 0.$$

Therefore, for all $v \in \Omega$ the expression

$$f_{r,R}v(t) = f(t, v(t) - \phi(t), v'(t) - \phi'(t)) + q(t) \tag{2.4}$$

is well defined.

Let $v \in \Omega$, for all $s > 0$ we have

$$\begin{aligned} f_{r,R}v(s) &= f(s, e^{ks}\left(e^{-ks}(v(s) - \phi(s))\right), e^{ks}\left(e^{-ks}(u'(s) - \phi'(s))\right)) + q(s) \\ &\leqslant \Phi(s) + q(s), \end{aligned}$$

then

$$\int_0^{+\infty} G(t,s) f_{r,R}v(t)\, ds \leqslant \frac{1}{k^2}\int_0^{+\infty} f_{r,R}v(s)\, ds \leqslant \frac{1}{k^2}\int_0^{+\infty} (\Phi(s) + q(s))\, ds < \infty$$

and

$$\int_0^{+\infty} \widetilde{G}(t,s) f_{r,R}v(s) ds \leqslant \frac{1}{2k}\int_0^{+\infty} f_{r,R}v(s) ds \leqslant \frac{1}{2k}\int_0^{+\infty} (\Phi(s) + q(s))\, ds < \infty.$$

Thus, let w and z be the function defined by

$$w(t) = \textstyle\int_0^{+\infty} G(t,s) f_{r,R}v(s) ds \quad , \quad z(t) = \int_0^{+\infty} \widetilde{G}(t,s) f_{r,R}v(s) ds.$$

Since for all $t > 0$,

$$\begin{aligned} w(t) &= -\frac{e^{-kt}}{k^2}\int_0^t \sinh(ks) f_{r,R}v(s) ds + \frac{1}{k^2}\int_0^t (1 - e^{-ks}) f_{r,R}v(s) ds \\ &\quad + \frac{\cosh(kt) - 1}{k^2}\int_0^t e^{-ks} f_{r,R}v(s) ds, \end{aligned}$$

we see that w is differentiable on $\mathbb{R}^+$ and for all $t \geqslant 0$

$$\begin{aligned} w'(t) &= \frac{e^{-kt}}{k}\int_0^t \sinh(ks) f_{r,R}v(s) ds + \frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks} f_{r,R}v(s) ds \\ &= \int_0^{+\infty} \widetilde{G}(t,s) f_{r,R}v(s) ds = z(t) \end{aligned}$$

with z continuous on $\mathbb{R}^+$.

At this stage, we have proved that w belongs to $C^1(\mathbb{R}^+,\mathbb{R})$ and we need to prove that $w\in E$. Thus, we have to prove that $\lim_{t\to+\infty} e^{-kt}v(t)=\lim_{t\to+\infty} e^{-kt}w'(t)=0$.. Clearly for all $t>0$, $w(t),w'(t)>0$ and we have

$$\begin{array}{rcl} e^{-kt}w(t) & = & e^{-kt}\displaystyle\int_0^{+\infty} G(t,s)f_{r,R}v(s)ds \leqslant \frac{e^{-kt}}{k^2}\int_0^{+\infty}\left(\Phi(s)+q(s)\right)ds \\ e^{-kt}w'(t) & = & e^{-kt}\displaystyle\int_0^{+\infty} \widetilde{G}(t,s)f_{r,R}v(s)ds \leqslant \frac{e^{-kt}}{2k}\int_0^{+\infty}\left(\Phi(s)+q(s)\right)ds. \end{array}$$

The above two estimates show that $\lim_{t\to+\infty} e^{-kt}w(t)=\lim_{t\to+\infty} e^{-kt}w'(t)=0$.

