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$F-$INDEX AND HYPER-ZAGREB INDEX OF k^{th}GENERATED TRANSFORMATION GRAPHS

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Abstract. Transformation graphs plays important role in the field of chemical graph theory. In this paper, we consider the k^{th} generalized transformation graphs G_k^{ab} and their complements and obtain expressions for F-index, hyper-Zagreb index and their coindices.

1. Introduction

Chemical compounds are often modeled as graphs (molecular graphs). Vertices on the molecular graph represent an atom and the edges between the vertices represent a covalent bond between the corresponding atoms. This kind of representation of chemical compounds is useful for the study of QSPR/QSAR by using molecular descriptors like topological indices. A topological index is a graph invariant which maps each molecular graph to a real number.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex u in $V(G)$ denoted by $deg_G(u)$, is the number of edges which are incident with u. The complement $\overline{G}$ of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

The first and second Zagreb indices introduced by Gutman and Trinajstic are two of the most important topological graph indices. They are denoted by $M_1(G)$ and $M_2(G)$ and were defined as [**6**],

$$M_1(G) = \sum_{uv\in E(G)} [deg_G(u) + deg_G(v)] = \sum_{u\in V(G)} deg_G^2(u)$$

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$$M_2(G) = \sum_{uv\in E(G)} deg_G(u)deg_G(v).$$

One of the well known topological index is the "forgotten topological index (F-index)", denoted by $F(G)$ and is defined as:

$$F(G) = \sum_{uv\in E(G)} [deg_G^2(u) + deg_G^2(v)] = \sum_{u\in V(G)} deg_G^3(u).$$

Though, the F-index is introduced with the first Zagreb index in [**6**], it gets rebirth by B. Furtula and I. Gutman in [**5**].

Another degree based topological index is the hyper-Zagreb index which is introduced by Shirdel et al. [**12**], and is defined as:

$$HM(G) = \sum_{uv\in E(G)} \big(deg_G(u) + deg_G(v)\big)^2.$$

All of the above indices sum over the pairs of adjacent vertices of a graph. T. Došlić [**4**] takes into account the contribution of pairs of non-adjacent vertices when computing vertex-weighted Wiener polynomials of certain composite graphs and called such contributions Zagreb coindices.

The first and second Zagreb coindex is defined as:

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} [deg_G(u) + deg_G(v)].$$

and

$$\overline{M}_2(G) = \sum_{uv\notin E(G)} deg_G(u)deg_G(v)$$

respectively.

In similar way the F-coindex[**8**] and the hyper-Zagreb coindex[**3, 9**] are defined as:

$$\overline{F}(G) = \sum_{uv\notin E(G)} [deg_G^2(u) + deg_G^2(v)] \text{ and}$$

$$\overline{HM}(G) = \sum_{uv\notin E(G)} \big(deg_G(u) + deg_G(v)\big)^2,$$

respectively.

Proposition 1.1 ([**3**]). *Let G be a graph of oreder n and size m. Then,*

$$F(\overline{G}) = n(n-1)^3 - 6(n-1)^2m - F(G) + 3(n-1)M_1(G).$$

Proposition 1.2 ([**2**]). *Let G be a simple graph. Then*

$$HM(G) = F(G) + 2M_2(G).$$

Proposition 1.3 ([**3**]). *Let G be a graph of order n and size m. Then,*

(1) $HM(\overline{G}) = F(\overline{G}) + 2M_2(\overline{G})$.

(2) $\overline{HM}(G) = 2\overline{M_2}(G) + (n-1)M_1(G) - F(G)$.

Proposition 1.4 ([**8**]). *Let G be a graph of order n and size m. Then,*

(1) $\overline{F}(G) = (n-1)M_1(G) - F(G)$.

(2) $\overline{F}(\overline{G}) = 2m(n-1)^2 - (n-1)M_1(G) - \overline{F}(G)$.

2. k^{th}-generalized transformation graphs G_k^{ab}

The chemical applications of transformation graphs are explained in [**1**]. In 1973, Sampathkumar and Chikkodimath introduced the new graph valued function, called as semitotal-point graph $T_2(G)$ of a graph G and is defined as follows [**11**]:

Definition 2.1. The semitotal-point graph $T_2(G)$ of a graph G is a graph whose vertex set is $V(T_2(G)) = V(G) \cup E(G)$ and two vertices are adjacent in $T_2(G)$ if and only if
(i) they are adjacent vertices of G or
(ii) one is a vertex of G and other is an edge of G incident with it.

