BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(2)(2021), 255-260
DOI: 10.7251/BIMVI2102255B

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

GRADINGS ON QUATERNION BLOCKS WITH TWO SIMPLE MODULES

Dusko Bogdanic

Abstract. In this paper we show that tame blocks of group algebras with quaternion defect groups and two isomorphism classes of simple modules can be non-trivially graded. We prove this by using the transfer of gradings via derived equivalences. The gradings that we transfer are such that the arrows of the quiver are homogeneous elements. We also show that, up to graded Morita equivalence, on such a block there exists a unique grading.

1. Introduction and preliminaries

This paper is a continuation of a series of papers [**6, 7, 8**] in which we study existence of gradings on blocks of group algebras. In our previous paper [**8**], we proved that tame blocks of group algebras with quaternion defect groups and three isomorphism classes of simple modules can be non-trivially graded. We use the same techniques of transfer of gradings between symmetric algebras via derived equivalences (see [**12**]) in order to construct non-trivial gradings on quaternion blocks of group algebras with two simple modules.

Let A be an algebra over a field k. We say that A is a graded algebra if A is the direct sum of subspaces $A = \bigoplus_{i\in\mathbb{Z}} A_i$, such that $A_iA_j \subset A_{i+j}$, $i, j \in \mathbb{Z}$. The subspace A_i is said to be the homogeneous subspace of degree i. It is obvious that we can always trivially grade A by setting $A_0 = A$. A given grading is positive if $A_i = 0$ for all $i < 0$. In this paper we study the problem of existence of non-trivial gradings on blocks of group algebras with quaternion defect groups and two simple modules. Also, for a given pair of non-trivial gradings on these blocks, we study

2010 *Mathematics Subject Classification.* Primary 16A03; Secondary 16W50, 20C20.
Key words and phrases. Graded algebras, derived equivalence, quaternion defect groups.
Communicated by Daniel A. Romano.

their relationship and prove that, up to graded Morita equivalence, these gradings are equal. We refer the reader to [**1**] for details on defect groups of blocks of group algebras.

Tame blocks of group algebras appear only for group algebras over fields of characteristic 2, so for the remainder of this paper we will assume that the field k we work over is of characteristic 2. All algebras in this paper are finite dimensional algebras over the field k, and all modules will be left modules. The category of finite dimensional A–modules is denoted by A–mod and the full subcategory of finite dimensional projective A–modules is denoted by P_A. The derived category of bounded complexes over A–mod is denoted by $D^b(A)$, and the homotopy category of bounded complexes over P_A will be denoted by $K^b(P_A)$.

We refer the reader to [**5, 8**] for introductory remarks about graded algebras and modules. The aim of this paper is to show how one can use transfer of gradings via derived equivalences to grade tame blocks with quaternion defect groups and two simple modules. These complex methods of constructing gradings on associative algebras have previously been studied in [**12, 5, 6, 7**].

1.1. Quaternion blocks with two simple modules. If B is any block with a quaternion defect group of order 2^n and with two isomorphism classes of simple modules, then B is Morita equivalent to the algebra $Q(2A)^{2^{n-2},c}$ or to the algebra $Q(2B)_1^{2^{n-2},c}$ for some $c \in k$ (see [**9**]):

(1) For any integer $r \geqslant 2$ and $c \in k$ let $Q(2A) := Q(2A)^{r,c}$ be the algebra defined by the quiver and relations

$$\begin{tikzcd} 0\bullet \arrow[loop left, "\alpha"] \arrow[r, bend left=15, "\beta"] & \bullet 1 \arrow[l, bend left=15, "\gamma"] \end{tikzcd}$$

$$\begin{aligned} \gamma\beta\gamma &= (\gamma\alpha\beta)^{r-1}\gamma\alpha, \\ \beta\gamma\beta &= (\alpha\beta\gamma)^{r-1}\alpha\beta,\ \alpha^2\beta = 0, \\ \alpha^2 &= (\beta\gamma\alpha)^{r-1}\beta\gamma + c(\beta\gamma\alpha)^r. \end{aligned}$$

