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A NOTE ON POSITIVE PLURIHARMONIC FUNCTIONS IN THE UNIT BALL IN $\mathbb{C}^n$

Miloš Arsenović[1] and Jelena Gajić

Abstract. We prove Schwarz-Pick lemma for strictly positive pluriharmonic functions in the unit ball in $\mathbb{C}^n$. We give a distance estimate in terms of Bergman metric as well as an estimate on the gradient of such functions.

1. Introduction

We use standard notation: the open unit ball in $\mathbb{C}^n$ is denoted by $\mathbb{B}^n$, the Hermitian inner product of $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ is denoted by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}.$$

This inner product induces the standard Euclidean norm $|z| = (\sum_{j=1}^{n} |z_j|^2)^{\frac{1}{2}}$ of a vector z in $\mathbb{C}^n$. The operator norm is denoted by $\| \cdot \|$. The open unit disc in the complex plane is denoted by $\mathbb{D}$ and the right half-plane $\{z \in \mathbb{C} : \Re z > 0\}$ in $\mathbb{C}$ is denoted by $\mathbb{K}$.

Let $\Omega \subset \mathbb{C}^n$ be a domain (an open, connected and nonempty set). A function $u \in C^2(\Omega, \mathbb{C})$ is said to be pluriharmonic if for every complex line

$$l = l_{a,b} = \{a + \xi b : \xi \in \mathbb{C}\}, \qquad a \in \mathbb{C}^n, \quad b \in \mathbb{C}^n, \quad b \neq 0$$

the function $\xi \mapsto u(a + \xi b)$ is harmonic on the set $\Omega_l = \{\xi \in \mathbb{C} : a + \xi b \in \Omega\}$.

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The set of all holomorphic maps between domains $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ is denoted by $H(\Omega_1, \Omega_2)$. The following proposition establishes connection between real valued pluriharmonic functions and holomorphic functions.

PROPOSITION 1.1 ([**2**]). *A function $u \in C^2(\mathbb{B}^n, \mathbb{R})$ is pluriharmonic if and only if it is the real part of a holomorphic function on $\mathbb{B}^n$.*

Let, for a in $\mathbb{B}^n$, P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a and let $Q_a = I - P_a$ be the projection onto the orthogonal complement of $[a]$. Set $s_a = (1-|a|^2)^{\frac{1}{2}}$ and define

$$\phi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B}^n. \tag{1.1}$$

Then $\phi_a'(0) = -s_a^2 P_a - s_a Q_a$ and $\phi_a'(a) = -\frac{1}{s_a^2} P_a - \frac{1}{s_a} Q_a$. More details about these automorphisms of the unit ball $\mathbb{B}^n$ can be found in [**6**]. Note that for $n = 1$ they are well known automorphisms of the unit disc.

Regarding Bergman metrics, we refer the reader to [**1**]. The Bergman distance on a domain $\Omega \subset \mathbb{C}^n$ is denoted by $d_\Omega(z, w)$, the Bergman norm of a (tangent) vector $\xi \in T_z\Omega \cong \mathbb{C}^n$ is denoted by $|\xi|_{B,\Omega,z}$ or, when there is no ambiguity, by $|\xi|_{B,z}$. The following proposition is a fundamental invariance property of the Bergman distance.

PROPOSITION 1.2 ([**1**]). *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains and let $f : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Then f induces an isometry of Bergman metrics:*

$$|\xi|_{B,z} = |(J_{\mathbb{C}} f)\xi|_{B,f(z)}, \qquad z \in \Omega_1, \quad \xi \in \mathbb{C}^n.$$

Equivalently, f induces an isometry with respect to Bergman distances:

$$d_{\Omega_2}(f(z), f(w)) = d_{\Omega_1}(z, w), \qquad z, w \in \Omega_1. \tag{1.2}$$

The Bergman distance on $\mathbb{B}^n$ is given by

$$d_{\mathbb{B}^n}(z, w) = \ln \frac{1 + |\phi_w(z)|}{1 - |\phi_w(z)|}, \qquad z, w \in \mathbb{B}^n, \tag{1.3}$$

where ϕ_w denotes automorphism of $\mathbb{B}^n$ introduced in (1.1). A multiplicative factor often appears in literature in the above formula; we choose to drop it in order to conform to [**4**]. The upshot is that we can use Theorem 1.1 without modifications.

Applying Proposition 1.2 to a conformal map $z \mapsto (z-1)/(z+1)$ of $\mathbb{K}$ onto $\mathbb{D}$ after a simple calculation one obtains Bergman distance on $\mathbb{K}$:

$$d_{\mathbb{K}}(z, w) = d_{\mathbb{D}}\left(\frac{z-1}{z+1}, \frac{w-1}{w+1}\right) = \ln \frac{|w + \bar{z}| + |z - w|}{|w + \bar{z}| - |z - w|}, \qquad z, w \in \mathbb{K}. \tag{1.4}$$

In particular, for real $x, y > 0$ we have

$$d_{\mathbb{K}}(x, y) = \left| \ln \frac{x}{y} \right|. \tag{1.5}$$

In general, $f \in H(\Omega_1, \Omega_2)$ does not imply that

$$d_{\Omega_2}(f(z), f(w)) \leqslant d_{\Omega_1}(z, w) \qquad \text{for all} \quad z, w \in \Omega_1. \tag{1.6}$$

However, in some special cases the above contraction property does hold. This is the content of the following theorem.