In 2012, S. R. Jog put forward k^{th} semitotal-point graph of G and is defined as follows [**7**]:

Definition 2.2. The k^{th} semitotal-point graph T_2^k of G is the graph obtained by adding k vertices to each edge of G and joining them to the end vertices of the respective edge. Obviously, this is equivalent to adding k triangles to each edge of G.

Later, Basavanagoud et al. [**1**] introduced some new graphical transformations namely *generalized transformation graphs* G^{xy} which generalizes the concept of semitotal -point graph.

Definition 2.3. The *generalized transformation graph* G^{xy} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:
(a) $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.
(b) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

In view of definitions 2.1-2.3, Raju B. et al. [**10**] introduced k^{th}-*Generalized transformation graphs* G_k^{ab} which generalizes the concept of *generalized transformation graph* G^{xy} [**1**].

Let e_i be the edge of G and let $E_1, E_2, ..., E_m$ be the distinct edge set, and each E_i is correspondent to the edge e_i and $|E_i| = k, i = 1, 2...m.$ for some positive integer k.

Definition 2.4. The k^{th}-*generalized transformation graph* G_k^{xy} is a graph whose vertex set is $V(G_k^{xy}) = V(G) \cup (E' = \cup_{i=1}^k E_i)$ and $\alpha, \beta \in V(G_k^{xy})$. The vertices α and β are adjacent in G_k^{xy} if and only if (a) or (b) holds:
(a) $\alpha, \beta \in V(G)$ α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.
(b) $\alpha \in V(G)$ and $\beta \in E_i$, for some $i \in m'$ (where $m' = 1, 2, ..m$), α, e_i are incident in G if $y = +$ and α, e_i are not incident in G if $y = -$.

Since there are four distinct 2-permutations of $\{+-\}$ we can obtain four-graphical transformations of G as G_k^{++}, G_k^{+-}, G_k^{-+} and G_k^{--}.

In this paper, we consider the $k^{th}-$ generalized transformation graphs G_k^{xy} and obtain expressions for F-index, hyper-Zagreb index and their coindices of k^{th} generalized graphs and their complements.

Proposition 2.1. [**10**] *Let G be a graph of order n and size m. Then degrees of the vertices $u \in V(G_k^{xy}) \cap V(G)$ and $e \in V(G_k^{xy}) \cap V_E$ in G_k^{xy}, are;*

(1) $deg_{G_k^{++}}(u) = deg_G(u)(k+1)$ *and* $deg_{G_k^{++}}(e_i) = 2$.

(2) $deg_{G_k^{+-}}(u) = km - (k-1)deg_G(u)$ *and* $d_{G_k^{+-}}(e_i) = n-2$.

(3) $deg_{G_k^{-+}}(u) = (n-1) + (k-1)deg_G(u)$ *and* $deg_{G_k^{-+}}(e_i) = 2$.

(4) $deg_{G_k^{--}}(u) = (n+km-1) - (k+1)deg_G(u)$ *and* $deg_{G_k^{--}}(e_i) = n-2$.

Proposition 2.2. [**10**] *Let G be a graph of order n and size m. Then the order of the graphs $G_k^{xy}, x, y \in \{-,+\}$, is $(n+km)$ and their sizes are;*

(1) $|E(G_k^{++})| = (2k+1)m$.

(2) $|E(G_k^{+-})| = (k(n-2)+1)m$.

(3) $|E(G_k^{-+})| = \binom{n}{2} + (2k-1)m$.

(4) $|E(G_k^{--})| = \binom{n}{2} + (k(n-2)-1)m$.

3. Results

Lemma 3.1 ([**10**]). *Let G be a graph with n vertices and m edges. Then,*

$$M_1(G_k^{++}) = (k+1)^2M_1(G) + 4km$$
$$M_1(G_k^{+-}) = km\big(kmn - 4m(k-1) + (n-2)^2\big) + (k-1)^2M_1(G)$$
$$M_1(G_k^{-+}) = n(n-1)^2 + 4m\big(k - (k-1)(n-1)\big) + (k-1)^2M_1(G)$$
$$M_1(G_k^{--}) = n(n+km-1)^2 - 4m(k+1)(n+km-1) + (k+1)^2M_1(G) + km(n-2)^2.$$

Theorem 3.1. *Let G be a graph with n vertices and m edges. Then,*

(1) $F(G_k^{++}) = (k+1)^3F(G) + 8km$.

(2) $F(G_k^{+-}) = km(nk^2m^2 + (n-2)^3)$.

(3) $F(G_k^{-+}) = n(n-1)^3 + 6m(k-1)(n-1)^2 + 3(k-1)^2(n-1)M_1(G) + (k-1)^3F(G) + 8km$.

