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ENERGY OF MONAD GRAPHS

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ABSTRACT. Let $\mathcal{G}$ be a digraph of order n. The energy of $\mathcal{G}$ is the sum of absolute values of the real parts of the eigenvalues of its adjacency matrix. The monad graph is a digraph pertaining to the finite group G, where every vertex of the elements g of G is adjacent to its image by a directed edge under the action of the map $f(g) = g^2$. In this paper, we compute the energy of monad graphs.

1. Introduction

Since 1978, when the concept of graph energy based on the eigenvalues of the adjacency matrix was conceived [**5**], a large number of other "graph energies" has been put forward. Nowadays, their number is near to 200 [**6, 7**]. Almost all of these "graph energies" are based on the eigenvalues of various graph matrices, different from the adjacency matrix. In the present paper we consider one more "graph energy", which – in contrast to the earlier ones – has its roots from group theory and uses the eigenvalues of the adjacency matrix.

Let $\mathcal{G}$ be a digraph (directed graph) of order n. Let $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$ be the vertex set and $E(\mathcal{G})$ the edge set of $\mathcal{G}$. By e_{ij} is denoted the directed edge of $\mathcal{G}$ starting at vertex v_i and ending at vertex v_j. The adjacency matrix of $\mathcal{G}$ is the $n \times n$ matrix $A(\mathcal{G})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in E(\mathcal{G}) \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $\mathcal{G}$ is defined as

$$\phi(G, \lambda) = \det(\lambda\, I_n - A(\mathcal{G}))$$

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where I_n is the unit matrix of order n. Its zeros $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $A(\mathcal{G})$. In the case of digraphs, some of the eigenvalues may be complex numbers. Therefore, the *energy of digraphs* is defined as the sum of absolute values of the real parts of the eigenvalues [**8, 9, 10**], i.e.,

$$E(\mathcal{G}) = \sum_{i=1}^{n} |Re\lambda_i| \, .$$

In this paper, we are interested in the energy of the so-called *monad graphs.*

Dynamical systems defined by finite groups are one of the most active areas of research in group theory, allowing us to understand the behavior of each element of the given group. In 2003, Arnold introduced a very interesting new concept, that he named *monad* [**1**].

Let G_n be finite group of order n. A *monad map* is a mapping of each element of G_n into itself, i.e., $f : G_n \to G_n$ for all $g \in G_n$, where $f(g) = g^2$. The *monad* is a digraph in which the vertices correspond to the elements of the group G_n. The (directed) edges of the monad connect a vertex with its image under the action of monad map. Thus, a monad is a dynamical system containing a finite action group G, the monad map f, and a respective digraph, denoted by $\Gamma(G_n)$.

In [**1**], Arnold considered the combinatorics of squaring monad graph, i.e. $f(g) = g^2$, and showed that even the simplest choice of the map f leads to non-trivial topological structures. A characteristic result of this kind is the following:

Theorem 1.1. [**1**] *Each connected component of a monad graph consists of a directed cycle to which rooted trees are attached, directed towards their roots.*

Details on the structure of monad graphs are seen from the below Table 1.

A directed cycle O_n of size n is formed by the directed edges

$$e_{1,2},\ e_{2,3},\ \dots,\ e_{n-1,n},\ e_{n,1}.$$

The edge e_{ii}, represented by a loop on the vertex v_i, is considered as a cycle of size 1. The two edges e_{ij} and e_{ji} are considered to form a directed cycle of size 2.

In Table 1 we use additive notation for the group operation and show the monad graphs of the first few residue class cyclic groups.

2. Spectral properties of monad graphs

In order to obtain our main results, we present here some necessary definitions and auxiliary results.