Now for all $t,\tau>0$, we have from Assertion (c) in Lemma 2.2

$$\begin{array}{rcl} w'(t) & = & e^{kt}\displaystyle\int_0^{+\infty} e^{-kt}\widetilde{G}(t,s)f_{r,R}v(s)ds \\ & \geqslant & e^{kt}\gamma_1(t)\displaystyle\int_0^{+\infty} e^{-k\tau}\widetilde{G}(\tau,s)f_{r,R}v(s)ds \\ & = & e^{kt}\gamma_1(t)e^{-k\tau}w'(\tau). \end{array}$$

Passing to the supremum on τ, we obtain

$$w'(t)\geqslant e^{kt}\gamma_1(t)\left\|w'\right\|_k \text{ for all } t>0. \tag{2.5}$$

Since for all $t>0$

$$w(t)=\int_0^t e^{k\xi}\left(e^{-k\xi}w'(\xi)\right)d\xi \leqslant \int_0^t e^{k\xi}d\xi\left\|w'\right\|_k \leqslant \frac{e^{k\xi}}{k}\left\|w'\right\|_k$$

we have

$$\left\|w'\right\|_k \geqslant k\left\|w\right\|_k. \tag{2.6}$$

Therefore, (2.5) Combined with (2.6) leads to

$$w'(t)\geqslant ke^{kt}\gamma_1(t)\left\|w'\right\|_k \text{ for all } t>0$$

then to

$$w'(t)\geqslant \widetilde{\gamma}(t)\left\|w\right\| \text{ for all } t>0. \tag{2.7}$$

Integrating (2.7), yields $w(t)\geqslant\gamma(t)\left\|w\right\|$ for all $t>0$.

Thus, we have proved that $w\in P$ and the operator $T_{r,R}:\Omega\to P$ where for $v\in\Omega$

$$T_{r,R}u(t)=\int_0^{+\infty} G(t,s)f_{r,R}v(s)ds,$$

is well defined.

Step 2. In this step we prove that the operator $T_{r,R}$ is continuous. Let (v_n) be a sequence in Ω such that $\lim_{n\to\infty} v_n=v$ in E. For all $n\geqslant 1$, we have

$$\begin{array}{rcl} \left\|T_{r,R}v_n - T_{r,R}v\right\|_k & = & \displaystyle\sup_{t>0} e^{-kt}\left|T_{r,R}v_n(t)-Tv(t)\right| \\ & \leqslant & \displaystyle\frac{1}{k^2}\int_0^{+\infty}\left|f_{r,R}v_n(s)-f_{r,R}v(s)\right|ds \end{array}$$

and

$$\left\|(T_{r,R}v_n)' - (T_{r,R}v)'\right\|_k = \sup_{t>0} e^{-kt}\left|(T_{r,R}v_n)'(t) - (T_{r,R}v)'(t)\right| \leqslant \frac{1}{2k}\int_0^{+\infty} |f_{r,R}v_n(s) - f_{r,R}v(s)|\, ds.$$

Because of

$$|f_{r,R}v_n(s) - f_{r,R}v(s)| \to 0, \text{ as } n \to +\infty$$

for all $s > 0$ and

$$|f_{r,R}v_n(s) - f_{r,R}v(s)| \leqslant 2\Phi(s) \text{ with } \int_0^{+\infty} \Phi(s)\, ds < \infty,$$

Lebesgue dominated convergence theorem guarantees that
$\lim_{n\to\infty} \|T_{r,R}v_n - T_{r,R}v\| = 0$. Hence, we have proved that T is continuous.

Step 3. In this step, we prove that $T_{r,R}$ is compact. For all $v \in \Omega$, we have

$$\|T_{r,R}u\| \leqslant \max\left(\frac{1}{k^2}, \frac{1}{2k}\right)\int_0^{+\infty} (\Phi(s) + q(s)ds) < \infty,$$

proving that $T(\Omega)$ is bounded in E.

Now, let $t_1, t_2 \in [\eta, \xi] \subset \mathbb{R}^+$, for all $v \in \Omega$, we have from (2.2) and (2.3) the estimates

$$|e^{-kt_1}T_{r,R}v(t_1) - e^{-kt_2}T_{r,R}v(t_2)| \leqslant \int_0^{+\infty} |e^{-kt_1}G(t_1,s) - e^{-kt_2}G(t_2,s)|\Phi(s)\, ds \leqslant \frac{3}{2k}|t_2 - t_1|\int_0^{+\infty} \Phi(s)\, ds$$

and

$$|e^{-kt_1}(T_{r,R}v)'(t_1) - e^{-kt_2}(T_{r,R}v)'(t_2)| \leqslant \int_0^{+\infty} |e^{-kt_1}\widetilde{G}(t_1,s) - e^{-kt_2}\widetilde{G}(t_2,s)|\Phi(s)\, ds \leqslant |t_2 - t_1|\int_0^{+\infty} \Phi(s)\, ds.$$

Proving the equicontinuity on bounded intervals of $T(\Omega)$.