(4) $F(G_k^{--}) = n(n+km-1)^3 - 6m(k+1)(n+km-1)^2 + 3(k+1)^2(n+km-1)M_1(G) - (k+1)^3F(G) + km(n-2)^3$.

Proof. Using the definition of F-index, Proposition 2.1 and Lemma 3.1 we have,

(1)

$$\begin{aligned} F(G_k^{++}) &= \sum_{\alpha \in V(G_k^{++})} \big(deg_{G_k^{++}}(\alpha)\big)^3 = \sum_{u \in V(G)} \big(deg_{G_k^{++}}(u)\big)^3 + \sum_{e \in V_E} \big(deg_{G_k^{++}}(e)\big)^3 \\ &= \sum_{u \in V(G)} \big((k+1)deg_G(u)\big)^3 + \sum_{e \in V_E} \big(2\big)^3 \end{aligned}$$

$$F(G_k^{++}) = (k+1)^3 F(G) + 8km.$$

(2)

$$\begin{aligned}
F(G_k^{+-}) &= \sum_{\alpha \in V(G_k^{+-})} \left(deg_{G_k^{+-}}(\alpha)\right)^3 = \sum_{u\in V(G)} \left(deg_{G_k^{+-}}(u)\right)^3 + \sum_{e\in V_E} \left(deg_{G_k^{+-}}(e)\right)^3 \\
&= \sum_{u\in V(G)} \left(deg_G(u) + k(m - deg_G(u)\right)^3 + \sum_{e\in V_E} \left(n-2\right)^3 \\
&= \sum_{u\in V(G)} \left[deg_G^3(u) + 3deg_G^2(u)k(m - deg_G(u))\right. \\
&\quad \left. +3deg_G(u)(k(m-deg_G(u))^2 + (k(m-deg_G(u)))^3\right] + km(n-2)^3 \\
&= \sum_{u\in V(G)} \left[deg_G^3(u) + 3kmdeg_G^2(u) - 3deg_G^3(u)\right. \\
&\quad +3deg_G(u)(k^2m^2 - 2kmdeg_G^(u) \\
&\quad \left. +deg_G^2(u)) + k^3m^3 - 3k^2m^2deg_G(u) + 3kmdeg_G^2(u) - deg_G^3(u)\right] \\
&\quad +km(n-2)^3 \\
F(G_k^{+-}) &= km(nk^2m^2 + (n-2)^3).
\end{aligned}$$

(3)

$$\begin{aligned}
F(G_k^{-+}) &= \sum_{\alpha \in V(G_k^{-+})} \left(deg_{G_k^{-+}}(\alpha)\right)^3 = \sum_{u\in V(G)} \left(deg_{G_k^{-+}}(u)\right)^3 + \sum_{e\in V_E} \left(deg_{G_k^{-+}}(e)\right)^3 \\
&= \sum_{u\in V(G)} \left((n-1) + (k-1)deg_G(u)\right)^3 + \sum_{e\in V_E} \left(2\right)^3 \\
&= \sum_{u\in V(G)} \left[(n-1)^3 + 3(n-1)^2(k-1)deg_G(u)\right. \\
&\quad \left. +3(n-1)(k-1)^2deg_G^2(u) + (k-1)^3deg_G^3(u)\right] + 8km \\
F(G_k^{-+}) &= n(n-1)^3 + 6m(k-1)(n-1)^2 + 3(k-1)^2(n-1)M_1(G) \\
&\quad +(k-1)^3F(G) + 8km.
\end{aligned}$$

(4)

$$\begin{aligned}
F(G_k^{--}) &= \sum_{\alpha \in V(G_k^{--})} \left(deg_{G_k^{--}}(\alpha)\right)^3 = \sum_{u\in V(G)} \left(deg_{G_k^{--}}(u)\right)^3 \\
&+ \sum_{e\in V_E} \left(deg_{G_k^{--}}(e)\right)^3 \\
&= \sum_{u\in V(G)} \left((n+km-1) - (k+1)deg_G(u))\right)^3 + \sum_{e\in V_E} \left(n-2\right)^3 \\
&= \sum_{u\in V(G)} \left[(n+km-1)^3 - 3(n+km-1)^2(k+1)deg_G(u)\right. \\
&\quad \left. +3(n+km-1)(k+1)^2deg_G^2(u) - (k+1)^3deg_G^3(u)\right]
\end{aligned}$$

$$
\begin{aligned}
& +km(n-2)^3 \\
F(G_k^{--}) &= n(n+km-1)^3 - 6m(k+1)(n+km-1)^2 \\
& +3(k+1)^2(n+km-1)M_1(G) - (k+1)^3F(G) + km(n-2)^3.
\end{aligned}
$$

