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ON SOME MATHEMATICAL PROPERTIES OF SOMBOR INDICES

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Abstract. Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple graph with n vertices, m edges and the vertex degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$, $d_i = d(v_i)$. Recently, a family of three Sombor indices was introduced. They are named the Sombor index, the reduced Sombor index and the average Sombor index and defined, respectively, as

$$SO(G) = \sum_{i\sim j} \sqrt{d_i^2 + d_j^2},\ SO_{red}(G) = \sum_{i\sim j} \sqrt{(d_i-1)^2 + (d_j-1)^2}$$

and

$$SO_{avr}(G) = \sum_{i\sim j} \sqrt{\left(d_i - \frac{2m}{n}\right)^2 + \left(d_j - \frac{2m}{n}\right)^2},$$

where $i \sim j$ denotes the adjacency of vertices v_i and v_j in G. In this paper some new bounds of these indices and their relationship with other degree–based indices are determined. Two inequalities of the Nordhaus–Gaddum type for the $SO(G)$ are proved.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple graph of order n and size m with vertex degree sequence $\Delta = d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n = \delta$, $d_i = d(v_i)$. Denote by $\overline{G}$ a complement of G. If vertices v_i and v_j are adjacent in G, we write $i \sim j$, whereas if v_i and v_j are adjacent i $\overline{G}$, we write $i \nsim j$. By $L(G)$ we denote a line graph associated to graph G, and by $\Delta_e = \max_{i\sim j}\{d_i + d_j\}$ and $\delta_e = \min_{i\sim j}\{d_i + d_j\}$.

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The first and the second Zagreb indices are vertex-degree-based graph invariants introduced in [**12**] and [**13**], respectively, and defined as

$$M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2(G) = \sum_{i\sim j} d_i d_j .$$

Both $M_1(G)$ and $M_2(G)$ were recognized to be a measure of the extent of branching of the carbon–atom skeleton of the underlying molecule.

In [**7**] (see also [**26**]) it was observed that the first Zagreb index can be also represented as

$$M_1(G) = \sum_{i\sim j} (d_i + d_j) .$$

In [**12**], another quantity, the sum of cubes of vertex degrees

$$F(G) = \sum_{i=1}^{n} d_i^3 ,$$

was encountered as well. This quantity is also a measure of branching and it was found that its predictive ability is quite similar to that of $M_1(G)$. However, for the unknown reasons, it did not attracted any attention until 2015 when it was reinvented in [**11**] and named the forgotten topological index. It can be also represented as

$$F(G) = \sum_{i\sim j} (d_i^2 + d_j^2) .$$

More on the mathematical properties of these indices one can found in [**2**–**4**, **11**, **14**, **22**, **25**, **26**] and the references cited therein.

The inverse degree index was introduced in [**10**]. It is defined to be

$$ID(G) = \sum_{i=1}^{n} \frac{1}{d_i} = \sum_{i\sim j} \left(\frac{1}{d_i^2} + \frac{1}{d_j^2} \right) .$$

The reverse Randić index is defined as [**15**, **21**]

$$RR(G) = \sum_{i\sim j} \sqrt{d_i d_j} .$$

A family of 148 discrete Adriatic indices was introduced and analyzed in [**32**]. An especially interesting subclass of these indices consists of 20 indices which are useful for predicting certain physicochemical properties of chemical compounds. One of them is the symmetric division deg index, $SDD(G)$. It is defined as

$$SDD(G) = \sum_{i\sim j} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right) .$$

In [**1**] Albertson introduced the quantity called the *imbalance* of an edge e as $imb(e) = |d_i - d_j|$, and used it to define the irregularity measure of a graph as

$$Alb(G) = \sum_{i\sim j} |d_i - d_j| ,$$

which is sometimes referred to as *Albertson index* [**17**, **18**] or the *third Zagreb index* [**9**].