(2) For any integer $r \geqslant 3$ and $c \in k$ let $Q(2B)_1 := Q(2B)_1^{r,c}$ be the algebra defined by the quiver and relations

$$\begin{tikzcd} 0\bullet \arrow[loop left, "\alpha"] \arrow[r, bend left=15, "\beta"] & \bullet 1 \arrow[l, bend left=15, "\gamma"] \arrow[loop right, "\eta"] \end{tikzcd}$$

$$\begin{aligned} &\gamma\beta = \eta^{r-1},\ \beta\eta = \alpha\beta\gamma\alpha\beta, \\ &\eta\gamma = \gamma\alpha\beta\gamma\alpha,\ \alpha^2\beta = 0, \\ &\alpha^2 = \beta\gamma\alpha\beta\gamma + c(\beta\gamma\alpha)^2. \end{aligned}$$

We recommend [**2**] and [**4**] as a good introduction to path algebras of quivers.

For fixed n and c, Holm proved in [**10**] that $Q(2A)^{2^{n-2},c}$ and $Q(2B)_1^{2^{n-2},c}$ are derived equivalent. It is not known whether for different scalars c we get algebras that are derived equivalent.

2. Transfer of gradings via derived equivalences

We prove the following theorem by using the transfer of gradings via derived equivalences.

THEOREM 2.1. *Let A be a tame block of a group algebra with quaternion defect groups and two isomorphism classes of simple modules. There exists a non-trivial grading on A.*

We recall from [**8**] the procedure of transfer of gradings via derived equivalences. Let A and B be two symmetric algebras over a field k and let us assume that A is a graded algebra, and that A and B are derived equivalent. For a given tilting complex T of A-modules, which is a bounded complex of finitely generated projective A-modules, there exists a structure of a complex of graded A-modules T' on T. If T is a tilting complex that tilts from A to B, then $\mathrm{End}_{K^b(P_A)}(T) \cong B^{op}$ (see [**5**] or [**12**] for more details). Viewing T as a graded complex T', and by computing its endomorphism ring as a graded object, we get a graded algebra which is isomorphic to the opposite algebra of the algebra B. The choice of a grading on T' is unique up to shifting the grading of each indecomposable summand of T', because any two different gradings on an indecomposable module (bounded complex) differ only by a shift (see [3, Lemma 2.5.3]). We refer the reader to [**11, 6, 7**] for details on categories of graded modules and their derived categories, and [**13**] for basics of homological algebra.

We will use tilting complexes given in [**10**] to transfer gradings between quaternion blocks with two simple modules.

For the remainder of this paper, if we say that an algebra given by a quiver and relations is graded, we will assume that it is graded in such a way that the arrows and the vertices of its quiver are homogeneous.

REMARK 2.1. By Example 2.5 in [**7**], we can always assume that the vertices of the quiver are homogeneous elements of degree 0. Also, for any arrow of the quiver we can assume that it is a sum of homogeneous paths, each with the same start and end point as the given arrow. Moreover, using the same arguments as in Lemma 2.12 in [**7**], we have that for any arrow α of the quiver, there exists a homogeneous element of the form $\alpha + t$, where t is a linear combination of paths of length strictly greater than 1. For the simplicity of our calculations, we assume that the arrows are homogeneous.

Let us now fix an integer r and an element c from the field k. Let us assume that $Q(2A)$ is graded in such a way that $d_1 = \deg(\alpha)$, $d_2 = \deg(\beta)$ and $d_3 = \deg(\gamma)$.

Since the relations of $Q(2A)$ are homogeneous, the following equations hold

$$\begin{aligned} r(d_2+d_3) + (r-3)d_1 &= 0, \\ (r-2)(d_2+d_3) + \quad rd_1 \quad &= 0. \end{aligned}$$

It follows that $d_1 = 0$ and $d_2 + d_3 = 0$. We write d and $-d$ instead of d_2 and d_3 respectively. Hence, this grading only depends on our choice of integer d_2. Thus, non-trivial gradings on $Q(2A)$ such that the arrows of the quiver of $Q(2A)$ are homogenous are parameterized by $\mathbb{Z} \setminus \{0\}$. With respect to this grading, the graded quiver of $Q(2A)$ is

$$\begin{tikzcd} 0\bullet \arrow[loop left, "0"] \arrow[r, bend left, "d"] & \bullet 1 \arrow[l, bend left, "-d"] \end{tikzcd}$$

REMARK 2.2. From $d_3 = -d_2$ follows that d_3 and d_2 cannot be positive simultaneously. Hence, there does not exist a non-trivial positive grading on $Q(2A)$ such that the arrows of its quiver are homogeneous elements.