□

Lemma 3.2 ([**10**]). *Let G be a graph with n vertices and m edges. Then,*

$$
\begin{aligned}
M_2(G_k^{++}) &= (k+1)^2M_2(G) + 2k(k+1)M_1(G). \\
M_2(G_k^{+-}) &= k^2m^3 + (n-2)\big(k^2m^2n - 2m(km+k-1)\big) \\
& \quad + k(k-1)(n-m-2)M_1(G) + (k-1)^2M_2(G). \\
M_2(G_k^{-+}) &= \frac{(n-1)}{2}\big(n(n-1)^2 - 2m(n-1) + 8km\big) \\
& \quad + (k-1)\big(2kM_1(G) - (n-1)\overline{M}_1(G) + (k-1)\overline{M}_2(G)\big). \\
M_2(G_k^{--}) &= \left(\binom{n}{2} - m\right)(n+km-1)^2 + \big(km(n-2)^2 \\
& \quad - (k+1)\overline{M}_1(G)\big)(n+km-1) \\
& \quad - k(k+1)(n-2)\big(2m^2 - M_1(G)\big) + (k+1)^2\overline{M}_2(G).
\end{aligned}
$$

Substituting expressions in Theorem 3.1 and 3.2 into the equation in Proposition 1.2, then the following results hold.

Theorem 3.2. *Let G be a simple graph with n vertices and m edges. Then,*

$$
\begin{aligned}
HM(G_k^{++}) &= (k+1)^3F(G) + 4k(k+1)M_1(G) + 2(k+1)^2M_2(G) + 8km. \\
HM(G_k^{+-}) &= k^2m^3(kn+2) + km(n-2)^2(n+2km-2) - 4k(k-1)m(n-2) \\
& +2k(k-1)(n-m-2)M_1(G) + 2k(k-1)^2mM_2(G). \\
HM(G_k^{-+}) &= 2n(n-1)^3 + m(3k-5)(n-1)^2 + 8kmn + (k-1)^3F(G) \\
& +(k-1)(n-1)\big[3(k-1)M_1(G) + 2M_1(\overline{G})\big] + 2(k-1)^2M_2(\overline{G}). \\
HM(G_k^{--}) &= 2(k+1)^2M_2(G) - (k+1)^3F(G) + km(n-2)^3 \\
& -2k(k+1)(n-2)(2m^2 - M_1(G)) \\
& +\big[2km(n-2)^2 - 2(k+1)M_1(\overline{G}) \\
& +3(k+1)^2M_1(G)\big](n+km-1) \\
& +\big[n^2 - n - (3k-5)m\big](n+km-1)^2 + n(n+km-1)^3.
\end{aligned}
$$

The F-index of the complement of $kth-$generalized transformation graphs graphs are given as follows by applying Proposition 2.2, Lemma 3.1 and Theorem 3.1 onto Proposition 1.1.

Theorem 3.3. *Let G be a graph of order n and size m. Then,*

$$
\begin{aligned}
F(\overline{G_k^{++}}) &= (n+km)(n+km-1)^3 - \big((k+1)^3F(G) - 8km\big) \\
& -6(n+km-1)^2(m(1+2k)) + 3(n+km-1)\big((k+1)^2M_1(G) + 4km\big). \\
F(\overline{G_k^{+-}}) &= (n+km)(n+km-1)^3 - \big(km(nk^2m^2 + (n-2)^3)\big)
\end{aligned}
$$

$$\begin{aligned}
&-6(n+km-1)^2(m(kn-1))+3(n+km-1)\big(mk(nmk+4m-4mk)\\
&\quad+(k-1)^2M_1(G)+km(n-2)^2\big).\\
F(\overline{G_k^{-+}}) &= (n+km)(n+km-1)^3-\big(n(n-1)^3+6m(k-1)(n-1)^2\\
&\quad+3(k-1)^2(n-1)M_1(G)+(k-1)^3F(G)+8km\big)\\
&\quad-6(n+km-1)^2\Big(\binom{n}{2}+(2k-1)m\Big)+3(n+km-1)\\
&\quad\big(4m(nk-n-k+1)(k+1)^2M_1(G)+n(n-1)^2+4km\big).\\
F(\overline{G_k^{--}}) &= (k+1)^3F(G)-km(n-2)^3+3km(n-2)^2(n+km-1)\\
&\quad+3(-n^2+n-2knm+2km)\\
&\quad(n+km-1)^2+(3n+km)(n+km-1)^3.
\end{aligned}$$