Theorem 1.1 ([**4**]). *Every holomorphic function $f : \mathbb{B}^n \to \mathbb{D}$ is a contraction with respect to Bergman metric on $\mathbb{B}^n$ and $\mathbb{D}$.*

For a function $f \in C^1(\mathbb{B}^n)$ $\mathcal{M}$-invariant real gradient is defined by

$$\tilde{\nabla} f(z) = \nabla(f \circ \phi_z)(0),$$

where ϕ_z denotes automorphism of $\mathbb{B}^n$. Then $|\tilde{\nabla} f(0)| = |\nabla f(0)|$ and $|\tilde{\nabla} f(z)| = |\tilde{\nabla}(f \circ \phi_z)(0)|$. More details about $\mathcal{M}-$ invariant real gradient can be found in [**5**].

2. Main results

We begin with an elementary lemma.

Lemma 2.1. *For all z_1 and z_2 in $\mathbb{K}$ we have*

$$\left| \frac{\Re z_1 - \Re z_2}{\Re z_1 + \Re z_2} \right| \leqslant \left| \frac{z_1 - z_2}{\overline{z_1} + z_2} \right|. \tag{2.1}$$

Equality holds if and only if $\Im z_1 = \Im z_2$.

Proof. We have $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where $x_1, x_2 > 0$. Our inequality is equivalent to $|x_1 - x_2| \cdot |\overline{z}_1 + z_2| \leqslant |x_1 + x_2| \cdot |z_1 - z_2|$ and therefore to

$$(x_1 - x_2)^2((x_1 + x_2)^2 + (y_2 - y_1)^2) \leqslant (x_1 + x_2)^2((x_1 - x_2)^2 + (y_1 - y_2)^2).$$

The last inequality is equivalent to an obviously valid inequality

$$(y_1 - y_2)^2((x_1 + x_2)^2 - (x_1 - x_2)^2) \geqslant 0.$$

Clearly, equality holds if and only if $y_1 = y_2$. □

The method of using estimates for analytic functions to obtain results of Schwarz-Pick type for harmonic functions was developed by M. Mateljević and his collaborators, see [**3**].

Theorem 2.1. *Let $\boldsymbol{u}$ be a pluriharmonic function from $\mathbb{B}^n$ to $(0, +\infty)$. Then we have*

$$d_{\mathbb{K}}(u(z), u(w)) \leqslant d_{\mathbb{B}^n}(z, w), \qquad z, w \in \mathbb{B}^n. \tag{2.2}$$

Proof. By Proposition 1.1 there is a holomorphic function $f : \mathbb{B}^n \to \mathbb{K}$ such that $u = \Re f$. Therefore $g(z) = \frac{f(z)-1}{f(z)+1} \in H(\mathbb{B}^n, \mathbb{D})$.

By Proposition 1.2 and Theorem 1.1 we have

$$d_{\mathbb{K}}(f(z), f(w)) = d_{\mathbb{D}}(g(z), g(w)) \leqslant d_{\mathbb{B}^n}(z, w) \qquad z, w \in \mathbb{B}^n. \tag{2.3}$$

Since $\frac{1+x}{1-x}$, $0 < x < 1$, and $\ln x$, $x > 0$, are increasing functions we have, by Lemma 2.1

$$d_{\mathbb{K}}(u(z), u(w)) = d_{\mathbb{K}}(\Re f(z), \Re f(w)) \leqslant d_{\mathbb{K}}(f(z), f(w)),$$

which, combined with (2.3), gives (2.2). □

In view of (1.5) inequality (2.2) can be written in the following form

$$e^{-d_{\mathbb{B}^n}(z,w)} \leqslant \frac{u(z)}{u(w)} \leqslant e^{d_{\mathbb{B}^n}(z,w)}, \qquad z, w \in \mathbb{B}^n. \tag{2.4}$$

Corollary 2.1. *If f is a holomorphic function on $\mathbb{B}^n$ and $0 < |f(z)| < 1$, then*

$$|f(w)|^{e^{d_{\mathbb{B}^n}(z,w)}} \leqslant |f(z)| \leqslant |f(w)|^{e^{-d_{\mathbb{B}^n}(z,w)}}, \qquad z, w \in \mathbb{B}^n. \tag{2.5}$$