Lemma 2.1. *Let $\mathcal{G}$ be a digraph.*

(a) *If the directed edge e does not belong to any cycle of $\mathcal{G}$, then e does not contribute to the spectrum of $\mathcal{G}$. In other words, by deleting e from $\mathcal{G}$, neither the spectrum nor the energy of $\mathcal{G}$ will change.*

(b) *If the vertex v does not belong to any cycle of $\mathcal{G}$, then v contributes to the spectrum of $\mathcal{G}$ by a zero. Therefore, by deleting v from $\mathcal{G}$, the energy of $\mathcal{G}$ will not change.*

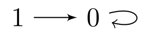
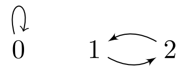
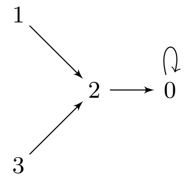
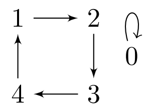
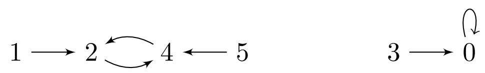
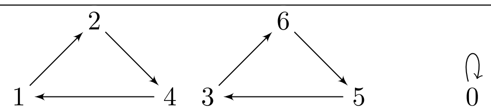
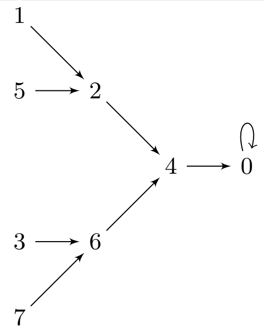
$\Gamma(\mathbb{Z}_2)$		A_1
$\Gamma(\mathbb{Z}_3)$		$O_1 + O_2$
$\Gamma(\mathbb{Z}_4)$		T_4
$\Gamma(\mathbb{Z}_5)$		$O_1 + O_4$
$\Gamma(\mathbb{Z}_6)$		$A_1 + A_2$
$\Gamma(\mathbb{Z}_7)$		$O_1 + 2O_3$
$\Gamma(\mathbb{Z}_8)$		T_8

Table 1. Monad graphs of the residue class cyclic groups for $n \leqslant 8$. O_n is the directed cycle on n vertices. A_n is the connected graph on $2n$ vertices, consisting of a directed cycle of length n to which n one-edge branches are attached, each for every vertex of the cycle. D_n is the $4n$-vertex graph consisting of the cycle O_n, to each of its vertices a three-edge branch is attached; for examples see the 16-vertex digraph on Fig. 1 and the 80-vertex digraph on Fig. 2. T_{2^n} is the rooted binary tree on 2^n vertices and n leaves. For more details see [**2, 11, 12**].

Lemma 2.1 is an immediate consequence of the Sachs coefficient theorem [**4**]. Recall that for digraphs, this theorem reads:

Lemma 2.2. *Let $\mathcal{G}$ be a digraph with characteristic polynomial*

$$\phi(\mathcal{G}) = \sum_{k=0}^{n} a_k \, x^{n-k} \, .$$

Then $a_0 = 1$ and for $k \geqslant 1$,

$$a_k = \sum_{S \in L_k} (-1)^{\omega(S)}$$

where L_k denotes the set of k-vertex subgraphs of $\mathcal{G}$, in which every component is a directed cycle. $\omega(S)$ is the number of connected components of S.

According to Lemma 2.2, the characteristic polynomial of the directed cycle O_n is

$$\phi(O_n, \lambda) = \lambda^n - 1 \, .$$

Thus the eigenvalues of O_n are

$$\lambda_j = e^{2\pi i j/n} \ , \quad j = 0, 1, 2, \ldots, n-1$$

implying

$$E(O_n) = \sum_{j=0}^{n-1} \left| \cos \frac{2\pi j}{n} \right| . \tag{2.1}$$

By direct calculation we get $E(O_1) = 1, \ E(O_2) = 2, \ E(O_3) = 2, \ E(O_4) = 2, \ E(O_5) = 1 + \sqrt{5} \approx 3.236, \ E(O_6) = 4$.

By an immediate application of Lemma 2.1 we obtain:

Theorem 2.1. *Let A_n, D_n, and T_{2^n} be the digraphs described in the caption of Table 1. Then*

$$\begin{aligned} E(A_n) &= E(O_n) \\ E(D_n) &= E(O_n) \\ E(T_{2^n}) &= 1 \, . \end{aligned}$$

3. Calculating the energy of monad graphs

Bearing in mind Theorem 2.1 and Eq. (2.1), in order to calculate the energy of a monad graph, one only has to determine the type and number of its connected components. In order to achieve this goal, we need some preparation.