LEMMA 3.3. [**10**] *Let G be a graph with n vertices and m edges. Then,*

$$\begin{aligned}
M_2(\overline{G_k^{++}}) &= \Big(\binom{n+km}{2}-3m(2k+1)\Big)(n+km-1)^2\\
&\quad+\big((k+1)^2M_1(G)+4km\big)\\
&\quad(n+km-1)+2m\big(((2k+1)^2+2k^2)m+2kn-3k\big)-\frac{(k+1)}{2}\\
&\quad(5k+1)M_1(G)-(k+1)^2M_2(G).\\
M_2(\overline{G_k^{+-}}) &= \frac{1}{2}\Big[n^4+k^4m^4-3n^3+k^2(9k-8)m^3+4k(3k-2)nm^2\\
&\quad+2(2k-3)n^2m\\
&\quad+2(-4k^2+2k+6)mn+3n^2-(8k^3-k^2+4)m^2\\
&\quad+(8k^2-9k-6)m-n\Big]\\
&\quad+k(k-1)(n-m-2)M_1(G)\\
&\quad-(k-1)^2\big(\frac{1}{2}(2n+2km-3)M_1(G)+M_2(G)\big).\\
M_2(\overline{G_k^{-+}}) &= \frac{1}{2}\Big[4k^3nm^3+3(knm)^2-k(22k^2-4)nm^2\\
&\quad-(3k-4)n^2m+(8-2k)nm\\
&\quad-k^2(15k-6)m^3+(km)^4+(43k^2-20k+4)m^2-(5k+4)m\Big]\\
&\quad-(k-1)\Big[\frac{1}{2}\big((k-1)(n+km-3)-4k\big)M_1(G)\\
&\quad+(n-1)\overline{M}_1(G)+(k-1)\overline{M}_2(G)\Big].\\
M_2(\overline{G_k^{--}}) &= \frac{m}{2}\Big[(7k^2+12k+4)m-4(k+1)\big(knm+n^2-2n\big)-(5k+4)\\
&\quad+k^3m^2(1+km)\Big]+(k+1)\Big[\big((n+k(k+1)m-1)\\
&\quad+\frac{(k-3)}{2}\big)M_1(G)(n+km-1)\overline{M}_1(G)-(k+1)\overline{M}_2(G)\Big].
\end{aligned}$$

The following expressions of the hyper-Zagreb index of the complement of $k^{th}-$generalized transformation graphs are found by applying the expressions of Theorem 3.3 and Lemma 3.3 onto Proposition 1.3(1).

THEOREM 3.4. *Let G be a simple graph with n vertices and m edges. Then,*

$$\begin{aligned}
HM(\overline{G_k^{++}}) &= 2(n+km)(n+km-1)^3 - 12m(2k+1)(n+km-1)^2 \\
&\quad +3\big((k+1)^2M_1(G)+4km\big)(n+km-1) - (k+1)^2F(G) \\
&\quad +4\big(2km+m((2k+1)^2+2k^2)m+2kn-3k\big) \\
&\quad +2(k+1)\big((k+1)(n+km-\frac{3}{2})-2k\big)M_1(G) - (k+1)^2M_2(G). \\
HM(\overline{G_k^{+-}}) &= (n+km)(n+km-1)^3 - 6m(kn-1)(n+km-1)^2 \\
&\quad +3\big[mk(nmk+4m-4mk)+(k-1)^2M_1(G)+km(n-2)^2\big] \\
&\quad (n+km-1)+n^4+k^4m^4-3n^3+(9k^3-8k^2-n)m^3 \\
&\quad +kn(12k-8)m^2+4kmn(m^2-(2k-1))+3n^2 \\
&\quad -(8k^3+k^2-4)m^2+(8k^2-9k+2)m-n(mn^2+1) \\
&\quad +[(k-1)(2k^2m-2n)+(k+1)^2-4]M_1(G) - 2(k-1)^2M_2(G). \\
HM(\overline{G_k^{-+}}) &= (n+km)(n+km-1)^3 - \big(n(n-1)^3+6m(k-1)(n-1)^2 \\
&\quad +3(k-1)^2(n-1)M_1(G)+(k-1)^3F(G)\big) \\
&\quad -6(n+km-1)^2\Big(\binom{n}{2}+(2k-1)m\Big)+3(n+km-1) \\
&\quad \big(4m(nk-n-k+1)(k+1)^2M_1(G)+n(n-1)^2+4km\big) \\
&\quad +\big[(8-2k)n+(3k-4)n^2-(13k+4)\big]m+\big[4k^3nm \\
&\quad +3k^2n^2+(4k-22)n-(6-15k^2)m+k^2m^2 \\
&\quad +(43k^2-20k+4)\big]m^2 \\
&\quad +\big[(2(n+km-1)-5)(k-1)^2-4(k-1)\big]M_1(G) \\
&\quad -2(k-1)(n-1)M_1(\overline{G}) - 2(k-1)^2M_2(\overline{G}). \\
HM(\overline{G_k^{--}}) &= [2(3n+km)](n+km-1)^3 - \big[n(-8n+7) \\
&\quad +(k(10-12n)+18)m\big] \\
&\quad (n+km-1)^2+\big[3km(n-2)^2+2((n+k-3)(-n+k+1) \\
&\quad +(k+1)((k+1)M_1(G)-M_1(\overline{G})+2m))\big](n+km-1)+ \\
&\quad \Big[-(k+1)(2k(n-2)+(k+1))M_1(G)+2(k+1)^2M_2(\overline{G}) \\
&\quad +4\Big(\binom{n}{2}+(k(n-2)-1)m\Big)^2 \\
&\quad +2(n-2)\big(2k(k+1)m^2-\frac{1}{2}km(n-2)\big)\Big].
\end{aligned}$$