Recently, in [**16**] a family of Sombor indices was introduced. The (ordinary) Sombor index is defined as

$$SO(G) = \sum_{i\sim j} \sqrt{d_i^2 + d_j^2}\,.$$

The reduced Sombor index is defined as

$$SO_{red}(G) = \sum_{i\sim j} \sqrt{(d_i-1)^2 + (d_j-1)^2}\,,$$

and the average Sombor index, as

$$SO_{avr}(G) = \sum_{i\sim j} \sqrt{\left(d_i - \frac{2m}{n}\right)^2 + \left(d_j - \frac{2m}{n}\right)^2}\,.$$

In this paper we determine upper and lower bounds on Sombor indices and their relationship with some of the above mentioned indices.

2. Preliminaries

In this section we recall some analytical inequalities for real number sequences that will be used in proofs of theorems.

Let $p = (p_i)$, $i = 1, 2, \dots, n$, be a sequence of non–negative real numbers, and $a = (a_i)$, $i = 1, 2, \dots, n$, a sequence of positive real numbers. In [**20**, **29**] it was proven that for any real r, $r \leqslant 0$ or $r \geqslant 1$, holds

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geqslant \left(\sum_{i=1}^{n} p_i a_i\right)^r . \tag{2.1}$$

For $0 \leqslant r \leqslant 1$ the opposite inequality is valid.

Equality holds if and only if either $r = 0$, or $r = 1$, or $a_1 = \cdots = a_n$, or $p_1 = p_2 = \cdots = p_t = 0$ and $a_{t+1} = \cdots = a_n$, for some t, $1 \leqslant t \leqslant n-1$.

Let $x = (x_i)$, and $a = (a_i)$, $i = 1, 2, \dots, n$ be two positive real number sequences. In [**31**] it was proven that for any $r \geqslant 0$ holds

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geqslant \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r} . \tag{2.2}$$

Equality holds if and only if either $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

3. Main results

THEOREM 3.1. *Let G be a simple graph with n vertices and m edges. Then*

$$\frac{\sqrt{2}}{2} M_1(G) \leqslant SO(G) \leqslant M_1(G)\,. \tag{3.1}$$

Equality in the left–hand side of (3.1) *holds if and only if* $d_i = d_j$ *for any pair of adjacent vertices* v_i *and* v_j *in* G. *Equality on the right–hand side holds if and only if* $G \cong \overline{K}_n$.

Proof. For any two non–negative real numbers a and b the following inequality is valid

$$\sqrt{a+b} \geqslant \frac{\sqrt{2}}{2}\left(\sqrt{a}+\sqrt{b}\right), \tag{3.2}$$

with equality if and only if $a = b$. For $a = d_i^2$ and $b = d_j^2$, the above inequality transforms into

$$\sqrt{d_i^2+d_j^2} \geqslant \frac{\sqrt{2}}{2}(d_i+d_j)\,. \tag{3.3}$$

Now, by the definition of Sombor index,we have that

$$SO(G) = \sum_{i\sim j}\sqrt{d_i^2+d_j^2} \geqslant \frac{\sqrt{2}}{2}\sum_{i\sim j}(d_i+d_j) = \frac{\sqrt{2}}{2}M_1(G)\,.$$

Equality in (3.3), and consequently in the left-hand side of (3.1), holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G.

Also, by the definition of Sombor index, we have that

$$\begin{aligned} SO(G) = \sum_{i\sim j}\sqrt{d_i^2+d_j^2} &= \sum_{i\sim j}\sqrt{(d_i+d_j)^2-2d_id_j} \leqslant \\ &\leqslant \sum_{i\sim j}\sqrt{(d_i+d_j)^2} = \sum_{i\sim j}(d_i+d_j) + M_1(G)\,, \end{aligned} \tag{3.4}$$

Equality in (3.4) holds if and only if $d_id_j = 0$, which implies that equality in the right-hand side of (3.1) holds if and only if $G \cong \overline{K}_n$. □

The relation (3.4) implies that Sombor index is closely related to the First Zagreb index. Therefore it is expected that its applicability and predictive ability will be quite similar to that of $M_1(G)$.

In [**28**] the following inequality was proven

$$M_1(G) \geqslant \frac{4m^2}{n} + \frac{1}{2}(\Delta-\delta)^2\,,$$

with equality holding if and only if $d_2 = \cdots = d_{n-1} = \frac{d_1+d_n}{2}$ [**3**, **4**, **23**]. Having this in mind, we obtain the following corollary of Theorem 3.1.

