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SOME FIXED POINT THEOREMS UNDER IMPLICIT RELATION ON S-METRIC SPACES

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ABSTRACT. The aim of this paper is to establish some fixed point and common fixed points theorems in the setting of S-metric space under implicit relation. Our results extend, unify and generalize several results from the current existing literature.

1. Introduction

Metric space is one of the most useful and important space in mathematics. Its wide area provides a powerful tool to the study of variational inequalities, optimization and approximation theory, computer sciences and so many other mathematics fields. As it is well-known, one of the most useful result in nonlinear analysis is the Banach contraction mappings principle [**2**]. Many authors generalized this famous result in different ways. Recently the study of fixed point theory in metric space is very interesting field and attract many researchers to investigated different results on it.

In 2006, Mustafa and Sims [**8**] introduced a new structure of generalized metric space, called G-metric space and gave a modification to the contraction principle of Banach. After then, some authors [**3, 9, 14**] have proved some fixed point results in these spaces. In 1992, B.C. Dhage [**4**] introduced the notion of D-metric space and proved some fixed point theorems. In 2007, Sedghi et al. [**11**] introduced D^*-metric space which is a modification of D-metric spaces and proved some fixed point theorems in D^*-metric spaces. Later on many authors have studied the fixed point theorems in generalized metric spaces (see, for example [**1, 6, 7, 15**]).

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In 2012, Sedghi et al. [**12**] introduced the concept of S-metric space which is a generalization of a G-metric space and D^*-metric space and obtained some fixed point theorems in S-metric space. They also give some examples of S-metric space which shows that S-metric space is different from other spaces.

In 2013, Gupta [**5**] introduced the concept of cyclic contraction on S-metric space and proved some fixed point theorems on S-metric spaces which generalized the results of Sedghi et al. [**12**]. In 2014, Sedghi and Dung [**13**] have proved a general fixed point theorem in S-metric space using implicit relation and as application they obtained many analogous of fixed point theorems in metric spaces for S-metric spaces.

In 2015, Prudhvi [**10**] proved some fixed point theorems on S-metric spaces which extend and improve the results of Sedghi and Dung [**13**].

Motivated by Gupta [**5**], Prudhvi [**10**] and some others, the main purpose of this paper is to study and establish some fixed point and common fixed point theorems in S-metric space satisfying ϕ-implicit relation. Our results extend, generalize and unify several results from the existing literature.

2. Preliminaries

We need the following definitions and lemmas in the sequel.

Definition 2.1. ([**12**]) Let X be a nonempty set and $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $t \in X$:

$(SM1)$ $S(x,y,z) = 0$ if and only if $x = y = z$;

$(SM2)$ $S(x,y,z) \leqslant S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 2.1. ([**15**]) Let X be a nonempty set and d be the ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Example 2.2. ([**12**]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on X.

Example 2.3. ([**12**]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an S-metric on X.

Example 2.4. ([**13**]) Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Lemma 2.1 ([**12**], Lemma 2.5). *If (X,S) is an S-metric space, then we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Lemma 2.2 ([**12**], Lemma 2.12). *Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.*

Definition 2.2. ([**12**]) Let (X,S) be an S-metric space.

($a1$) A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

($a2$) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

($a3$) The S-metric space (X, S) is called complete if every Cauchy sequence in X is convergent in X.

Definition 2.3. Let T be a self mapping on an S-metric space (X, S). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.4. ([**12**]) Let (X, S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \leqslant \alpha < 1$ such that

$$S(Tx, Tx, Ty) \leqslant \alpha\, S(x, x, y)$$

for all $x, y \in X$. If the S-metric space (X, S) is complete then the mapping defined as above has a unique fixed point.

Proposition 2.1. *Let (X, S) be an S-metric space. Then the following statements are equivalent.*

(1) *The sequence $\{x_n\}$ is Cauchy.*

(2) *For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$, for all $n, m \geqslant n_0$.*

Proposition 2.2. *Let (X, S) be an S-metric space. Then, for any $x, y, z \in X$ it follows that:*

(1) *if $S(x, y, z) = 0$, then $x = y = z$;*

(2) $S(x, x, y) \leqslant 2S(x, x, z) + S(y, y, z)$.