Though some coindex of a graph and the index of the complement of the same graph seems similar, but they are not identical. The following results show that this statement is true for F-index and hyper-Zagreb index.

By applying Proposition 2.2, Lemma 3.1 and Theorem 3.1 onto proposition 1.4(1), the following results follow.

THEOREM 3.5. *Let G be a simple graph with n vertices and m edges. Then,*

$$\overline{F}(G_k^{++}) = (k+1)^2\big((n+km-1)M_1(G) - (k+1)F(G)\big) + 4km(n-km-1).$$

$$\overline{F}(G_k^{+-}) = km\big[(kmn - 4m(k-1) + (n-2)^2)(n+km-1) - (nk^2m^2 + (n-2)^3)\big] + (k-1)^2(n+km-1)M_1(G).$$

$$\overline{F}(G_k^{-+}) = km\big(n(n-1)^2 + 4(n+km-3)\big) - 2m(k-1)(n-1)(5n+km-5) - (k-1)^2(2n-km-2)M_1(G) - (k-1)^3F(G).$$

$$\overline{F}(G_k^{--}) = 2(k+1)(n+km-1)\big(m(n+km-1) - (k+1)M_1(G)\big) + km(km-1)(n-2)^2 + (k+1)^3F(G).$$

LEMMA 3.4 ([**10**]). *Let G be a graph with n vertices and m edges. Then,*

$$\overline{M}_2(G_k^{++}) = 2m\big(m(2k+1)^2 - k\big) - \big(\frac{(k+1)^2}{2} + k(k+1)\big)M_1(G) - (k+1)^2M_2(G).$$

$$\overline{M}_2(G_k^{+-}) = 2m^2(k(n-2)+1)^2 - k^2m^3 - (n-2)\big(k^2m^2n - 2m(km+k-1)\big) - \frac{1}{2}\big(km(kmn-4m(k-1)+(n-2)^2)\big) - \big(\frac{(k-1)^2}{2} + k(k-1)(n-m-2)\big)M_1(G) - (k-1)^2M_2(G).$$

$$\overline{M}_2(G_k^{-+}) = 2\Big(\binom{n}{2} + (2k-1)m\Big)^2 - (n-1)\Big(n\binom{n}{2} + (2k+3)m - nm\Big) - 2km - (k-1)\Big[\frac{1}{2}(k+3)M_1(G) + (n-1)\overline{M}_1(G) + (k-1)\overline{M}_2(G)\Big].$$

$$\overline{M}_2(G_k^{--}) = \frac{1}{2}\Big[(kmn)^2 + (2(k+1)m)^2 - (2km)^2n - (k+2)mn^2 + 4nm - 2m + 2k^2m^3\Big] - (k+1)\Big[\frac{1}{2}(2k(n-1)-k+1)M_1(G) - (n+km-1)\overline{M}_1(G) + (k+1)\overline{M}_2(G)\Big].$$

Using the results of Proposition 2.2, Lemma 3.1, 3.4 and theorem 3.1 into Proposition 1.3(2), we get the following expressions.