For a given algebra A, if we have two different gradings on A, we will denote them by (A, π) and (A, π').

DEFINITION 2.1 ([**12**, Section 5]). Let (A, π) and (A, π') be gradings on a finite dimensional k-algebra A, and let $S_1, S_2, \ldots, S_r$ be the isomorphism classes of simple A-modules. We say that (A, π) and (A, π') are graded Morita equivalent if there exist integers d_{ij}, where $1 \leqslant j \leqslant \dim S_i$ and $1 \leqslant i \leqslant r$, such that the graded algebras (A, π') and $\mathrm{Endgr}_{(A,\pi)}(\bigoplus_{i,j} P_i\langle d_{ij}\rangle)^{\mathrm{op}}$ are isomorphic, where P_i denotes the projective cover of S_i.

Note that graded algebras (A, π) and (A, π') are graded Morita equivalent if and only if their categories of graded modules are equivalent.

THEOREM 2.2. *Up to graded Morita equivalence, there exists a unique non-trivial grading on $Q(2A)$ such that the arrows of its quiver are homogeneous.*

PROOF. We have seen above that the grading on $Q(2A)$ is completely determined by our choice of degree d_2 of the arrow β. If d_2 and d_2' are two integers giving us two non-trivial gradings on $Q(2A)$ (respectively denoted by $(Q(2A), \pi)$ and $(Q(2A), \pi')$), then by shifting P_0, the projective indecomposable module that we see as a graded module in such a way that its top is in degree 0, by $d_2' - d_2$ we get that $(Q(2A), \pi')$ and $\mathrm{Endgr}_{(Q(2A),\pi)}(P_0\langle d_2' - d_2\rangle \oplus P_1)^{\mathrm{op}}$ are isomorphic as graded algebras. $\square$

The graded radical layers, with respect to the above grading where $d_2 = d$, of the projective indecomposable $Q(2A)$-modules are:

$$
\begin{array}{ccccc}
 & & S_0 & & \\
0 & S_0 & & S_1 & -d \\
-d & S_1 & & S_0 & 0 \\
0 & S_0 & & S_0 & 0 \\
 & \vdots & & \vdots & \\
0 & S_0 & & S_1 & -d \\
-d & S_1 & & S_0 & 0 \\
0 & & S_0 & &
\end{array}
,\qquad
\begin{array}{ccccc}
 & & S_1 & & \\
 & & S_0 & d & \\
d & S_0 & & & \\
0 & S_1 & & & \\
 & \vdots & & S_1 & 0 \\
d & S_0 & & & \\
d & & S_0 & & \\
0 & & S_1 & &
\end{array}
.
$$

The integers that appear to the left and right of simple modules in the above radical layers denote homogeneous degrees of these simple modules.

With respect to this grading, the graded Cartan matrix of $Q(2A)$ is

$$\begin{pmatrix} 4r & 2rq^{d} \\ 2rq^{-d} & r+2 \end{pmatrix}$$

and its determinant is $8r$.

Let $T := T_0 \oplus T_1$ be a complex of $Q(2A)$-modules, where

$$T_0 \,:\, 0 \longrightarrow P_1\langle d\rangle \oplus P_1\langle d\rangle \xrightarrow{(\gamma,\gamma\alpha)} P_0 \longrightarrow 0$$

has non-zero entries in degrees 0 and 1, and T_1 is the stalk complex with P_1 in degree 0. Holm proved in [**10**] that T tilts from $Q(2A)$ to $Q(2B)_1$. Viewing T as a graded complex and calculating $\mathrm{End}_{K^b(P_{Q(2A)})}(T)$ in the category of graded complexes gives us $Q(2B)_1$ as a graded algebra.

