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ON SOLVABILITY OF A PARABOLIC THIRD-ORDER DIFFERENTIAL EQUATION SET ON SINGULAR DOMAIN

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Abstract. We give regularity results for solutions of a third order parabolic equation set in nonsmooth domain. the existence and uniqueness of solutions is discussed via an abstract point of view via the study of a third order abstract differential equation with variable operator coefficients.

1. Introduction

Recently, many works have dealt with the resolution of the complete abstract differential equation of third order

$$\frac{d^3u\left(t\right)}{dt^3}+A\frac{d^2u\left(t\right)}{dt^2}+B\frac{du\left(t\right)}{dt}+Cu\left(t\right)=h(t),$$

set on unbounded intervals with variable operator coefficients A, B, C in a Hilbert space H. For example, in [**1**], we found a complete study of the following problem

$$\left(\frac{d}{dt}+A\right)^3 u\left(t\right)+A_1\frac{d^2u\left(t\right)}{dt^2}+A_2\frac{du\left(t\right)}{dt}=h(t), \tag{1.1}$$

some optimal results about the existence and uniqueness have been established, where

- $h\in L^2\left(\mathbb{R},H\right)$,
- $(A, D(A))$ is a self-adjoint positive definite in some Hilbert space H,
- $\left(A_1, D(A_1)\right), \left(A_2, D(A_2)\right)$ are in general linear unbounded operators.

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In [**2**], problem

$$\left(\frac{d}{dt}+A\right)^3 u\left(t\right)+A_1\frac{d^2u\left(t\right)}{dt^2}+A_2\frac{du\left(t\right)}{dt}+A_3u\left(t\right)=h(t),$$

was considered but under the following assumptions

- $h\in L^2\left(\mathbb{R}^+,H\right)$,
- $(A,D(A))$ is an a self-adjoint positive definite in some Hilbert space H,
- $\left(A_1,D(A_1)\right),\left(A_2,D(A_2)\right)\ \left(A_3,D(A_3)\right)$ are in general linear unbounded operators.
- $\dfrac{d^2u\left(t\right)}{dt^2}=\dfrac{du\left(t\right)}{dt}=u(0)=0.$

The techniques of investigation are based essentially on the use of some classical tools of harmonic analysis such the Fourier transform and the well known variational method.

In this work, we moved away from the classical hilbertian framework. The starting point of our study is a concrete problem set on non smooth cylindrical domain $\Pi\subset\mathbb{R}^3$. This problem will be written later as a third order abstract differential equation involving the principal part of (1.1). To be more precise, we consider a cylinderical domain

$$\Pi=\mathbb{R}^+\times\Omega,$$

but its base is a cusp domain defined by the set

$$\Omega=\left\{(x,y)\in\mathbb{R}^2:0<x<a,\ \ -\psi\left(x\right)<y<\psi\left(x\right)\right\}, \tag{1.2}$$

where $\psi\left(x\right)=x^{\alpha}$, $1<\alpha\leqslant 2$ and a is a given positive number small enough.

This work deals with the following equation

$$\left(\frac{\partial}{\partial t}+\Delta\right)^3 u\left(t,x,y\right)=h\left(t,x,y\right),\qquad (t,x,y)\in\Pi, \tag{1.3}$$

where $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ is the standard Laplace operator. We note that this equation appears in several interesting applications like heat conductivity in viscoelastic materials and problems of world population growth; see [**13, 14**]. This equation is viewed as a parabolic differential equations according to [**11**].

The right hand term h of (1.3) is taken in the anisotropic Lebesgue space

$$L^p\left(\mathbb{R}^+;h^{2\sigma}\left(\Omega\right)\right),1<p<+\infty,$$

endowed with its natural norm

$$\|f\|_{L^p(\mathbb{R}^+;h^{2\sigma}(\Omega))}=\left(\int\limits_0^{+\infty}\|f\left(t\right)\|^p_{h^{2\sigma}(\Omega)}\ dt\right)^{1/p}.$$

Throughout this work, we assume that for $\omega>0$

$$f\left(t,.,.\right)=f\left(t+\omega,.,\right). \tag{1.4}$$

For the reader convenience, we recall here that $h^{2\sigma}(\Omega)$ denotes the subset of $C^{2\sigma}(\Omega)$ consisting of the functions ϕ such that

$$\lim_{\delta\to 0^+}\ \sup_{0<\|(x,y)-(x',y')\|\leqslant\delta}\frac{|\phi(x,y)-\phi(x',y')|}{\|(x,y)-(x',y')\|^{2\sigma}}=0.$$