For all $v \in \Omega$ and $t > 0$, we have

$$|e^{-kt}Tv(t)| \leqslant \frac{e^{-kt}}{k^2}\int_0^{+\infty} (\Phi(s) + q(s))\, ds$$

and

$$|e^{-kt}(Tu)'(t)| \leqslant \frac{e^{-kt}}{k}\int_0^{+\infty} (\Phi(s) + q(s))\, ds.$$

Thus, the equiconvergence of $T(\Omega)$ follows from the fact that $\lim_{t\to\infty} e^{-kt} = 0$.

In view of Lemma 2.1, the operator is compact.

Step 4. In this step we prove that if v is a fixed point of T then $u = v - \phi$ is a positive solution to the bvp (1.1). Let $v \in \Omega$ be a fixed point of T, then for all $t > 0$

$$u(t) = v(t) - \phi(t) \geqslant (\|v\| - \phi^*)\gamma(t) \geqslant (r - \phi^*)\gamma(t) > 0,$$
$$u'(t) = v'(t) - \phi'(t) \geqslant (\|v\| - \phi^*)\widetilde{\gamma}(t) \geqslant (r - \phi^*)\widetilde{\gamma}(t) > 0,$$

and $u = v - \phi$ satisfies

$$\begin{aligned} u(t) &= -\phi\left(t\right) + \int_0^{+\infty} G(t,s)\left(f((s,u(s),u'(s)) + q(s)\right)ds \\ &= -\int_0^{+\infty} G(t,s)q(s)ds + \int_0^{+\infty} G(t,s)\left(f((s,u(s),u'(s)) + q\left(s\right)\right)ds \\ &= \int_0^{+\infty} G(t,s)f((s,u(s),u'(s))ds \end{aligned}$$

and

$$u'(t) = \int_0^{+\infty} \widetilde{G}(t,s)f((s,u(s),u'(s))ds.$$

These with (2.1) lead to $u(0) = u'(0) = 0$.

Differentiating twice in

$$\begin{aligned} u'(t) &= \int_0^{+\infty} \widetilde{G}(t,s)\phi\left(s\right)f((s,u(s),u'(s))ds \\ &= \frac{e^{-kt}}{k}\int_0^t \sinh(ks)\phi\left(s\right)f(s,u(s),u'(s))ds \\ &\quad + \frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\phi\left(s\right)f(s,u(s),u'(s))ds, \end{aligned}$$

we see that $-u'''\left(t\right) + ku'(t) = f(t,u(t),u'(t))$ for all $t > 0$.

It remains to prove that $\lim\limits_{t\to+\infty} u'(t) = 0$. We have

$u'(t) \leqslant$
$\frac{1}{ke^{kt}}\int_0^t \sinh(ks)\left|f(s,u(s),u'(s))\right|ds + \frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\left|f(s,u(s),u'(s))\right|ds \leqslant$
$\frac{1}{ke^{kt}}\int_0^t \sinh(ks)\Phi_R\left(s\right)ds + \frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\Phi_R\left(s\right)ds,$

$$\lim_{t\to+\infty}\frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\Phi_R\left(s\right)ds \leqslant \frac{1}{k}\lim_{t\to+\infty}\frac{\sinh(kt)}{e^{-kt}}\int_t^{+\infty}\Phi_R\left(s\right)ds = 0$$

and the L'Hopital's rule leads to

$$\begin{aligned} \lim_{t\to+\infty}\frac{1}{ke^{kt}}\int_0^t \sinh(ks)\Phi_R\left(s\right)ds &= \lim_{t\to+\infty}\frac{\sinh(kt)}{ke^{kt}}\frac{\int_t^{+\infty} e^{-ks}\Phi_R\left(s\right)ds}{e^{-kt}} \\ &= \frac{1}{k}\lim_{t\to+\infty}\frac{\sinh(kt)}{ke^{kt}}\lim_{t\to+\infty}\Phi_R\left(t\right) = 0. \end{aligned}$$