Corollary 3.1. *Let* G *be a simple connected graph of order* $n \geqslant 2$ *and size* m. *Then*

$$SO(G) \geqslant \frac{\sqrt{2}}{2}\left(\frac{4m^2}{n} + \frac{1}{2}(\Delta-\delta)^2\right) \geqslant \frac{2\sqrt{2}m^2}{n}\,.$$

Equality holds if and only if G *is regular.*

Since

$$M_1(G) = \sum_{i\sim j}(d_i + d_j) \geqslant m\delta_e \geqslant 2m\delta\,,$$

we have the following corollary of Theorem 3.1.

Corollary 3.2. *Let G be a simple connected graph of order $n \geqslant 2$ and size m. Then*

$$SO(G) \geqslant \frac{\sqrt{2}}{2} m\delta_e \geqslant \sqrt{2} m\delta\,.$$

Equality holds if and only if G is regular.

When G is unicyclic graph, $G \cong U$, we have the following corollary of Theorem 3.1.

Corollary 3.3. *Let U be a unicyclic graph with n vertices. Then*

$$SO(U) \geqslant 2\sqrt{2}n + \frac{\sqrt{2}}{4}(\Delta - \delta)^2 \geqslant 2\sqrt{2}n\,,$$

and

$$SO(U) \geqslant \frac{\sqrt{2}}{2} n\delta_e \geqslant \sqrt{2} n\delta\,.$$

Equalities hold if and only if $U \cong C_n$.

In the next two theorems the results analogous to the one obtained in Theorem 3.1 for the reduced and average Sombor indices are derived.

Theorem 3.2. *Let G be a simple graph with n vertices and m edges. Then*

$$\frac{\sqrt{2}}{2}\big(M_1(G - 2m\big) \leqslant SO_{red}(G) \leqslant M_1(G) - 2m\,.$$

Equality in the left–hand side of (3.1) *holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G. Equality on the right–hand side holds if and only if $G \cong \frac{n}{2}K_2$, for even n.*

Theorem 3.3. *Let G be a simple graph with n vertices and m edges. Then*

$$SO_{avr}(G) \geqslant \frac{\sqrt{2}}{2}\left(M_1(G - \frac{4m^2}{n}\right)\,.$$

Equality holds if and only if $d_i + d_j = \frac{4m}{n}$ for any pair of adjacent vertices v_i and v_j in G.

The next theorem reveals a relationship between Sombor, Albertson and reverse Randić index.

Theorem 3.4. *Let G be a simple graph of order $n \geqslant 2$ and size m. Then*

$$\frac{\sqrt{2}}{2}\left(Alb(G) + \sqrt{2}RR(G)\right) \leqslant SO(G) \leqslant Alb(G) + \sqrt{2}RR(G)\,. \tag{3.5}$$

Equality on the left–hand side holds if and only if $G \cong \overline{K_n}$. Equality on the right–hand side holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G.

PROOF. For $a := (d_i - d_j)^2$ and $b := 2d_id_j$, the inequality (3.2) becomes

$$\sqrt{(d_i - d_j)^2 + 2d_id_j} \geqslant \frac{\sqrt{2}}{2}\left(\sqrt{(d_i - d_j)^2} + \sqrt{2d_id_j}\right) = \tag{3.6}$$
$$= \frac{\sqrt{2}}{2}\left(|d_i - d_j| + \sqrt{2}\sqrt{d_id_j}\right) .$$

Thus, we have

$$SO(G) = \sum_{i\sim j}\sqrt{d_i^2 + d_j^2} = \sum_{i\sim j}\sqrt{(d_i - d_j)^2 + 2d_id_j} \geqslant$$
$$\geqslant \frac{\sqrt{2}}{2}\left(\sum_{i\sim j}|d_i - d_j| + \sqrt{2}\sum_{i\sim j}\sqrt{d_id_j}\right)$$

from which left–hand side of (3.5) is obtained.