Now, we introduce an implicit relation to investigate some fixed point and common fixed point theorems in S-metric spaces.

Definition 2.5. (**Implicit Relation**) Let Φ be the family of all real valued continuous functions $\phi\colon \mathbb{R}_+^4 \to \mathbb{R}_+$, non-decreasing in the first argument for four variables. For some $k \in [0, 1)$, we consider the following conditions.

($R1$) For $x, y \in \mathbb{R}_+$, if $x \leqslant \phi(y, y, x, \frac{4x+y}{3})$, then $x \leqslant ky$.

($R2$) For $x \in \mathbb{R}_+$, if $x \leqslant \phi(0, x, 0, 0)$, then $x = 0$.

($R3$) For $x \in \mathbb{R}_+$, if $x \leqslant \phi(x, 0, 0, \frac{x}{3})$, then $x = 0$ since $k \in [0, 1)$.

Example 2.5. Let $\phi(r, s, t, u) = r - \mu \min\{s, t, u\} + (2 + \mu)u$, where $\mu > 0$.

Example 2.6. Let $\phi(r, s, t, u) = r^2 + a\, r \max\{s, t, u\} - bs$, where $a > 0, b > 0$.

Example 2.7. Let $\phi(r, s, t, u) = r + c \max\{s, t, u\}$, where $c > 0$.

3. Main Results

In this section, we shall prove some fixed point and common fixed point theorems satisfying an implicit relation in the setting of S-metric spaces.

THEOREM 3.1. *Let $\mathcal{T}$ be a self-map on a complete S-metric space (X, S) and*

$$\begin{aligned} S(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \;\; &\leqslant \;\; \phi\Big(S(x, y, z), S(z, z, \mathcal{T}z), S(y, y, \mathcal{T}y), \\ &\qquad \frac{1}{3}\big[S(x, x, \mathcal{T}y) + S(z, z, \mathcal{T}y) + S(y, y, \mathcal{T}x)\big]\Big) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$ and some $\phi \in \Phi$. If ϕ satisfies the conditions $(R1)$, $(R2)$ and $(R3)$, then $\mathcal{T}$ has a unique fixed point in X.

PROOF. For each $x_0 \in X$ and define a sequence $\{x_n\}$ in X such that $x_{n+1} = \mathcal{T}x_n$ for any $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $x_{n+1} = x_n$, then $x_n = \mathcal{T}x_n$, that is, $\mathcal{T}$ has a fixed point. Thus, we may assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. It follows from (3.1), $(SM2)$ and Lemma 2.1 that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) \;\; &= \;\; S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}x_{n-1}) \\ &\leqslant \;\; \phi\Big(S(x_n, x_n, x_{n-1}), S(x_{n-1}, x_{n-1}, \mathcal{T}x_{n-1}), S(x_n, x_n, \mathcal{T}x_n), \\ &\quad \frac{1}{3}\big[S(x_n, x_n, \mathcal{T}x_n) + S(x_{n-1}, x_{n-1}, \mathcal{T}x_n) + S(x_n, x_n, \mathcal{T}x_n)\big]\Big) \\ &\quad \phi\Big(S(x_n, x_n, x_{n-1}), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ &\quad \frac{1}{3}\big[S(x_n, x_n, x_{n+1}) + S(x_{n-1}, x_{n-1}, x_{n+1}) + S(x_n, x_n, x_{n+1})\big]\Big) \\ &\quad \phi\Big(S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), \\ &\quad \frac{1}{3}\big[S(x_{n+1}, x_{n+1}, x_n) + S(x_{n+1}, x_{n+1}, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_n)\big]\Big) \\ &\leqslant \;\; \phi\Big(S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), \\ &\quad \frac{1}{3}\big[2S(x_{n+1}, x_{n+1}, x_n) + 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)\big]\Big) \\ &= \;\; \phi\Big(S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), \\ &\quad \frac{1}{3}\big[4S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_{n-1})\big]\Big). \end{aligned} \tag{3.2}$$

Since ϕ satisfies the condition $(R1)$, there exists $k \in [0, 1)$ such that

$$S(x_{n+1}, x_{n+1}, x_n) \leqslant kS(x_n, x_n, x_{n-1}) \leqslant k^n S(x_1, x_1, x_0). \tag{3.3}$$