We easily conclude that $\mathrm{Endgr}_{K^b(P_{Q(2A)})}(T_1)$ is isomorphic to the direct sum of $r+2$ copies of $k\langle 0\rangle$. It follows that $\deg(\eta) = 0$ in the quiver of $Q(2B)_1$. From the relations of $Q(2B)_1$ we get that $\deg(\beta) + \deg(\gamma) = 0$ and $\deg(\alpha) = 0$. It is obvious that the only summand of

$$\mathrm{Homgr}_{K^b(P_{Q(2A)})}(T_1, T_0) \cong \mathrm{Homgr}_{Q(2A)}(P_1, \ker(\gamma, \gamma\alpha))$$

is $k\langle d\rangle$. This means that the paths from the vertex corresponding to T_1 to the vertex corresponding to T_0 are in degree $-d$. Since γ is one of these paths we have that $\deg(\gamma) = -d$ and $\deg(\beta) = d$.

With respect to this grading, the graded quiver of $Q(2B)_1$ is given by

$$\begin{tikzcd} 0\bullet \arrow[loop left, "0"] \arrow[r, bend left, "d"] & \bullet 1 \arrow[l, bend left, "-d"] \arrow[loop right, "0"] \end{tikzcd}$$

The graded Cartan matrix of $Q(2B)_1$ with respect to this grading is

$$\begin{pmatrix} 8 & 4q^{d} \\ 4q^{-d} & r+2 \end{pmatrix}$$

and its graded determinant is $8r$.

We see that for each quaternion block the resulting grading obtained by transfer of gradings via derived equivalences is a non-trivial grading for an appropriate choice of the homogeneous degrees of the arrows of an appropriate quiver. Thus, we have proved Theorem 2.1. Also, since d and $-d$ can not simultaneously be greater than zero, it follows that this grading is never positive. Using the same arguments as with $Q(2A)$, it can be shown that up to graded Morita equivalence there exists a unique grading on $Q(2B)_1$.

References

[1] J. L. Alperin. *Local representation theory.* in: Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge Univ. Press, 1986.

[2] I. Assem, D. Simson and A. Skowronski. *Elements of the representation theory of associative algebras, Vol. 1.* in: LMS Student Texts, vol. 65, Cambridge Univ. Press, 2006.

[3] A. Beilinson, V. Ginzburg, W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.* , **9**(2)(1996), 473–527.

[4] D. J. Benson. *Representations and cohomology I: Basic representation theory of finite groups and associative algebras*, second edition. in: Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge Univ. Press, 1998.

[5] D. Bogdanic. *Graded blocks of group algebras.* University of Oxford, Ph.D. Thesis, 2010.

[6] D. Bogdanic. Graded Brauer tree algebras. *J. Pure Appl. Algebra*, **214**(9)(2010), 1534–1552.

[7] D. Bogdanic. Graded blocks of group algebras with dihedral defect groups. *Colloq. Math.*, **122**(2)(2011), 149–176.

[8] D. Bogdanic. Gradings on quaternion blocks with three simple modules. Submitted to *Bull. Int. Math. Virtual Inst.,*

[9] K. Erdmann. *Blocks of tame representation type and related algebras.* in: Lecture Notes in Mathematics, vol. 1428. Springer–Verlag, Berlin, 1990.

[10] T. Holm. Derived equivalent tame blocks. *J. Algebra*, **194**(1)(1997), 178–200.

[11] C. Nastasescu and F. Van Oystaeyen. *Methods of graded rings.* in: Lecture Notes in Mathematics, vol. 1836. Springer, 2004.

[12] R. Rouquier. Automorphismes, graduations et catégories triangulées. *J. Inst. Math. Jussieu*, **10**(3)(2011), 713–751.

[13] C. A. Weibel. *An introduction to homological algebra.* in: Cambridge studies in advanced mathematics, vol. 38. Cambridge Univ. Press, 1994.

Received by editors 03.04.2020; Revised version 15.09.2020; Available online 23.11.2020.

Department of Mathematics and Computer Science,
University of Banja Luka, Banja Luka, Bosnia and Herzegovina
E-mail address: dusko.bogdanic@pmf.unibl.org