In the sequel, we assume also that

$$h|_{\mathbb{R}^+\times(a,\pm\psi(a))}=0. \tag{1.5}$$

The solvability of (1.3) is discussed under the following boundary conditions

$$\begin{aligned} u|_{\mathbb{R}^+\times(\Gamma_1\cup\Gamma_2)}&=0,\\ u|_{\mathbb{R}^+\times\Gamma_3}&=0. \end{aligned} \tag{1.6}$$

Here

$$\begin{aligned}\Gamma_1&=\{(x,y)\in\partial\Omega: y=\psi(x)\},\\ \Gamma_2&=\{(x,y)\in\partial\Omega: y=-\psi(x)\},\\ \Gamma_3&=\{(a,y): -\psi(a)<y<\psi(a)\}.\end{aligned}$$

Our purpose is to establish some results about the existence and uniqueness of a ω-periodic solution

$$u(t,.,.)=u(t+\omega,.,.), \tag{1.7}$$

to (1.3)-(1.6).

Our strategy is based on the same argument used in [**4, 5, 6, 7**] and [**8**]. The main idea to solve Problem (1.3)-(1.6) consists in transforming the original problem set non-regular domain Π to a new one set on regular domain. Next, the transformed problem is written as an abstract differential equation set on some suitable Banach spaces. This approach needs the use of the sum's operator theory as in [**5**]. As consequence, we establish some interesting regularity results for our problem (1.3)-(1.7).

2. Change of variables

We use the following natural change of variables

$$T:\Pi\to Q,\quad (t,x,y)\mapsto(t,\xi,\eta)=\left(t,\frac{x^{1-\alpha}}{\alpha-1},\frac{y}{\psi(x)}\right),$$

where

$$Q=\mathbb{R}^+\times D, \tag{2.1}$$

with

$$D=\left]\xi_0,+\infty\right[\times\left]-1,1\right[,$$

and

$$\xi_0=\frac{1}{\alpha-1}a^{1-\alpha}>0.$$

One has

$$T^{-1}:Q\to\Pi\quad(t,\xi,\eta)\mapsto(x,y)=\left(t,e^{\frac{\ln[(\alpha-1)\xi]}{1-\alpha}},\eta e^{\frac{\alpha\ln[(\alpha-1)\xi]}{1-\alpha}}\right).\tag{2.2}$$

Note that for any $x\in]0,a[$

$$\Pi(t,x,\psi(x))=\left(t,\frac{x^{1-\alpha}}{\alpha-1},1\right),\quad\Pi(t,x,-\psi(x))=\left(t,\frac{x^{1-\alpha}}{\alpha-1},-1\right),$$

then

$$\begin{cases}\Pi(\Gamma_1)=\{(\xi,1):\xi\in\,]\xi_0,+\infty[\},\\ \Pi(\Gamma_2)=\{(\xi,-1):\xi\in\,]\xi_0,+\infty[\},\\ \Pi(\Gamma_3)=\{(\xi_0,\eta):\eta\in\,]-1,+1[\},\end{cases}\tag{2.3}$$

and

$$\begin{aligned}\lim_{x\to0^+}\Pi(t,x,\psi(x))&=\lim_{x\to0^+}\left(t,\frac{x^{1-\alpha}}{\alpha-1},1\right)=(t,+\infty,1),\\ \lim_{x\to0^+}\Pi(t,x,-\psi(x))&=\lim_{x\to0^+}\left(t,\frac{x^{1-\alpha}}{\alpha-1},-1\right)=(t,+\infty,-1),\end{aligned}$$

which means that the axis $\{t\in\mathbb{R}^+:(t,0,0)\}$ is transformed in

$$D_\infty=\{(t,+\infty,\eta):\eta\in\,]-1,+1[\}=\mathbb{R}^+\times\{+\infty\}\times\,]-1,+1[\,.$$

Now, define the following change of functions

$$\begin{cases}v(t,\xi,\eta):=(v\circ\Pi)(t,x,y)=u(t,x,y)=u\left(t,e^{\frac{\ln[(\alpha-1)\xi]}{1-\alpha}},\eta e^{\frac{\alpha\ln[(\alpha-1)\xi]}{1-\alpha}}\right)\\ \\ g(t,\xi,\eta):=(g\circ\Pi)(t,x,y)=h(t,x,y)=h\left(t,e^{\frac{\ln[(\alpha-1)\xi]}{1-\alpha}},\eta e^{\frac{\alpha\ln[(\alpha-1)\xi]}{1-\alpha}}\right).\end{cases}\tag{2.4}$$