Thus for all $n < m$, by using $(SM2)$, Lemma 2.1 and equation (3.3), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leqslant 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\cdots \\ &\leqslant 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1) \\ &\leqslant \Big(\frac{2k^n}{1-k}\Big)S(x_0, x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to \infty$, we get $S(x_n, x_n, x_m) \to 0$ since $0 < k < 1$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now we prove that x is a fixed point of $\mathcal{T}$. Again by using inequality (3.1), we obtain

$$\begin{aligned} S(x_{n+1}, x_{n+1}, \mathcal{T}u) &= S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}u) \\ &\leqslant \phi\Big(S(x_n, x_n, u), S(u, u, \mathcal{T}u), S(x_n, x_n, \mathcal{T}x_n), \\ &\qquad \frac{1}{3}\big[S(x_n, x_n, \mathcal{T}x_n) + S(u, u, \mathcal{T}x_n) + S(x_n, x_n, \mathcal{T}x_n)\big]\Big) \\ &= \phi\Big(S(x_n, x_n, u), S(u, u, \mathcal{T}u), S(x_n, x_n, x_{n+1}), \\ &\qquad \frac{1}{3}\big[S(x_n, x_n, x_{n+1}) + S(u, u, x_{n+1}) + S(x_n, x_n, x_{n+1})\big]\Big). \end{aligned}$$

Note that $\phi \in \Phi$, then using Lemma 2.2 and taking the limit as $n \to \infty$, we get

$$S(u, u, \mathcal{T}u) \leqslant \phi\Big(0, S(u, u, \mathcal{T}u), 0, 0\Big).$$

Since ϕ satisfies the condition $(R2)$, then $S(u, u, \mathcal{T}u) \leqslant k.0 = 0$. This shows that $u = \mathcal{T}u$. Thus u is a fixed point of $\mathcal{T}$.

Now, we have to show that the fixed point of $\mathcal{T}$ is unique. For this, let u_1, u_2 be fixed points of $\mathcal{T}$ with $u_1 \neq u_2$. We shall prove that $u_1 = u_2$. It follows from equation (3.1) and Lemma 2.1 that

$$\begin{aligned} S(u_1, u_1, u_2) &= S(\mathcal{T}u_1, \mathcal{T}u_1, \mathcal{T}u_2) \\ &\leqslant \phi\Big(S(u_1, u_1, u_2), S(u_2, u_2, \mathcal{T}u_2), S(u_1, u_1, \mathcal{T}u_1), \\ &\qquad \frac{1}{3}\big[S(u_1, u_1, \mathcal{T}u_1) + S(u_2, u_2, \mathcal{T}u_1) + S(u_1, u_1, \mathcal{T}u_1)\big]\Big) \\ &= \phi\Big(S(u_1, u_1, u_2), S(u_2, u_2, u_2), S(u_1, u_1, u_1), \\ &\qquad \frac{1}{3}\big[S(u_1, u_1, u_1) + S(u_2, u_2, u_1) + S(u_1, u_1, u_1)\big]\Big) \\ &= \phi\Big(S(u_1, u_1, u_2), 0, 0, \frac{1}{3}S(u_1, u_1, u_2)\Big). \end{aligned}$$

Since ϕ satisfies the condition $(R3)$, then we get

$$\begin{aligned} S(u_1,u_1,u_2) &\leqslant k\,S(u_1,u_1,u_2) \\ &\Rightarrow S(u_1,u_1,u_2)=0, \text{ since } 0<k<1. \end{aligned}$$

This shows that $u_1 = u_2$. Thus the fixed point of $\mathcal{T}$ is unique. This completes the proof. □

Common Fixed Point Theorems

THEOREM 3.2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two self-maps on a complete S-metric space (X,S) and*

$$\begin{aligned} S(\mathcal{T}_1x,\mathcal{T}_1y,\mathcal{T}_2z) \leqslant\ & \phi\Big(S(x,y,z),S(z,z,\mathcal{T}_2z),S(y,y,\mathcal{T}_1y), \\ & \frac{1}{3}\big[S(x,x,\mathcal{T}_1y)+S(z,z,\mathcal{T}_1y)+S(y,y,\mathcal{T}_1x)\big]\Big) \end{aligned} \tag{3.4}$$

for all $x,y,z\in X$ and some $\phi\in\Phi$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique common fixed point in X.