Definition 3.1 ([**1**]). The graph of the product monad will be called the product of the graphs of the factors. The multiplication of graphs will be denoted by the same symbol $*$:

$$[graph(A * B)] = [graph(A)] * [graph(B)] \, .$$

Lemma 3.1 ([**1**]). *The monad graph of multiplication by 2 (i.e., adding an element to itself) in an additive cyclic group of odd order is a disjoint union of directed cycles.*

LEMMA 3.2 ([**1**]). *The monad graph of multiplication by 2 (adding to itself) in the additive cyclic group of residues modulo* 2^n *is the binary rooted tree* T_{2^n}*.*

Directly from Lemma 3.1 we get:

THEOREM 3.1. *The energy of the monad graph pertaining to an additive cyclic group* G_n *of odd order* n *is given by*

$$E(\Gamma(G_n)) = \sum_m O_m = \sum_m \sum_{j=0}^{m-1} \left| \cos \frac{2\pi j}{m} \right|$$

for some (not necessarily mutually distinct) values of m , $1 \leqslant m \leqslant n-1$. *For details see Table 2.*

Directly from Theorem 2.1 and Lemma 2.2 we get:

THEOREM 3.2. *The energy of the monad graph pertaining to an additive cyclic group of residues modulo* 2^n *is* $E(\Gamma(T_{2^n})) = 1$.

If $\gamma > 0$ and m are odd numbers, then Lemmas 3.1 and 3.2 imply that some of monad graphs pertaining to finite commutative groups G_n of order $n = 2^\gamma \cdot m$ are sums of products of monads of the form T_{2^γ} and O_m. In this case, for fixed positive 2^γ, we express the energy as

$$E(\Gamma(G_n)) = E\left(\sum_m (T_{2^\gamma} * O_m)\right), \tag{3.1}$$

for some (not necessarily mutually distinct) values of m , $1 \leqslant m \leqslant n-1$.

In order to illustrate Eq. (3.1), we calculate the energy of the monad graph of $\mathbb{Z}_{100} = \mathbb{Z}_4 * \mathbb{Z}_{25}$. The monad graph of $\Gamma(\mathbb{Z}_4)$ is T_{2^2}. For $\mathbb{Z}_{25}$, we begin by calculating the graphs of the factors. $\mathbb{Z}_{25}$ is of odd order and consists of directed cycles of orders 1, 4, and 20. Hence, $\mathbb{Z}_{25} = O_1 + O_4 + O_{20}$. Therefore, the product is of the form

$$\Gamma(\mathbb{Z}_{100}) = T_4 + (T_4 * O_4) + (T_4 * O_{20}). \tag{3.2}$$

The graphs T_4, $T_4 * O_4$, and $T_4 * O_{20}$ are depicted in Figs. 1 and 2.

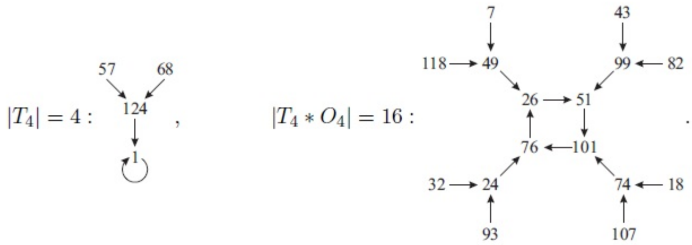


Fig. 1. The monad graphs T_4 and $T_4 * O_4 \cong D_4$.

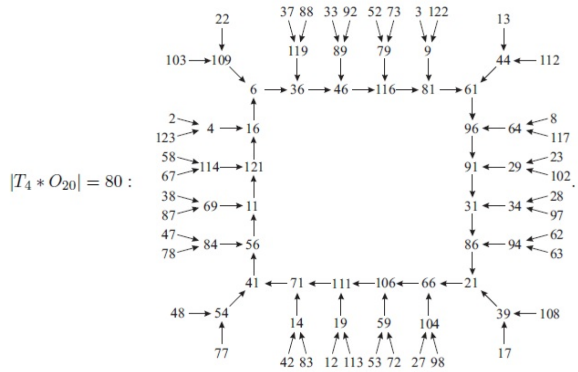


Fig. 2. The monad graph $T_4 * O_{20} \cong D_{20}$.