Put

$$\beta=\alpha/(\alpha-1)\ \text{ and }\ \theta=(\alpha-1)^{\beta}.\tag{2.5}$$

Then, Equation (1.3) becomes

$$\left(\frac{\partial}{\partial t}+\Delta\right)^3v(t,\xi,\eta)+\frac{1}{\xi}[Pv](t,\xi,\eta)=f(t,\xi,\eta),\quad(t,\xi,\eta)\in Q\tag{2.6}$$

where

$$f(t,\xi,\eta)=\theta^{-2}\xi^{-2\beta}g(t,\xi,\eta),\tag{2.7}$$

and

$$Pv=\sum_{k=1}^{3}\frac{1}{\xi^{k-1}}\frac{\partial^{3-k}}{\partial t^{3-k}}\ L^kv.\tag{2.8}$$

Here L is the second differential operator with C^{∞}-bounded coefficients on Q given by

$$\begin{aligned}
&[Lv](t,\xi,\eta)\\
=\;& \alpha^2\theta^{-2/\beta}\frac{\eta^2}{\xi}\frac{\partial^2 v}{\partial\eta^2}(t,\xi,\eta)+2\alpha\theta^{-1/\beta}\eta\frac{\partial^2 v}{\partial\xi\partial\eta}(t,\xi,\eta)\\
&+\alpha\theta^{-1/\beta}\frac{\partial v}{\partial\xi}(t,\xi,\eta)+\alpha(\alpha+1)^2\theta^{-2/\beta}\frac{\eta}{\xi}\frac{\partial v}{\partial\eta}(t,\xi,\eta).
\end{aligned}$$

Since the solvability of our problem (1.3) is discussed near the singular part of our domain, the operator P defined (2.8) is viewed in the sequel as a perturbation of the following one given by

$$\left\{\begin{array}{ll}
\left(\dfrac{\partial}{\partial t}+\Delta\right)^3 v(t,\xi,\eta)=f(t,\xi,\eta), & (t,\xi,\eta)\in Q,\\
v(t+\omega,\xi,\eta)=v(t,\xi,\eta), & (t,\xi,\eta)\in Q,\\
v(t,\xi_0,\eta)=0, & (t,\eta)\in\mathbb{R}^+\times\left]-1,1\right[,\\
v(t,\xi,-1)=v(t,\xi,1)=0, & (t,\xi)\in\mathbb{R}^+\times\left]\xi_0,+\infty\right[.
\end{array}\right. \tag{2.9}$$

At this level, we note that condition (1.5) becomes

$$f(.,+\infty,.)=0 \text{ and } f(.,0,\pm1)=0. \tag{2.10}$$

Now, it is necessary to specify the impact of the change of variables on the functional framework of our study:

Lemma 2.1. *Let $0<\sigma<\frac{1}{2}$ and $1<\alpha\leqslant 2$. Then*

(1) $h\in L^p_\omega\left(\mathbb{R}^+;h^{2\sigma}(\Omega)\right)\Rightarrow f\in L^p_\omega\left(\mathbb{R}^+;h^{2\sigma}(D)\right)$

(2) $f\in L^p_\omega\left(\mathbb{R}^+;h^{2\sigma}(D)\right)\Rightarrow h\in L^p_\omega\left(\mathbb{R}^+;h^{2\sigma}_{\alpha,\sigma}(\Omega)\right)$ *with*

$$h^{2\sigma}_{\alpha,\sigma}(\Omega)=\left\{\phi\in h^{2\sigma}(\Omega):x^{2\alpha(\sigma+1)}\phi\in h^{2\sigma}(\Omega)\right\}.$$

Proof. See Proposition 3.1 in [**4**]. □

3. The abstract formulation of the principal problem (2.9)

Set $E=h^{2\sigma}(D)$ and define the functions :

$$\begin{aligned}
v &: \mathbb{R}^+\to E\ ;\ t\longrightarrow v(t)\ ;\quad v(t)(\xi,\eta)=v(t,\xi,\eta),\\
f &: \mathbb{R}^+\to E\ ;\ t\longrightarrow f(t)\ ;\quad f(t)(\xi,\eta)=f(t,\xi,\eta).
\end{aligned}$$