Equality in (3.6) holds if and only if $(d_i - d_j) = 2d_id_j$, for any pair of adjacent vertices v_i and v_j in G. On the other hand, this is true if and only if $d_i = d_j = 0$, which implies than equality in the left–hand side of (3.5) holds if and only if $G \cong \overline{K_n}$.

To prove the right–hand side of (3.5) we use the fact that for any two non–negative real numbers a and b holds

$$\sqrt{a + b} \leqslant \sqrt{a} + \sqrt{b}\,, \tag{3.7}$$

with equality if and only if $a = 0$ or $b = 0$.

For $a := (d_i - d_j)^2$ and $b := 2d_id_j$, the above inequality becomes

$$\sqrt{(d_i - d_j)^2 + 2d_id_j} \leqslant \sqrt{(d_i - d_j)^2} + \sqrt{2d_id_j} = |d_i - d_j| + \sqrt{2}\sqrt{d_id_j}\,. \tag{3.8}$$

Thus, we have

$$SO(G) = \sum_{i\sim j}\sqrt{d_i^2 + d_j^2} = \sum_{i\sim j}\sqrt{(d_i - d_j)^2 + 2d_id_j} \leqslant \sum_{i\sim j}\left(|d_i - d_j| + \sqrt{2}\sqrt{d_id_j}\right) ,$$

from which the right-hand side of (3.5) is obtained.

Equality in (3.8) holds if and only if either $d_i - d_j = 0$, or $d_id_j = 0$. Thus we conclude that equality in the right–hand side of (3.5) holds if and only if $d_i = d_j$. □

In the next theorem we exhibit a relationship between $SO(G)$, $Alb(G)$ and $M_1(G)$.

THEOREM 3.5. *Let G be a simple graph with $n \geqslant 2$ vertices. Then*

$$SO(G) \leqslant \frac{\sqrt{2}}{2}\Big(Alb(G) + M_1(G)\Big)\,. \tag{3.9}$$

Equality holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G.

PROOF. For $a := \frac{1}{2}(d_i - d_j)^2$ and $b := \frac{1}{2}(d_i + d_j)^2$, the inequality (3.7) transforms into

$$\begin{aligned}\sqrt{\frac{1}{2}(d_i-d_j)^2+\frac{1}{2}(d_i+d_j)^2} &\leqslant \frac{\sqrt{2}}{2}\left(\sqrt{(d_i-d_j)^2}+\sqrt{(d_i+d_j)^2}\right) = \\ &= \frac{\sqrt{2}}{2}\big(|d_i-d_j|+(d_i+d_j)\big)\,.\end{aligned} \tag{3.10}$$

Thus, we have

$$\begin{aligned}SO(G) = \sum_{i\sim j}\sqrt{d_i^2+d_j^2} = \sum_{i\sim j}\sqrt{\frac{1}{2}\big((d_i-d_j)^2+(d_i+d_j)^2\big)} &\leqslant \\ \leqslant \frac{\sqrt{2}}{2}\left(\sum_{i\sim j}|d_i-d_j|+\sum_{i\sim j}(d_i+d_j)\right)&,\end{aligned}$$

from which (3.9) is obtained.

Equality in (3.10) holds if and only if either $(d_i - d_j)^2 = 0$, or $(d_i + d_j)^2 = 0$, which implies that equality in (3.9) holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G. □

COROLLARY 3.4. *Let G be a simple graph with $n \geqslant 2$ vertices. Then*

$$SO(G) \leqslant \frac{\sqrt{2}}{2}\Big(M_1(G) + m(\Delta - \delta)\Big)\,.$$

Equality holds if and only if $d_i = d_j$ for any pair of adjacent vertices in G.

Analogous results can be proven for the reduced and average Sombor indices. These are given in the next two theorems.

THEOREM 3.6. *Let G be a simple connected graph of order $n \geqslant 2$ and size m. Then*

$$SO_{red}(G) \leqslant \frac{\sqrt{2}}{2}\left(Alb(G) + M_1(G) - 2m\right)\,.$$

Equality holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G.