The above calculations show that $\lim_{t\to+\infty} u'(t) = 0$, completing the proof of the lemma. □

3. Main result

The main result of this paper needs to introduce the following notations. For $\alpha \in L^1\left(I\right)$ with $\alpha\left(t\right) \geqslant 0$ a.e. $t > 0$ and $\sigma > 1$, we let

$$\Gamma\left(\alpha\right) = \sup_{t>0} e^{-kt}\int_0^{+\infty} G\left(t,s\right)\alpha\left(s\right)ds + \sup_{t>0} e^{-kt}\int_0^{+\infty}\widetilde{G}\left(t,s\right)\alpha\left(s\right)ds,$$

$$\Delta(\alpha,\sigma)=\sup_{t>0} e^{-kt}\int_{1/\sigma}^{\sigma} G(t,s)\alpha(s)\,ds+\sup_{t>0} e^{-kt}\int_{1/\sigma}^{\sigma} \widetilde{G}(t,s)\alpha(s)\,ds.$$

THEOREM 3.1. *Suppose that Hypotheses (1.2) and (1.3) hold and*

(a): *there exist a function $\alpha \in L^1(I)$ and $R_1 > \max(\phi^*, \Gamma(\alpha))$ such that*

$$f(t,e^{kt}u,e^{kt}v)+q(t)\leqslant \alpha(t)$$

for a.e. $t>0$ and all $u,v\in I$ with $|(u,v)|\leqslant R_1$,

(b): *there exist $\sigma>1$, a function $\beta\in L^1(I)$ and $R_2\in(\phi^*,\Delta(\beta,\sigma))$ with $R_2\neq R_1$ such that*

$$f(t,e^{kt}u,e^{kt}v)+q(t)\geqslant \beta(t),$$

for a.e. $t\in[1/\sigma,\sigma]$, all $u\in[\gamma_\sigma(R_2-\phi^*),R_2]$ and all $v\in[\widetilde{\gamma}_\sigma(R_2-\phi^*),R_2]$, where $\gamma_\sigma=\min_{s\in[1/\sigma,\sigma]}\left(e^{-ks}\gamma(s)\right)$ and $\widetilde{\gamma}_\sigma=\min_{s\in[1/\sigma,\sigma]}\left(e^{-ks}\widetilde{\gamma}(s)\right)$.

Then, the bvp (1.1) admits a bounded positive solution.

PROOF. Without loss of generality, assume that $R_1<R_2$ and let $T=T_{R_1,R_2}$ be the operator given by Lemma 2.4. The following estimates hold, for all $v\in P\cap\partial B(0,R_1)$ and all $t>0$,

$$\begin{aligned}&e^{-kt}Tv(t)=\\&e^{-kt}\textstyle\int_0^{+\infty}G(t,s)f(s,e^{ks}(v(s)-\phi(s))e^{-ks},e^{ks}(v'(s)-\phi'(s))e^{-ks})ds\\&+e^{-kt}\textstyle\int_0^{+\infty}G(t,s)q(s)ds\\&\leqslant e^{-kt}\textstyle\int_0^{+\infty}G(t,s)\alpha(s)\,ds\\&\leqslant \sup_{t>0}e^{-kt}\textstyle\int_0^{+\infty}G(t,s)\alpha(s)\,ds.\end{aligned}$$

Passing to the supremum in the above estimates, we get

$$\|Tv\|_k\leqslant \sup_{t>0}e^{-kt}\int_0^{+\infty}G(t,s)\alpha(s)\,ds. \tag{3.1}$$

Similarly, we have

$$\begin{aligned}e^{-kt}(Tv)'(t)&=e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\left(f(s,v(s)-\phi(s),v'(s)-\phi'(s))+q(s)\right)ds\\&\leqslant e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\alpha(s)\,ds\leqslant \sup_{t>0}e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\alpha(s)\,ds,\end{aligned}$$

leading to

$$\|(Tv)'\|_k\leqslant \sup_{t>0}e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\alpha(s)\,ds. \tag{3.2}$$

Summing (3.1) with (3.2), we obtain

$$\begin{aligned}\|Tv\|&\leqslant \sup_{t>0}e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\alpha(s)\,ds+\sup_{t>0}e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\alpha(s)\,ds\\&=\Gamma(\alpha)\leqslant R_1=\|v\|.\end{aligned}$$