Proof. Let $u(z) = \ln \frac{1}{|f(z)|}$. Then u is a pluriharmonic function from $\mathbb{B}^n$ into $(0, +\infty)$ and according to (2.4) we have

$$\ln \left(\frac{1}{|f(w)|}\right)^{e^{-d_{\mathbb{B}^n}(z,w)}} \leqslant \ln \frac{1}{|f(z)|} \leqslant \ln \left(\frac{1}{|f(w)|}\right)^{e^{d_{\mathbb{B}^n}(z,w)}}$$

which gives (2.5). □

Theorem 2.2. *If u is a pluriharmonic function from $\mathbb{B}^n$ to $(0, +\infty)$, then*

$$|\nabla u(z)| \leqslant \frac{2u(z)}{1 - |z|^2}, \qquad z \in \mathbb{B}^n.$$

Proof. As in the proof of Theorem 2.1, there exists a holomorphic function $f \in H(\mathbb{B}^n, \mathbb{K})$ such that $u = \Re f$. Using Cauchy-Riemann equations we obtain

$$|f'(z)| = \sqrt{\sum_{i=1}^n \left|\frac{\partial f}{\partial z_i}(z)\right|^2} = \sqrt{\sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(z)\right|^2 + \sum_{i=1}^n \left|\frac{\partial u}{\partial y_i}(z)\right|^2} = |\nabla u(z)|. \tag{2.6}$$

Since $g(z) = \frac{f(z)-1}{f(z)+1} \in H(\mathbb{B}^n, \mathbb{D})$ we can apply classical Schwarz lemma to $h_\xi(\lambda) = g(\lambda\xi)$, where $\xi \in \mathbb{C}^n$ and $|\xi| = 1$, to obtain $|g'(0)\xi| \leqslant 1 - |g(0)|^2$. Since this is valid for all $|\xi| = 1$, we obtain $|g'(0)| \leqslant 1 - |g(0)|^2$. Therefore

$$|\nabla u(0)| = |f'(0)| = \frac{2}{|1 - g(0)|^2}|g'(0)| \leqslant 2\frac{1 - |g(0)|^2}{|1 - g(0)|^2}.$$

This, combined with $1 - |g|^2 = \frac{4u}{|f+1|^2}$ and $|1 - g|^2 = \frac{4}{|f+1|^2}$, gives

$$|\nabla u(0)| \leqslant 2u(0),$$

which is desired estimate at the origin. The function $v = u \circ \phi_z$ is also pluriharmonic, see [**6**], so we can apply the obtained estimate at the origin and get

$$|\nabla v(0)| = |\nabla(u \circ \phi_z)(0)| \leqslant 2(u \circ \phi_z)(0) = 2u(z). \tag{2.7}$$

Since $\nabla v(0) = \nabla u(z) \cdot \phi_z'(0)$ and $\phi_z'(z)\phi_z'(0) = I$, we have $\nabla u(z) = \nabla v(0)\phi_z'(z)$. Using $\|\phi_z'(z)\| = \frac{1}{1-|z|^2}$ and (2.7) we obtain

$$|\nabla u(z)| = |\nabla v(0)\phi_z'(z)| \leqslant |\nabla v(0)| \cdot \|\phi_z'(z)\| \leqslant \frac{2u(z)}{1 - |z|^2}$$

and complete the proof. □

At the origin, based on the previous theorem, we have $|\tilde{\nabla} u(0)| \leqslant 2u(0)$. Then,

$$|\tilde{\nabla}(u \circ \phi_z)(0)| \leqslant 2(u \circ \phi_z)(0) = 2u(z).$$

Since $|\tilde{\nabla} u(z)| = |\tilde{\nabla}(u \circ \phi_z)(0)|$ we have obtained the following corollary.

COROLLARY 2.2. *If u is a pluriharmonic function from $\mathbb{B}^n$ to $(0, +\infty)$, then*

$$|\tilde{\nabla} u(z)| \leqslant 2u(z), \qquad z \in \mathbb{B}^n.$$

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References

[1] S. G. Krantz. *Geometric analysis of the Bergman kernel and metric.* Springer Science + Bussines Media, New York 2013.

[2] S. G. Krantz. *Function teory of several complex variables*, 2^{nd} ed. American Mathematical Society, Providence, Rhode Island 2001.

[3] M. Mateljević. Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions. *J. Math. Anal. Appl.*, **464**(1)(2018), 78–100.

[4] P. Melentijević. Invariant gradient in refinements of Schwarz lemma and Harnack inequalities. *Ann. Acad. Sci. Fenn., Math.*, **43**(2018), 391–399.

[5] M. Pavlović. Inequalities for the gradient of eigenfunctions of the invariant Laplacian in the unit ball. *Indag. Math.*, **2**(1)(1991), 89–98.

[6] W. Rudin. *Function theory in the unit ball of* $\mathbb{C}^n$. Springer-Verlag, New-York 1980.

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FACULTY OF MATHEMATICS, UNIVERSITY OF BELGRADE,
STUDENTSKI TRG 16, 11000 BELGRADE, SERBIA
E-mail address: arsenovic@matf.bg.ac.rs

FACULTY OF NATURAL SCIENCES AND MATHEMATICS, MLADEN STOJANOVIĆ STREET 2,
78000 BANJA LUKA, THE REPUBLIC OF SRPSKA, BOSNIA AND HERZEGOVINA
E-mail address: jelena.gajic.mm@gmail.com