THEOREM 3.7. *Let G be a simple connected graph of order $n \geqslant 2$ and size m. Then*

$$SO_{avr}(G) \geqslant \frac{1}{2}\left(Alb(G) + M_1(G) - \frac{4m^2}{n}\right)\,.$$

Equality holds if and only if $d_i = d_j$ for any pair of adjacent vertices v_i and v_j in G.

The next theorem gives a connection between the Sombor and the forgotten topological index.

THEOREM 3.8. *Let G be a simple connected graph of order $n \geqslant 2$ and size m. Then*

$$SO(G) \leqslant \sqrt{mF(G)}\,. \tag{3.11}$$

Equality holds if and only if $d_i^2 + d_j^2$ *is a constant for any pair of adjacent vertices* v_i *and* v_j *in* G.

PROOF. For $r = 2$, $p_i := 1$, $a_i := \sqrt{d_i^2 + d_j^2}$, where summation is performed over all edges of G, the inequality (2.1) becomes

$$\sum_{i\sim j} 1 \sum_{i\sim j} \left(\sqrt{d_i^2 + d_j^2}\right)^2 \geqslant \left(\sum_{i\sim j} \sqrt{d_i^2 + d_j^2}\right)^2 ,$$

that is

$$mF(G) \geqslant SO(G)^2 , \tag{3.12}$$

from which (3.11) is obtained.

Equality in (3.12) holds if and only if $\sqrt{d_i^2 + d_j^2}$ is a constant for any pair of adjacent vertices v_i and v_j in G, which implies that equality in (3.11) holds if and only if $d_i^2 + d_j^2$ is a constant for any pair of adjacent vertices v_i and v_j in G. □

The forgotten topological index is well elaborated in the literature. The inequality (3.12) enables deriving upper bounds on $SO(G)$ in terms of other topological indices and graph parameters. This is illustrated in the following corollaries of Theorem 3.8.

COROLLARY 3.5. *Let* G *be a simple connected graph of order* $n \geqslant 2$ *and size* m. *Then*

$$SO(G) \leqslant \sqrt{m((\Delta + \delta)M_1(G) - 2m\Delta\delta)} . \tag{3.13}$$

Equality holds if and only if either G *is regular, or semiregular bipartite graph.*

PROOF. In [**19**] it was proven that

$$F(G) \leqslant (\Delta + \delta)M_1(G) - 2m\Delta\delta .$$

From the above and (3.11), the inequality (3.13) immediately follows. □

In [**5**] the following inequality was proven

$$M_1(G) \leqslant 2m(\Delta + \delta) - n\Delta .$$

From the above and (3.13) the following result is obtained:

COROLLARY 3.6. *Let* G *be a simple connected graph with* $n \geqslant 2$ *vertices and* m *edges. Then*

$$SO(G) \leqslant \sqrt{m(2m(\Delta^2 + \delta^2 + \Delta\delta) - n\Delta\delta(\Delta + \delta))} .$$

Equality holds if and only if G *is regular or semiregular bipartite graph.*

In [**24**] it was proven that

$$F(G) \leqslant 2m(\Delta^2 + \delta^2) - ID(G)(\Delta\delta)^2 .$$

Combining the above and (3.11) gives the following result:

Corollary 3.7. *Let G be a simple connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$SO(G) \leqslant \sqrt{m(2m(\Delta^2+\delta^2) - ID(G)(\Delta\delta)^2)}\,.$$

Equality holds if and only if G is regular or semiregular bipartite graph.

In [**27**] it was proven that for the irregular graphs holds

$$F(G) \leqslant \Delta M_1(G) - \frac{(2m\Delta - M_1(G))^2}{n\Delta - 2m}\,.$$

From the above and inequality (3.11) we obtain the following result:

Corollary 3.8. *Let G be a simple connected irregular graph with $n \geqslant 2$ vertices and m edges. Then*

$$SO(G) \leqslant \sqrt{m\left(\Delta M_1(G) - \frac{(2m\Delta - M_1(G))^2}{n\Delta - 2m}\right)}\,.$$

Equality holds if and only if $d_i \in \{\delta, \Delta\}$, $\delta \neq \Delta$, for every $i = 1, 2, \dots, n$.