Consider the operator A defined by

$$\left\{\begin{array}{l}
D(A)=\{v\in L^p(\mathbb{R}^+;E):v(t)\in D(C)\},\\
(Av)(t)=C(v(t)),
\end{array}\right. \tag{3.1}$$

with

(3.2)

$$\left\{\begin{array}{ll}
D(C) & =\left\{w\in W^{2,p}(D)\cap C^2(D),p>2:\ w|_{\xi=\xi_0}=0,\ \ w|_{\eta=\pm1}=0\right\},\\
(Cw)(\xi,\eta) & =\Delta w(\xi,\eta).
\end{array}\right.$$

The abstract version of the principal problem (2.9) is given by

$$\begin{aligned} &\frac{d^3 v(t)}{dt^3} + 3A\frac{d^2 v(t)}{dt^2} + 3A^2\frac{dv(t)}{dt} + A^3 v(t) = f(t), \qquad t \in \mathbb{R}^+, \\ &v(t+\omega) = v(t). \end{aligned} \tag{3.3}$$

4. Spectral study of the operator $(A, D(A))$

This section gives some useful spectral properties of the operator $(A, D(A))$ defined by (3.1). The operator $(A, D(A))$ has the same properties as its realization $(C, D(C))$ defined by (3.2). Then, in the framework of the little Hölder space $h^{2\sigma}(D)$, we consider the following problem

$$\begin{cases} (\Delta - \lambda) v(\xi, \eta) = f(\xi, \eta), & (\xi, \eta) \in D, \ \lambda > 0, \\ v(\xi_0, \eta) = 0, \quad v(+\infty, \eta) = 0, & -1 < \eta < 1, \\ v(\xi, -1) = v(\xi, 1) = 0, & \xi > \xi_0. \end{cases} \tag{4.1}$$

The solvability of (4.1) is discussed by using the commutative version of the sum's operator technique developed in [**9**]. First, let us recall the essential of this method

4.1. On the sum of linear operators.

Let X a complex Banach space and M, N two closed linear operators with domains $D(M)$, $D(N)$. Let C be the operator defined by

$$\begin{cases} Cv = Mv + Nv, \\ v \in D(C) = D(M) \cap D(N), \end{cases}$$

where M and N verify the assumptions

$$(H.1) \begin{cases} i) \ \rho(M) \supset \sum_M = \{\mu : |\mu| \geqslant r, \ |Arg(\mu)| < \pi - \epsilon_M\}, \\ \quad \forall \mu \in \sum_M \quad \left\| (M - \mu I)^{-1} \right\|_{L(X)} \leqslant C_M / |\mu|. \\ ii) \ \rho(N) \supset \sum_N = \{\mu : |\mu| \geqslant r, \ |Arg(\mu)| < \pi - \epsilon_N\}, \\ \quad \forall \mu \in \sum_N \quad \left\| (N - \mu I)^{-1} \right\|_{L(X)} \leqslant C_N / |\mu|. \\ iii) \ \epsilon_M + \epsilon_N < \pi. \\ iv) \ \overline{D(M) + D(N)} = X. \\ v) \ \sigma(-M) \cap \sigma(N) = \emptyset, \end{cases}$$

and

$$(H.2)\begin{cases} \forall\mu_1 \in \rho(M), \forall\mu_2 \in \rho(N) \\ (M-\mu_1 I)^{-1}(N-\mu_2 I)^{-1} - (N-\mu_2 I)^{-1}(A-\mu_1 I)^{-1} \\ = \left[(M-\mu_1 I)^{-1};\ (N-\mu_2 I)^{-1}\right] = 0, \end{cases}$$

where $\rho(M)$ and $\rho(N)$ are the resolvent sets of M and N.
The application of the sum's theory needs to introduce some interpolation spaces. For any $\varrho \in]0,1[$, and thanks to $(H.1)$, we introduce the two families of real Banach interpolation spaces between $D(M)$ and E :

$$D_M(\varrho, +\infty) = \left\{\zeta \in E : \sup_{r>0} \left\| r^{\varrho} M (M-rI)^{-1}\zeta \right\|_X < \infty \right\},$$

and its subspace

$$D_M(\varrho) = \left\{\zeta \in E : \lim_{r\to+\infty} \left\| r^{\varrho} M (M-rI)^{-1}\zeta \right\|_X = 0 \right\}.$$

Among the main results proved in [**9**] one has

THEOREM 4.1. *Let $\varrho \in]0,1[$. Assume $(H.1)$, $(H.2)$ and $f \in D_{\varrho}(M)$ (resp. $f \in D_{\varrho}(N)$). Then, the problem*

$$Au + Bu - \lambda u = f,$$

has a unique strict solution

$$u \in D(M) \cap D(N),$$

given by

$$u = -\frac{1}{2i\pi}\int_{\gamma_1} (M+zI)^{-1}(N-zI-\lambda I)^{-1} f dz$$

where γ_1 is a suitable sectorial curve lying in $\rho(-M)\cap\rho(N)$.