THEOREM 3.6. *Let G be a simple graph with n vertices and m edges. Then,*

$$\begin{aligned}
\overline{HM}(G_k^{++}) &= 4m\big[kn + \big((2k+1)^2 + k^2\big)m - 4k\big] + (k+1)\Big[\big((k+1)(n+km-2)\\
&\quad - 4k\big)M_1(G) - (k+1)\big(2M_2(G) + (k+1)F(G)\big)\Big].\\
\overline{HM}(G_k^{+-}) &= km\big[km(2n^2 - 2m - 4 - n) - (n-2)^2 - k^2m^2n - (n-2)^3\big]\\
&\quad + km(n+km-1)\big[kmn - 4m(k-1) + (n-2)^2\big]\\
&\quad + (k-1)\big[4m\big((6k-1)m + 3mn - n - 2\big) + \big((k-1)(n+km-2)\\
&\quad - 2k(n-m-2)\big)M_1(G) + (k-1)M_2(G)\big].\\
\overline{HM}(G_k^{-+}) &= 4\Big(\binom{n}{2} + (2k-1)m\Big)^2 - 2(n-1)\Big(n\binom{n}{2} + (2k+3)m - nm\Big)\\
&\quad - 12km + (n+km-1)\big(n(n-1)^2 + 4km\big) - n(n-1)^3\\
&\quad - (k-1)\Big[(k-1)\big[(k-1)F(G) - (2n+km+1)M_1(G) + 2\overline{M}_2(G)\\
&\quad \big] + m(n-1)(-10n + km + 10) + 2(n-1)\overline{M}_1(G)\Big].\\
\overline{HM}(G_k^{--}) &= (k+1)\big[(k+1)^2F(G) - 2(k+1)\overline{M}_2(G) -\\
&\quad (2k(n-1) - k + 1)M_1(G)\big]\\
&\quad + (kmn)^2 + \big(2(k+1)m\big)^2 - (2km)^2n - (k+2)mn^2 + 4nm - 2m +\\
&\quad 2k^2m^3 - km(n-2)^3 + \big[2(k+1)\overline{M}_1(G) - 2(k+1)^2M_1(G)\\
&\quad + km(n-2)^2\big](n+km-1) + 2m(k+1)(n+km-1)^2.
\end{aligned}$$

By substituting results in Proposition 2.2, Lemma 3.1 and Theorem 3.1 in Proposition 1.4(2) we get the following expressions.

THEOREM 3.7. *Let G be a simple graph with n vertices and m edges. Then,*

$$\begin{aligned}
\overline{F}(\overline{G_k^{++}}) &= 2m(2k+1)(n+km-1)^2 - 2\big[(k+1)^2M_1(G) + 4km\big](n+km-1)\\
&\quad + 8km + (k+1)^3F(G).
\end{aligned}$$

$$\begin{aligned}
\overline{F}(\overline{G_k^{+-}}) &= 2m(k(n-2)+1)(n+km-1)^2 - 2\big[km(kmn - 4m(k-1) + (n-2)^2)\\
&\quad + (k-1)^2M_1(G)\big] + km(nk^2m^2 + (n-2)^3).\\
\overline{F}(\overline{G_k^{-+}}) &= 2m(2k-1)(n+km-1)^2 + k^2m^2n(n-1) + 2m(k-1)(n-1)\\
&\quad (7n + 4km - 1) - 8km(n+km-2) + (k-1)^2(n - 2km - 1)M_1(G)\\
&\quad + (k-1)^3F(G).\\
\overline{F}(\overline{G_k^{--}}) &= -(n+km-1)^3 + (n-1)(km+1)(n+km-1)^2 + [(k+1)^2M_1(G)\\
&\quad - 2km(n-2)^2](n+km-1) + km(n-2)^3 - (k+1)^3F(G).
\end{aligned}$$

LEMMA 3.5 ([**10**]). *Let G be a graph with n vertices and m edges. Then,*

$$M_1(\overline{G_k^{++}}) = (n+km)(n+km-1)^2 - 4m(2k+1)(n+km-1) + 4km + (k+1)^2 M_1(G).$$

$$M_1(\overline{G_k^{+-}}) = (n+km)(n+km-1)^2 - 4m(k(n-2)+1)(n+km-1) + km\big(kmn - 4m(k-1) + (n-2)^2\big) + (k-1)^2 M_1(G).$$

$$M_1(\overline{G_k^{-+}}) = (n+km)(n+km-1)^2 - 4\Big(\binom{n}{2} + (2k-1)m\Big)(n+km-1) + n(n-1)^2 + 4m\big(k-(k-1)(n-1)\big) + (k-1)^2 M_1(G).$$

$$M_1(\overline{G^{--}}_k) = km(km+1)^2 + (k+1)M_1(G).$$

LEMMA 3.6 ([**10**]). *Let G be a graph with n vertices and m edges. Then,*

$$\overline{M}_2(\overline{G_k^{++}}) = (k+1)^2 M_2(G) - \frac{(k+1)}{2}\big((k+1)(2n+2km-1) - (5k+1)\big)M_1(G) + m\big[k^2(2k+1)m^2 - 2k(4k+1)m + (2k+1)n^2 + 2k(2k+1)nm - 2(4k+1)n + (6k+1)\big] - 4km(n+km-1).$$