Corollary 3.9. *Let G be a simple connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$SO(G) \leqslant \sqrt{m\Big(M_1(G)(\Delta_e + \delta_e) - m\Delta_e\delta_e - 2M_2(G)\Big)}\,.$$

Equality holds if and only if $L(G)$ is regular or semiregular bipartite graph.

Proof. In [**26**] it was proven that

$$F(G) \leqslant M_1(G)(\Delta_e + \delta_e) - m\Delta_e\delta_e - 2M_2(G)\,.$$

The required result immediately follows from the above and inequality (3.11) □

In the next two theorems a connection between the reduced and average Sombor index, respectively, and the first Zagreb and forgotten topological index are given. The proofs are analogous to that of Theorem 3.8, thus omitted.

Theorem 3.9. *Let G be a simple graph with $n \geqslant 2$ vertices and m edges. Then*

$$SO_{red}(G) \leqslant \sqrt{m\big(F(G) - 2M_1(G) + 2m\big)}\,.$$

Equality holds if and only if $(d_i - 1)^2 + (d_j - 1)^2$ is a constant for any pair of adjacent vertices v_i and v_j in G.

Theorem 3.10. *Let G be a simple graph with $n \geqslant 2$ vertices and m edges. Then*

$$SO_{avr}(G) \leqslant \sqrt{m\left(F(G) - \frac{4m}{n}M_1(G) + \frac{8m^3}{n^2}\right)}\,.$$

Equality holds if and only if $\big(d_i - \frac{2m}{n}\big)^2 + \big(d_j - \frac{2m}{n}\big)^2$ is a constant for any pair of adjacent vertices v_i and v_j in G.

In the next theorem we give a relationship between $SO(G)$, $M_2(G)$ and $SDD(G)$.

THEOREM 3.11. *Let G be a simple connected graph with $n \geqslant 2$ vertices. Then*

$$SO(G) \leqslant \sqrt{M_2(G)SDD(G)}\,. \tag{3.14}$$

Equality holds if and only if $\frac{1}{d_i^2} + \frac{1}{d_j^2}$ is a constant for any pair of adjacent veritces v_i and v_j in G.

PROOF. For $r = 1$, $x_i := \sqrt{d_i^2 + d_j^2}$, $a_i := d_i d_j$, and summation performed over all edges of G, the inequality (2.2) becomes

$$\sum_{i\sim j} \frac{\left(\sqrt{d_i^2 + d_j^2}\right)^2}{d_i d_j} \geqslant \frac{\left(\sum_{i\sim j} \sqrt{d_i^2 + d_j^2}\right)^2}{\sum_{i\sim j} d_i d_j}\,,$$

that is

$$SDD(G) \geqslant \frac{SO(G)^2}{M_2(G)}\,, \tag{3.15}$$

from which (3.14) immediately follows.

Equality in (3.15) holds if and only if $\frac{\sqrt{d_i^2+d_j^2}}{d_i d_j} = \sqrt{\frac{1}{d_i^2} + \frac{1}{d_j^2}}$ is a constant for any pair of nonadjacent vertices v_i and v_j in G. Therefore we conclude that equality in (3.14) holds if and only if $\frac{1}{d_i^2} + \frac{1}{d_j^2}$ is a constant for any pair of adjacent vertices v_i and v_j in G. □

In [**6**] it was proven that

$$SDD(G) \leqslant n(\delta_e + \Delta_e) - \frac{m^2 \delta_e \Delta_e}{M_2(G)} - 2m\,,$$

with equality if and only if G is regular or semiregular bipartite graph. Therefore, we have the following corollary of Theorem 3.11

COROLLARY 3.10. *Let G be a simple connected graph of order n and size m. Then*

$$SO(G) \leqslant \sqrt{m\Big((n(\Delta_e + \delta_e) - 2m)M_2(G) - m^2\Delta_e\delta_e\Big)}\,.$$

Equality holds if and only if G is regular or semiregular bipartite graph.