4.2. The application of the sum's theory. Set $X = C(D)$ and let us introduce the two closed linear operators

$$(4.2)\qquad \begin{cases} D(M) &= \left\{w \in C^2([\xi_0,+\infty[) : w(\xi_0) = 0, w(+\infty) = 0\right\}, \\ (Mw)(\eta) &= w''(\xi), \end{cases}$$

and

$$(4.3)\qquad \begin{cases} D(N) &= \left\{w \in C^2([-1,1]) : w(-1) = w(1) = 0\right\}, \\ (Nw)(\eta) &= w''(\eta) \end{cases}$$

Our problem (4.1) is written as follows

$$(4.4)\qquad \begin{cases} Mv + Nv - \lambda v = f, \\ v \in D(M)\cap D(N). \end{cases}$$

One has

LEMMA 4.1. *The operators $(M, D(M))$ and $(N, D(N))$ satisfy Assumptions $(H.1)$ and $(H.2)$*

PROOF. First, a direct computation show that

$$(M+z)^{-1}\phi = -\int_{\xi_0}^{+\infty} k(\xi,s)\phi(s)ds,$$

where

$$k(\xi,s)=\begin{cases} \dfrac{e^{-\sqrt{-z}(\xi-\xi_0)}\sinh\sqrt{-z}s}{\sqrt{-z}} & \xi_0 \leqslant s \leqslant \xi, \\[2ex] \dfrac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}(\xi-\xi_0)}{\sqrt{-z}} & s \geqslant \xi. \end{cases}$$

Here $\sqrt{-z}$ is the analytic determination defined by $Re\sqrt{-z}>0$. Then

$$\left\| \int_{\xi_0}^{+\infty} k(\xi,s)\phi(s)ds \right\|_X \leqslant I+J,$$

and

$$\begin{aligned} I &= \int_{\xi_0}^{\xi} \frac{e^{-(Re\sqrt{-z})(\xi-\xi_0)}\left|\sinh\sqrt{-z}s\right|}{\left|\sqrt{-z}\right|} ds\,\|\phi\|_X \\ &\leqslant \frac{e^{-(Re\sqrt{-z})(\xi-\xi_0)}}{|z|^{1/2}} \frac{1}{Re\sqrt{-z}} \sinh(Re\sqrt{z}\xi)\,\|\phi\|_X \end{aligned}$$

and

$$\begin{aligned} J &\leqslant \frac{\left|\sinh\sqrt{-z}(\xi-\xi_0)\right|}{\left|\sqrt{-z}\right||z|} \int_{\xi}^{+\infty} e^{-(Re\sqrt{-z})s} ds\,\|\phi\|_X \\ &\leqslant \frac{\cosh(Re\sqrt{-z}(\xi-\xi_0))e^{-(Re\sqrt{-z})\xi}}{|z|^{1/2}\,Re\sqrt{-z}} \|\phi\|_X, \end{aligned}$$

then

$$\left\|(M+zI)^{-1}\right\|_{L(X)} = O\left(\frac{1}{|z|}\right). \tag{4.5}$$

In the general case, it is well known that estimate (4.5) implies the existence of $\delta_0 \in (0,\pi/2)$ and $\varepsilon_0 > 0$ such that the resolvent set of M contains a sectorial domain of the form

$$\Sigma = \{z\in\mathbb{C} : |z| \geqslant \varepsilon_0 \text{ and } |\arg z| = \delta_0\} \cup \{z=\varepsilon_0 e^{i\theta} : \delta_0 \leqslant \theta \leqslant 2\pi-\delta_0\}.$$

The operator $(N, D(N))$ can be treated by same techniques. This means that Hypothesis $(H.1)$ is handled by choosing a suitable ϵ_M and ϵ_N in $(0,\pi/2)$.
Now, Hypothesis $(H.2)$ is easily checked since the two operators are acting on different variables. □

REMARK 4.1. In our situation, thanks to (2.10), one has exactly

$$D_M(\sigma) = D_M(\sigma) = \left\{\phi \in h^{2\sigma}(D) : \phi(.,\pm 1)=0, \phi(\xi_0,.)=0\right\}, 0<2\sigma<1, \tag{4.6}$$

this justify the importance of the conditions (1.5).