For all $v \in P \cap \partial B(0, R_2)$ and $s \in [1/\sigma, \sigma]$,

$$(3.3) \qquad \begin{array}{l} R_2 \geqslant (v(s) - \phi(s))\, e^{-ks} \geqslant (R_2 - \phi^*)\, \gamma(s) e^{-ks} = (R_2 - \phi^*)\, \gamma_\sigma \\ R_2 \geqslant (v'(t) - \phi'(s))\, e^{-ks} \geqslant (R_2 - \phi^*)\, \widetilde{\gamma}(s) e^{-ks} = (R_2 - \phi^*)\, \widetilde{\gamma}_\sigma \end{array}$$

Assumption **(b)** and (3.3) lead to the following estimates

$$\begin{array}{l} \|Tu\|_k \geqslant \\ \quad \sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} G(t,s) f(s, e^{ks} (v(s) - \phi(s))\, e^{-ks}, e^{ks} (v'(s) - \phi'(s))\, e^{-ks}) ds \right. \\ \qquad\qquad \left. + e^{-kt} \int_{1/\sigma}^{\sigma} G(t,s) q(s) ds \right) \\ \qquad\qquad \geqslant \sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} G(t,s) \beta(s)\; ds \right) \end{array}$$

and similarly

$$\| (Tv)' \|_k \geqslant \sup_{t \in > 0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} \widetilde{G}(t,s) \beta(s)\; ds \right).$$

Summing the above inequalities, we obtain

$$\begin{array}{l} \|Tv\| \geqslant \sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} \widetilde{G}(t,s) \beta(s)\; ds \right) + \sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} \widetilde{G}(t,s) \beta(s)\; ds \right) \\ \qquad\qquad = \Delta(\beta, \sigma) \geqslant R_2 = \|v\|. \end{array}$$

Thus, it follows from Assertion 1 in Theorem 1.1 that T_{R_1,R_2} admits a fixed point v such that $R_1 \leqslant \|v\| \leqslant R_2$. Then by Lemma 2.4, $u = v - \phi$ is a positive solution to the bvp (1.1).

Now, we have to prove that u is bounded. Since for all $t > 0$,

$$\begin{array}{l} \|v\| + \|\phi\| \geqslant e^{-kt} u(t) = e^{-kt} (v(t) - \phi(t)) \geqslant (\|v\| - \phi^*)\, e^{-kt} \gamma(t), \\ \|v\| + \|\phi\| \geqslant e^{-kt} u'(t) = e^{-kt} (v'(t) - \phi'(t)) \geqslant (\|v\| - \phi^*)\, e^{-kt} \widetilde{\gamma}(t), \end{array}$$

we obtain from Hypothesis (1.3) for $\rho = \|u\|$,

$$\begin{array}{rcl} u(t) & = & Tu(t) \leqslant \displaystyle\int_0^{+\infty} G(t,s) \left| f(s, u(s), u'(s)) \right| ds \\ & \leqslant & \displaystyle\frac{1}{k^2} \int_0^{+\infty} \omega_\rho(s)\, \Psi_\rho(\left(e^{-ks} u(s)\right), \left(e^{-ks} u'(s)\right)) ds \\ & \leqslant & \displaystyle\frac{1}{k^2} \int_0^{+\infty} \omega_\rho(s)\, \Psi_R((\|v\| - \phi^*)\, e^{-ks} \gamma(s), (\|v\| - \phi^*)\, e^{-ks} \widetilde{\gamma}(s)) ds < \infty. \end{array}$$

The proof is complete. □

Set for $\sigma > 1$

$$f_\sigma = \liminf_{|(w,z)| \to +\infty} \left(\min_{t \in [1/\sigma, \sigma]} \frac{f(t, e^{kt} w, e^{kt} z)}{w + z} \right).$$

We obtain from Theorem 3.1 the following corlolary:

Corollary 3.1. *Suppose that Hypotheses (1.2) and (1.3) hold and*

(c): *there exists $R_1 > \phi^*$ such that $\Gamma(\alpha_1) < R_1$ where*

$$\alpha_1 = \omega_{R_1}(s)\, \Psi_{R_1}\left((R_1 - \phi^*)\, e^{-ks} \gamma(s), (R_1 - \phi^*)\, e^{-ks} \widetilde{\gamma}(s) \right) + q(s),$$

(d): *there exists* $\sigma > 1$, *such that* $f_\sigma \Delta(\beta_0, \sigma) > 1$, *where* $\beta_0(s) = e^{-ks}\gamma(s)$.