PROOF. For each $x_0\in X$. Put $x_{2n+1}=\mathcal{T}_1x_{2n}$ and $x_{2n+2}=\mathcal{T}_2x_{2n+1}$ for $n=0,1,2,\dots$. It follows from (3.4), $(SM2)$ and Lemma 2.1 that

$$\begin{aligned} & S(x_{2n+1},x_{2n+1},x_{2n}) = S(\mathcal{T}_1x_{2n},\mathcal{T}_1x_{2n},\mathcal{T}_2x_{2n-1}) \\ &\leqslant \phi\Big(S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n-1},x_{2n-1},\mathcal{T}_2x_{2n-1}),S(x_{2n},x_{2n},\mathcal{T}_1x_{2n}), \\ &\quad \frac{1}{3}\big[S(x_{2n},x_{2n},\mathcal{T}_1x_{2n})+S(x_{2n-1},x_{2n-1},\mathcal{T}_1x_{2n})+S(x_{2n},x_{2n},\mathcal{T}_1x_{2n})\big]\Big) \\ &= \phi\Big(S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n-1},x_{2n-1},x_{2n}),S(x_{2n},x_{2n},x_{2n+1}), \\ &\quad \frac{1}{3}\big[S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n-1},x_{2n-1},x_{2n+1})+S(x_{2n},x_{2n},x_{2n+1})\big]\Big) \\ &= \phi\Big(S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n+1},x_{2n+1},x_{2n}), \\ &\quad \frac{1}{3}\big[S(x_{2n+1},x_{2n+1},x_{2n})+S(x_{2n+1},x_{2n+1},x_{2n-1}) \\ &\quad +S(x_{2n+1},x_{2n+1},x_{2n})\big]\Big) \\ &\leqslant \phi\Big(S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n+1},x_{2n+1},x_{2n}), \\ &\quad \frac{1}{3}\big[2S(x_{2n+1},x_{2n+1},x_{2n})+2S(x_{2n+1},x_{2n+1},x_{2n})+S(x_{2n},x_{2n},x_{2n-1})\big]\Big) \\ &= \phi\Big(S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n},x_{2n},x_{2n-1}),S(x_{2n+1},x_{2n+1},x_{2n}), \\ &\quad \frac{1}{3}\big[4S(x_{2n+1},x_{2n+1},x_{2n})+S(x_{2n},x_{2n},x_{2n-1})\big]\Big). \end{aligned} \tag{3.5}$$

Since ϕ satisfies the condition $(R1)$, there exists $k\in[0,1)$ such that

$$S(x_{2n+1},x_{2n+1},x_{2n}) \leqslant kS(x_{2n},x_{2n},x_{2n-1}) \leqslant k^{2n}S(x_1,x_1,x_0). \tag{3.6}$$

Thus for all $n < m$, by using $(SM2)$, Lemma 2.1 and equation (3.6), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leqslant 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\cdots \\ &\leqslant 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1) \\ &\leqslant \Big(\frac{2k^n}{1-k}\Big)S(x_0, x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to \infty$, we get $S(x_n, x_n, x_m) \to 0$ since $0 < k < 1$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$. Now we have to prove that v is a common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$. For this, consider

$$\begin{aligned} S(x_{2n+1}, x_{2n+1} \quad , \quad & \mathcal{T}_1 v) = S(\mathcal{T}_1 x_{2n}, \mathcal{T}_1 x_{2n}, \mathcal{T}_1 v) \\ \leqslant \quad & \phi\Big(S(x_{2n}, x_{2n}, v), S(v, v, \mathcal{T}_1 v), S(x_{2n}, x_{2n}, \mathcal{T}_1 x_{2n}), \\ & \frac{1}{3}\big[S(x_{2n}, x_{2n}, \mathcal{T}_1 x_{2n}) + S(v, v, \mathcal{T}_1 x_{2n}) + S(x_{2n}, x_{2n}, \mathcal{T}_1 x_{2n})\big]\Big) \\ = \quad & \phi\Big(S(x_{2n}, x_{2n}, v), S(v, v, \mathcal{T}_1 v), S(x_{2n}, x_{2n}, x_{2n+1}), \\ & \frac{1}{3}\big[S(x_{2n}, x_{2n}, x_{2n+1}) + S(v, v, x_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1})\big]\Big) \end{aligned}$$