Since $T_4 * O_4 \cong D_4$, by Theorem 2.1, $E(T_4 * O_4) = E(D_4) = E(O_4) = 2$. Analogously, $E(T_4 * O_{20}) = E(D_{20}) = E(O_{20}) \approx 11.728$. This finally yields

$$E(\Gamma(\mathbb{Z}_{100})) = E(T_4) + E(T_4 * O_4) + E(T_4 * O_{20}) = 1 + 2 + 11.728$$

and thus

$$E(\Gamma(\mathbb{Z}_{100})) \approx 14.728\,.$$

In Table 2 are displayed the energies of the monad graphs for the squaring map, of the cyclic groups G_n for $n \leqslant 23$.

In what follows, for a positive integer n, the Euler group $\mathcal{E}(n)$ is the multiplicative group of coprime residue classes modulo n. The first few nontrivial Euler groups are

$$\mathcal{E}(2) = \{1\},\ \mathcal{E}(3) \equiv \mathbb{Z}_2,\ \mathcal{E}(5) \equiv \mathbb{Z}_4,\ \mathcal{E}(6) \equiv \mathbb{Z}_2,$$
$$\mathcal{E}(7) \equiv \mathbb{Z}_6,\ \mathcal{E}(8) \equiv \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathcal{E}(9) \equiv \mathbb{Z}_6,\ \mathcal{E}(13) \equiv \mathbb{Z}_{12}\,.$$

In [**3**], it was shown that the Euler group $\mathcal{E}(n)$ is cyclic whenever n is a prime or a degree of q prime, in which case

$$\mathcal{E}(p^q) \equiv \mathbb{Z}_{(p-1)\,p^{q-1}}\,.$$

In Table 3 are displayed the energies of the monad graphs pertaining to the Euler groups $\mathcal{E}(n)$, $n \leqslant 23$, with multiplication operation.

By comparing the data from Tables 2 and 3, it can be seen that the energies of the cyclic and Euler groups are related as

$$E(\Gamma(\mathbb{Z}_n)) \geqslant E(\Gamma(\mathcal{E}(n)))\,.$$

n	$\Gamma(\mathbb{Z}_n)$	$E(\Gamma(\mathbb{Z}_n))$
2	A_1	1
3	$O_1 + O_2$	3
4	T_4	1
5	$O_1 + O_4$	3
6	$A_1 + A_2$	3
7	$O_1 + 2O_3$	5
8	T_8	1
9	$O_1 + O_6$	5
10	$A_1 + A_4$	3
11	$O_1 + O_{10}$	7.472
12	$T_4 + (T_4 * O_2)$	3
13	$O_1 + O_{12}$	8.464
14	$A_1 + 2A_3$	5
15	$O_1 + O_2 + 3O_4$	9
16	T_{16}	1
17	$O_1 + 2O_8$	9.6568
18	$A_1 + A_2 + A_6$	7
19	$O_1 + O_{18}$	12.517
20	$T_4 + (T_4 * O_4)$	3
21	$O_1 + 2O_3 + 2O_6$	13
22	$A_1 + A_{10}$	7.472
23	$O_1 + 2O_{11}$	15.0536

Table 2. Energies of monad graphs pertaining to cyclic groups G_n for the first few values of n.

n	$\Gamma(\mathcal{E}(n))$	$E(\Gamma(\mathcal{E}(n)))$
2	O_1	1
3	A_1	1
4	A_1	1
5	T_4	1
6	A_1	1
7	$A_1 + A_2$	3
8	D_1	1
9	$A_1 + A_2$	3
10	T_4	1
11	$A_1 + A_4$	3
12	D_1	1
13	$T_4 + (T_4 * O_2)$	3
14	$A_1 + A_2$	3
15	$T_4 * A_1$	1
16	$T_4 * A_1$	1
17	T_{16}	1
18	$A_1 + A_2$	3
19	$A_1 + A_2 + A_6$	7
20	$T_4 * A_1$	1
21	$A_1 + A_2 + D_1 + D_2$	7
22	$A_1 + A_4$	3
23	$A_1 + A_{10}$	7.472

Table 3. Energies of monad graphs pertaining to Euler groups $\mathcal{E}(n)$ for the first few values of n.

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