Our main result concerning the operator $(C, D(C))$ is formulated as follows

LEMMA 4.2. *The operator $(C, D(C))$ defined by (3.2) is a closed operator satisfying the Krein-ellipticity property, that is: $\mathbb{R}^+ \subset \rho(A)$ and*

$$\exists\ K > 0 : \forall \lambda \geqslant 0 \quad \left\| (C - \lambda I)^{-1} \right\|_{L(h^{2\sigma}(D))} \leqslant \frac{K}{1 + |\lambda|}. \tag{4.7}$$

PROOF. As a direct consequence of the sum's theory, we can say that under conditions (1.5) Problem (4.1) has a unique solution v. More precisely $(C - \lambda I)^{-1}$ is well defined and one has

$$v = (C - \lambda I)^{-1} f = (M + N - \lambda)^{-1} f = \frac{-1}{2i\pi} \int_{\gamma_1} (M + z)^{-1} (N - z - \lambda)^{-1} f dz,$$

where γ_1 is the boundary of Σ. Then

$$\begin{aligned} & v(\xi, \eta) \\ = & -\frac{1}{2i\pi} \int_{\gamma_1} \int_0^{\xi-\xi_0} \frac{e^{-\sqrt{-z}(\xi-\xi_0)} \sinh \sqrt{-z} s}{\sqrt{-z}} \left((N - zI)^{-1} f(s) \right)(\eta) ds dz \\ & -\frac{1}{2i\pi} \int_{\gamma_1} \int_{\xi-\xi_0}^{+\infty} \frac{e^{-\sqrt{-z} s} \sinh \sqrt{-z}(\xi - \xi_0)}{\sqrt{-z}} \left((N - zI)^{-1} f(s) \right)(\eta) ds dz; \end{aligned}$$

but in our case, one has

$$\begin{aligned} & \left((N - zI)^{-1} f(s) \right)(\eta) \\ = & -\int_{-1}^{\eta} \frac{2 \sinh \sqrt{4z} \left(\frac{1+\tau}{2} \right) \sinh \sqrt{4z} \left(\frac{1-\eta}{2} \right)}{\sqrt{4z} \sinh \sqrt{4z}} f(s)(\tau) d\tau \\ & -\int_{\eta}^{1} \frac{2 \sinh \sqrt{4z} \left(\frac{1+\eta}{2} \right) \sinh \sqrt{4z} \left(\frac{1-\tau}{2} \right)}{\sqrt{4\,(z)} \sinh \sqrt{4\,(z)}} f(s)(\tau) d\tau \\ = & \int_{-1}^{1} K_z(\eta, \tau) f(s)(\tau) d\tau \\ = & \int_{-1}^{1} K_z(\eta, \tau) f(s + \xi_0, \tau) d\tau. \end{aligned}$$

We then obtain the formula

$$\begin{aligned} & v(\xi, \eta) \\ = & -\frac{1}{2i\pi} \int_{\gamma_1} \int_0^{\xi-\xi_0} \frac{e^{-\sqrt{-z}(\xi-\xi_0)} \sinh \sqrt{-z} s}{\sqrt{-z}} \left[{}_{-1}^{1} K_z(\eta, \tau) f(s + \xi_0, \tau) d\tau \right] ds dz \\ & -\frac{1}{2i\pi} \int_{\gamma_1} \int_{\xi-\xi_0}^{+\infty} \frac{e^{-\sqrt{-z} s} \sinh \sqrt{-z}(\xi - \xi_0)}{\sqrt{-z}} \left[{}_{-1}^{1} K_z(\eta, \tau) f(s + \xi_0, \tau) d\tau \right] ds dz. \end{aligned}$$

The estimate (4.7) is obtained via the same argument used in [**3**]. □

Consequently, one has

Lemma 4.3. *The operator $(A, D(A))$ defined by (3.1) is a closed operator satisfying the Krein-ellipticity property, that is:* $\mathbb{R}^+ \subset \rho(A)$ *and*

$$\exists\, C > 0 : \forall \lambda \geqslant 0 \quad \left\| (A - \lambda I)^{-1} \right\|_{L(L^p(\mathbb{R}^+; h^{2\sigma}(D)))} \leqslant \frac{C}{1 + |\lambda|}, \tag{4.8}$$

Remark 4.2. Assumption (4.8) implies that operator $B = -(-A)^{1/2}$ is well defined and it is the infinitesimal generator of the generalized analytic semigroup $\left(e^{tB}\right)_{t>0}$. More precisely, there exists a sector

$$\Pi_{\delta, r_0} = \{ z \in \mathbb{C}^* : |\arg z| \leqslant \delta + \pi/2 \} \cup \overline{B(0, r_0)}$$