Then, the bvp (1.1) admits a positive solution.

Proof. Clearly, Condition (a) of in Theorem 3.1 is satisfied for $\alpha = \alpha_1$. We have to prove that Condition (b) also is satisfied. Let $\varepsilon > 0$ be such that $(f_\sigma - \varepsilon)\Delta(\beta_0, \sigma) > 1$. There exists R_∞ such that $f(t, e^{kt}w, e^{kt}z) > (f_\sigma - \varepsilon)(w+z)$ for all $t \in [1/\sigma, \sigma]$ and all w, z with $|(w,z)| \geqslant R_\infty$. Let

$$R_2 = 1 + \sup\left(R_1, \phi^* + \frac{R_\infty}{\gamma_\sigma}, \frac{\phi^*(f_\sigma - \varepsilon)\Delta(\beta_0, \sigma)}{(f_\sigma - \varepsilon)\Delta(\beta_0,\sigma) - 1}\right)$$

and

$$\beta(t) = (f_\sigma - \varepsilon)(R_2 - \phi^*)\gamma(s)e^{-ks} + q(s).$$

where $\gamma_\sigma = \min_{s\in[1/\sigma,\sigma]}\left(e^{-ks}\gamma(s)\right)$ and notice that

$$(f_\sigma - \varepsilon)\Delta(\beta_0, \sigma)(R_2 - \phi^*) > R_2.$$

We have then

$$\begin{aligned}\Delta(\beta, \sigma) = &\sup\nolimits_{t>0}\left(e^{-kt}\int_{1/\sigma}^{\sigma} G(t,s)\left((f_\sigma - \varepsilon)(R_2 - \phi^*)\gamma(s)e^{-ks} + q(s)\right)ds\right)\\ &+\sup\nolimits_{t>0}\left(e^{-kt}\int_{1/\sigma}^{\sigma} \widetilde{G}(t,s)\left((f_\sigma - \varepsilon)(R_2 - \phi^*)\gamma(s)e^{-ks} + q(s)\right)ds\right)\\ &\geqslant (f_\sigma - \varepsilon)\Delta(\beta_0,\sigma)(R_2 - \phi^*) > R_2.\end{aligned}$$

The proof is complete. □

4. Example

Consider the case of the bvp (1.1) where

$$f(t,u,v) = e^{-\delta t}\left(\left(\frac{u+v}{e^{kt}}\right)^p + \frac{B(u+v)^2}{Be^{kt} + u + v} - 1\right)$$

where $\delta > (1-p)k$, $p \in (-1, 0)$ and $B > 0$.

Clearly, Hypothesis (1.2) is satisfied for $q(t) = e^{-\delta t}$ and we have

$$f(t, e^{kt}w, e^{kt}z) = e^{-\delta t}\left((w+z)^p + \frac{Be^{kt}(w+z)^2}{B+w+z} - 1\right),$$

leading to

$$\begin{aligned}\left|f(t, e^{kt}w, e^{kt}z)\right| &= \left|e^{-(\delta-k)t}\left(e^{-kt}(w+z)^p + \frac{B(w+z)^2}{B+w+z} - e^{-kt}\right)\right|\\ &\leqslant e^{-(\delta-k)t}\left((w+z)^p + \frac{B(w+z)^2}{B+w+z} + 1\right).\end{aligned}$$

Therefore, Hypothesis (1.3) is satisfied for all $\rho > 0$ with

$$\omega_\rho(s) = e^{-(\delta-k)s} \quad \text{and} \quad \Psi_\rho(w,z) = (w+z)^p + \frac{B\rho^2}{B+\rho} + 1$$

for all $s > 0$ and all $w, z > 0$ with $|(w,z)| = w + z \leqslant \rho$.