$$\overline{M}_2(\overline{G_k^{+-}}) = m\big[k^2n^2m + (k+1)n^2 - k(7k-4)mn + 2k(2k^2+k-2)m + (4k^2-7k-2)n - (4k^2-6k-1) - 2k^2(k-1)m^2\big] - k(k-1)(n-m-2)M_1(G) + (k-1)^2\big((n+km-2)M_1(G) - M_2(G)\big).$$

$$\overline{M}_2(\overline{G_k^{-+}}) = 2\Big(\binom{n+km}{2} - \Big(\binom{n}{2} + (2k-1)m\Big)\Big)^2 - \frac{1}{2}\big[4k^3nm^3 + 3n^2m^2 - 5k(7k-4)nm^2 - 2k^2(7k-3)m^3 + k^4m^4 + (49k^2-32k+4)m^2 - 6(k-1)(n-1)^2m\big] + \frac{1}{2}\big((k-1)^2(n+km-4) - 4k(k-1)\big)M_1(G) + (k-1)(n-1)\overline{M}_1(G) + (k-1)^2\overline{M}_2(G).$$

$$\overline{M}_2(\overline{G_k^{--}}) = (k+1)\Big[2(k^2m^3 + knm^2 + n^2m - 2nm + m) - \frac{1}{2}\big(2n + (k+1)(2km+k+1) + k - 5\big)M_1(G) - (n+km-1)\overline{M}_1(G) + (k+1)\overline{M}_2(G)\Big].$$

By applying expressions in Proposition 2.2, Lemma 3.5,3.6 and Theorem 3.3 into Proposition1.3(2), we get the following expressions.

THEOREM 3.8. *Let G be a simple graph with n vertices and m edges. Then,*

$$\overline{HM}(\overline{G_k^{++}}) = 2(k+1)^2 M_2(G) - 2(k+1)\big((k+1)(n+km-2) - (k-1)\big)M_1(G) + (k+1)^3 F(G) + 8km + 2m\big[k^2(2k+1)m^2 - 2k(4k+1)m + (2k+1)n^2 + 2k(2k+1)nm - 2(4k+1)n + (6k+1)\big].$$

$$\begin{aligned}\overline{HM}(\overline{G_k^{+-}}) = {} & 2m\big[k^2n^2m + (k+1)n^2 - k(7k-4)mn + 2k(2k^2+k-2)m \\ & + (4k^2-7k-2)n \\ & - (4k^2-6k-1) - 2k^2(k-1)m^2\big] + km\big(nk^2m^2 + (n-2)^3\big) \\ & + (n+km-1) \\ & \big(-2km(kmn+(n-2)^2)\big) + 2m(k(n-2)+1)(n+km-1)^2 \\ & - 2k(k-1)(n-m-2)M_1(G) - 2(k-1)^2[M_1(G)+M_2(G)].\end{aligned}$$

$$\begin{aligned}\overline{HM}(\overline{G_k^{-+}}) = {} & 4\Big(\binom{n}{2} + (2k-1)m\Big)^2 + n(n-1)^3 + 6m(k-1)(n-1)^2 + 8km \\ & - 4k^3nm^3 - 3n^2m^2 + 5k(7k-4)nm^2 + 2k^2(7k-3)m^3 - k^4m^4 \\ & - (49k^2-32k+4)m^2 + 6(k-1)(n-1)^2m - 2\big[n(n-1)(n-2) \\ & - 8km - 2m((k-1)(n-1)-1)\big](n+km-1) + \big[(n+km)^2 \\ & - 2\Big(\binom{n}{2} \\ & + (2k-1)m\Big)\big](n+km-1)^2 + (k-1)\big[\big(2(n+km-1)((k-1) \\ & - 6m(n-1)(k+1)^2) + 3(k-1)(n-2) - 4k\big)M_1(G) \\ & + 2(n-1)\overline{M}_1(G) \\ & + 2(k-1)\overline{M}_2(G) + (k-1)^2F(G)\big].\end{aligned}$$

$$\begin{aligned}\overline{HM}(\overline{G_k^{--}}) = {} & (k+1)^2[2\overline{M}_2(G) - (k+1)F(G)] - (k+1)\big(\big(2n \\ & + (k+1)(2km+k+1) \\ & + k - 5\big)\big)M_1(G) + 4(k+1)m[k^2m^2 + knm + n^2 - 2n + 1] \\ & + km(n-2)^3 + \big[(k+1)^2M_1(G) + km(km+1)^2 \\ & - 3km(n-2)^2\big](n+km-1) \\ & - 3[n - n^2 - 2kmn + 2km](n+km-1)^2 \\ & - (3n+km)(n+km-1)^3.\end{aligned}$$

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