(with some positive δ, r_0) and $K > 0$ such that $\rho(B) \supset \Pi_{\delta, r_0}$ and

$$\exists C > 0 : \forall z \in \Pi_{\delta, r_0}, \quad \left\| (B - zI)^{-1} \right\| \leqslant \frac{K}{1 + |z|}. \tag{4.9}$$

We recall that one has for all $t > 0$ and $\varphi \in E$

$$e^{Bt}\varphi = \frac{1}{2i\pi} \int_\gamma e^{zt} (B - zI)^{-1} \varphi dz, \tag{4.10}$$

where $\gamma = \partial \Pi_{\delta, r_0}$, (the sectorial boundary curve of Π_{δ, r_0} oriented from $\infty e^{i(\delta + \pi/2)}$ to $\infty e^{-i(\delta + \pi/2)}$).

Taking into account the definition of the operator A, we mention some usual properties of the semigroup $\left(e^{tB}\right)_{t>0}$. For more information, we refer the reader to [**10**] and [**16**].

Lemma 4.4. *Let E a complex Banach space. For $t \in \mathbb{R}^+$, $1 < p < +\infty$, $\Psi \in L^p(\mathbb{R}^+; E)$ and $\Phi \in L^p(\mathbb{R}^+; E)$, one has*

(1) $t \longmapsto B \int_0^t e^{(t-s)B} \Psi(s) ds \in L^p(\mathbb{R}^+; E)$.
(2) $t \longmapsto B \int_0^t e^{(t-s)B} \Phi(s) ds \in L^p(\mathbb{R}^+; E)$.
(3) $t \longmapsto B \int_t^{+\infty} e^{(s-t)B} \Phi(s) ds \in L^p(\mathbb{R}^+; E)$.
(4) $t \longmapsto B \int_0^{+\infty} e^{(s+t)B} \Phi(s) ds \in L^p(\mathbb{R}^+; E)$.

5. Some regularity results for the complete transformed problem

We look now an explicit representation of v. Classical considerations give the following representation, for a.e. $t \in \mathbb{R}^+$

$$v(t) = e^{tB} b_1 + t e^{tB} b_2 + t^2 e^{tB} b_3,$$

where b_1, b_2 and b_3 are arbitrary constants in E. Using condition (1.7), a formal computation show that

$$\begin{aligned} v(t) &= \frac{(1+2t)}{2\left(1-e^{B\omega}\right)^3}\int_t^{t+\omega}\left(\omega - s\left(I-e^{B\omega}\right)\right)^2 e^{B(t+\omega-s)}f(s)ds \\ &+\frac{\omega^2 e^{B\omega}}{2\left(1-e^{B\omega}\right)^3}\int_t^{t+\omega} e^{B(t+\omega-s)}f(s)ds \\ &+\frac{t^2}{2\left(1-e^{B\omega}\right)^3}\int_t^{t+\omega} e^{B(t+\omega-s)}f(s)ds. \end{aligned} \tag{5.1}$$

Remark 5.1. Note here that this solution is well defined. In fact, we know that (4.9) allow us to say that the operator $1-e^{B\omega}$ has a bounded inverse and

$$\left(1-e^{B\omega}\right)^{-1} = \frac{1}{2\pi i}\int_\gamma \frac{e^{z\omega}}{(1-e^{z\omega})}(B-zI)^{-1}dz + I,$$

where γ is a suitable curve in the complex plane, see [**15**], page 59.

Putting

$$Z = e^{B\omega},$$

then, (5.1) can be written in a compact form as follows

$$v(t) = \sum_{i=1}^{3} v_i(t),$$

where

$$\begin{aligned} v_1(t) &= \Phi_1(Z)\int_t^{t+\omega}(1+2t)(\omega - s(I-Z))^2 e^{B(t-s)}f(s)ds, \\ v_2(t) &= \Phi_2(Z)\int_t^{t+\omega} e^{B(t-s)}f(s)ds, \\ v_3(t) &= \Phi_1(Z)\int_t^{t+\omega} t^2 e^{B(t-s)}f(s)ds. \end{aligned}$$

Here

$$\begin{aligned} \Phi_1(Z) &= \frac{Z}{2(1-Z)^3}, \\ \Phi_2(Z) &= \frac{\omega^2 Z^2}{2(1-Z)^3}. \end{aligned}$$

Thereafter, keeping in mind Assumption (2.10) and Lemma 4.4 we give here some interesting regularity results. The proofs of these results is straightforward and will be omitted.