We have then

$$\omega_\rho(s)\,\psi_\rho\left(\rho e^{-ks}\gamma(s),\rho e^{-ks}\widetilde{\gamma}(s)\right)=e^{-(\delta-k)s}\left(1+\tfrac{B\rho^2}{B+\rho}\right)\\ +(k^*\rho)^p\,e^{-(\delta+pk-2k)s}\left(1-e^{-ks}\right)^p\theta(s)$$

where

$$\theta(s)=\left(\frac{1}{3k}\left(1-e^{-ks}\right)\left(2+e^{-ks}\right)+e^{-ks}\left(1+e^{-ks}\right)\right)^p$$

and satisfies

$$\left(2+\frac{1}{k}\right)^p\leqslant\rho(s)\leqslant 2^p<1.$$

Because of $\delta>(1-p)\,k$ and $p\in(-1,0)$, we have

$$\lim_{s\to+\infty}\omega_R(s)\psi_R\left(Re^{-ks}\gamma(s),Re^{-ks}\widetilde{\gamma}(s)\right)=0\text{ and}\\ \int_0^{+\infty}\omega_R(s)\psi_R\left(Re^{-ks}\gamma(s),Re^{-ks}\widetilde{\gamma}(s)\right)ds<\infty.$$

For

$$\alpha_1(t)=\omega_R(t)\psi_R\left(\left(R-\phi^*\right)e^{-ks}\gamma(t),\left(R-\phi^*\right)e^{-ks}\widetilde{\gamma}(t)\right)+q(t)$$

straightforward computations lead to

$$\Gamma(\alpha_1)\leqslant\Lambda(R)=\widetilde{k}\left(\lambda(p,\delta,k)\left(R-\phi^*\right)^p+\mu(\delta,k)\,\frac{B\left(R-\phi^*\right)^2}{B+\left(R-\phi^*\right)}+\eta(\delta,k)\right)$$

where

$$\widetilde{k}=\frac{1}{k^2}+\frac{1}{2k},\quad \mu(\delta,k)=\int_0^{+\infty}\omega_R(s)ds=\frac{1}{\delta-k},\\ \eta=\int_0^{+\infty}\left(\omega_R(s)+q(s)\right)ds=\frac{1}{\delta-k}+\frac{1}{\delta}.$$

and

$$\begin{aligned}\lambda(p,\delta,k)&=(k^*)^p\int_0^{+\infty}e^{-(\delta+pk-k)s}\left(1-e^{-ks}\right)^p\theta(s)ds\\ &\leqslant(k^*)^p\int_0^{+\infty}e^{-(\delta+pk-k)s}\left(1-e^{-ks}\right)^p ds\\ &\leqslant(k^*)^p\left(\int_0^{1/k}\left(1-e^{-ks}\right)^p ds+\left(1-e^{-1}\right)^p\int_{1/k}^{+\infty}e^{-(\delta+pk-k)s}ds\right)\\ &\leqslant(k^*)^p\left(2^{-p}\int_0^{1/k}(ks)^p\left(2-ks\right)^p ds+\tfrac{\left(1-e^{-1}\right)^p}{(\delta+pk-k)}\right)\\ &\leqslant(k^*)^p\left(2^{-p}k^p\int_0^{1/k}s^p ds+\tfrac{\left(1-e^{-1}\right)^p}{(\delta+pk-k)}\right)\leqslant(k^*)^p\left(\tfrac{1}{2^pk(p+1)}+\tfrac{\left(1-e^{-1}\right)^p}{(\delta+pk-k)}\right).\end{aligned}$$

We have

$$\begin{aligned}\Lambda\left(1+\phi^*\right)&=\widetilde{k}\left(\lambda(p,\delta,k)+\mu(\delta,k)\,\frac{B}{B+1}+\eta(\delta,k)\right)\\ &\leqslant\widetilde{k}\left(\lambda(\delta,k)+\mu(\delta,k)+\eta(\delta,k)\right).\end{aligned}$$

The above calculations show that for k large enough we have

$$\Lambda\left(1+\phi^*\right)\leqslant 1\leqslant 1+\phi^*$$

and Condition (c) in Corollary 3.1 is satisfied for $R=1+\phi^*$.

Clearly, we have $f_\sigma = +\infty$ for all $\sigma > 1$. Therefore, we conclude from Corollary 3.1 and all the above calculations that if k is large enough then this case of the bvp (1.1) admits a positive solution.

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