In the next two theorems we prove the inequalities of Nordhaus–Gaddum type [**30**] for the Sombor index.

THEOREM 3.12. *Let G be a simple connected graph of order $n \geqslant 2$ and size m. Then*

$$SO(G) + SO(\overline{G}) \geqslant \frac{\sqrt{2}}{2} n(n-1)^2\,. \tag{3.16}$$

Equality holds if and only if G is $\left(\frac{n-1}{2}\right)$–regular graph, where n is odd.

Proof. For $a = (n-1-d_i)^2$ and $b = (n-1-d_j)^2$, the inequality (3.2) becomes

$$\sqrt{(n-1-d_i)^2+(n-1-d_j)^2} \geqslant \frac{\sqrt{2}}{2}\left(\sqrt{(n-1-d_i)^2}+\sqrt{(n-1-d_j)^2}\right) = \\ = \frac{\sqrt{2}}{2}\big((n-1-d_i)+(n-1-d_j)\big)\,.$$

Thus, we have

$$\begin{aligned} SO(\overline{G}) &= \sum_{i\nsim j}\sqrt{(n-1-d_i)^2+(n-1-d_j)^2} \geqslant \\ &\geqslant \frac{\sqrt{2}}{2}\sum_{i\nsim j}\Big((n-1-d_i)+(n-1-d_j)\Big) = \\ &= \frac{\sqrt{2}}{2}\sum_{i=1}^{n}(n-1-d_i)^2 = \frac{\sqrt{2}}{2}M_1(\overline{G})\,. \end{aligned} \tag{3.17}$$

From the above and left-hand side of (3.1) we obtain

$$\begin{aligned} SO(G)+SO(\overline{G}) &\geqslant \frac{\sqrt{2}}{2}\big(M_1(G)+M_1(\overline{G})\big) = \\ &= \frac{\sqrt{2}}{2}\sum_{i=1}^{n}\left(d_i^2+(n-1-d_i)^2\right)\,. \end{aligned} \tag{3.18}$$

It is well known that for any pair of non–negative real numbers a and b, the following inequality holds

$$a^2+b^2 \geqslant \left(\frac{a+b}{2}\right)^2\,.$$

For $a = d_i$ and $b = n-1-d_i$, from the above we obtain

$$d_i^2+(n-1-d_i)^2 \geqslant \frac{1}{2}(n-1)^2\,. \tag{3.19}$$

Now, from the above and (3.18) we obtain (3.16).

Equality in (3.11) holds if and only if G is regular. Equality in (3.19) holds if and only if $d_1 = d_2 = \cdots = d_n = \frac{n-1}{2}$, which implies that equality in (3.16) holds if and only if G is $\left(\frac{n-1}{2}\right)$–regular graph, where n is odd. □

Theorem 3.13. *Let G be a simple connected graph of order $n \geqslant 2$ and size m. Then*

$$SO(G)\cdot SO(\overline{G}) \geqslant \frac{2m^2(n(n-1)-2m)^2}{n^2}\,. \tag{3.20}$$

Equality holds if and only if G is regular.

Proof. From the left–hand side of (3.1) and (3.17) we have that

$$SO(G)\cdot SO(\overline{G}) \geqslant \frac{1}{2}M_1(G)\cdot M_1(\overline{G})\,. \tag{3.21}$$

Since, see e.g. [**8**],

$$M_1(G) \geqslant \frac{4m^2}{n}\,, \tag{3.22}$$

and

$$M_1(\overline{G}) \geqslant \frac{(n(n-1)-2m)^2}{n}\,, \tag{3.23}$$

from the above and (3.21) we arrive at (3.20).

Equalities in (3.22) and (3.23) hold if and only if G is regular, which implies that equality in (3.20) holds if and only if G is regular. □

When G is a unicyclic graph, $G \cong U$, we have the following corollary of THeorem 3.13

Corollary 3.11. *Let U be a unicyclic graph with $n \geqslant 3$ vertices. Then*

$$SO(U)\,SO(\overline{U}) \geqslant 2n^2(n-3)^2\,.$$

Equality holds if and only if $U \cong C_n$.

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