Lemma 5.1. *Let $f \in L^p([0,+\infty[;E)$ with $1<p<+\infty$. Then for $k \in \{0,1,2\}$ and $i \in \{1,2,3\}$, we get*

(1) $t \mapsto v_i^{(k)}(t) \in L^p(\mathbb{R}^+; E)$.
(2) $t \mapsto A^3 v_i(t) \in L^p(\mathbb{R}^+; E)$.
(3) $t \mapsto A^2 v_i(t) \in L^p(\mathbb{R}^+; E)$.
(4) $t \mapsto A v_i(t) \in L^p(\mathbb{R}^+; E)$.

Now, we are able to give our main results for problem (3.3)

PROPOSITION 5.1. *Problem (3.3) has unique $\omega-$periodic solution*

$$v \in W^{3,p}\left(\mathbb{R}^+; E\right)$$

satisfying the maximal regularity property

$$\frac{d^3 v}{dt^3}, A\frac{d^2 v}{dt^2}, A^2\frac{dv}{dt}, A^3 v \in L^p\left(\mathbb{R}^+; E\right).$$

So, our main result concerning Problem (2.9) is given by the following

THEOREM 5.1. *Let $f \in L^p\left([0,+\infty[; h^{2\sigma}(D)\right)$ with $1 < p < +\infty$, $0 < 2\sigma < 1$ satisfying*

$$f(.,+\infty,.) = 0 \text{ and } f(.,0,\pm 1) = 0.$$

Then, Problem (2.9) has unique $\omega-$periodic solution

$$v \in W^{3,p}\left(\mathbb{R}^+; h^{2\sigma}(D)\right),$$

such that

$$\frac{\partial^3 v}{\partial t^3},\ \Delta\frac{d^2 v}{dt^2},\ \Delta^2\frac{\partial v}{\partial t} \text{ and } \Delta^3 v \in L^p\left(\mathbb{R}^+; E\right).$$

Adapting the same classical perturbation argument used in [**12**], we obtain

THEOREM 5.2. *Let $f \in L^p\left([0,+\infty[; h^{2\sigma}(D)\right)$ with $1 < p < +\infty$, $0 < 2\sigma < 1$ satisfying*

$$f(.,+\infty,.) = 0 \text{ and } f(.,0,\pm 1) = 0.$$

Then, Problem

$$\left(\frac{\partial}{\partial t} + \Delta\right)^3 v(t,\xi,\eta) + \frac{1}{\xi}[Pv](t,\xi,\eta) = f(t,\xi,\eta), \quad (t,\xi,\eta) \in Q$$

$$\begin{array}{ll} v(t+\omega,\xi,\eta) = v(t,\xi,\eta), & (t,\xi,\eta) \in Q, \\ v(t,\xi_0,\eta) = 0, & (t,\eta) \in \mathbb{R}^+ \times]-1,1[, \\ v(t,\xi,-1) = v(t,\xi,1) = 0, & (t,\xi) \in \mathbb{R}^+ \times]\xi_0,+\infty[, \end{array}$$

has unique $\omega-$periodic solution

$$v \in W^{3,p}\left(\mathbb{R}^+; h^{2\sigma}(D)\right),$$

such that

$$\frac{\partial^3 v}{\partial t^3},\ \Delta\frac{d^2 v}{dt^2},\ \Delta^2\frac{\partial v}{\partial t} \text{ and } \Delta^3 v \in L^p\left(\mathbb{R}^+; E\right).$$

Then we can go back to the original problem (1.3) is ensured by the inverse change of variables (2.2) and Lemma 2.1. Taking into account all results obtained in preceding theorem, we are able to justify our main result, that is

THEOREM 5.3. *Let $f \in L^p\left([0,+\infty[;h^{2\sigma}(D)\right)$ with $1<p<+\infty$, $0<2\sigma<1$ satisfying condition (1.5). Then, Problem (1.3) -(1.7) has a unique $\omega-$periodic solution*

$$u \in W^{3,p}\left(\mathbb{R}^+; h^{2\sigma}_{\alpha,\sigma}(\Omega)\right),$$

such that

$$\frac{\partial^3 u}{\partial t^3},\ \Delta\frac{\partial^2 u}{\partial t^2},\ \Delta^2\frac{\partial u}{\partial t} \text{ and } \Delta^3 u \in L^p\left(\mathbb{R}^+; h^{2\sigma}_{\alpha,\sigma}(\Omega)\right).$$

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