(3.7)

Note that $\phi \in \Phi$, then using Lemma 2.2 and taking the limit as $n \to \infty$, we get

$$S(v, v, \mathcal{T}_1 v) \leqslant \phi\Big(0, S(v, v, \mathcal{T}_1 v), 0, 0\Big).$$

Since ϕ satisfies the condition $(R2)$, then $S(v, v, \mathcal{T}_1 v) \leqslant k.0 = 0$. This shows that $v = \mathcal{T}_1 v$ for all $v \in X$. Similarly, we can show that $v = \mathcal{T}_2 v$. This shows that v is a common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Now to show that the common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$ is unique. For this, let v_1 be another common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$, that is, $\mathcal{T}_1 v_1 = \mathcal{T}_2 v_1 = v_1$ with $v \neq v_1$. Then we have to show that $v = v_1$. It follows from equation (3.4) and Lemma 2.1 that

$$\begin{aligned} S(v, v, v_1) &= S(\mathcal{T}_1 v, \mathcal{T}_1 v, \mathcal{T}_2 v_1) \\ &\leqslant \phi\Big(S(v, v, v_1), S(v_1, v_1, \mathcal{T}_2 v_1), S(v, v, \mathcal{T}_1 v), \\ &\qquad \frac{1}{3}\big[S(v, v, \mathcal{T}_1 v) + S(v_1, v_1, \mathcal{T}_1 v) + S(v, v, \mathcal{T}_1 v)\big]\Big) \\ &= \phi\Big(S(v, v, v_1), S(v_1, v_1, v_1), S(v, v, v), \\ &\qquad \frac{1}{3}\big[S(v, v, v) + S(v_1, v_1, v) + S(v, v, v)\big]\Big) \\ &= \phi\Big(S(v, v, v_1), 0, 0, \frac{1}{3}S(v, v, v_1)\big]\Big). \end{aligned}$$

Since ϕ satisfies the condition $(R3)$, then we get

$$\begin{aligned} S(v,v,v_1) &\leqslant k\,S(v,v,v_1) \\ &\Rightarrow S(v,v,v_1)=0, \text{ since } 0<k<1. \end{aligned}$$

Thus, we have $v=v_1$. This shows that v is the unique common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$. This completes the proof. □

Theorem 3.3. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two continuous self-maps on a complete S-metric space (X,S) and*

$$\begin{aligned} S(\mathcal{T}_1^p x,\mathcal{T}_1^p y,\mathcal{T}_2^q z) &\leqslant \phi\Big(S(x,y,z),S(z,z,\mathcal{T}_2^q z),S(y,y,\mathcal{T}_1^p y), \\ &\quad \frac{1}{3}\big[S(x,x,\mathcal{T}_1^p y)+S(z,z,\mathcal{T}_1^p y) \\ &\quad +S(y,y,\mathcal{T}_1^p x)\big]\Big) \end{aligned} \tag{3.8}$$

for all $x,y,z\in X$, where p and q are some integers and some $\phi\in\Phi$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique common fixed point in X.

Proof. Since $\mathcal{T}_1^p$ and $\mathcal{T}_2^q$ satisfy the conditions of Theorem 3.2. So $\mathcal{T}_1^p$ and $\mathcal{T}_2^q$ have a unique common fixed point. Let z be the common fixed point. Then, we have

$$\begin{aligned} \mathcal{T}_1^p z = z \Rightarrow \mathcal{T}_1(\mathcal{T}_1^p z) &= \mathcal{T}_1 z \\ \Rightarrow \mathcal{T}_1^p(\mathcal{T}_1 z) &= \mathcal{T}_1 z. \end{aligned}$$

If $\mathcal{T}_1 z = z_0$, then $\mathcal{T}_1^p z_0 = z_0$. So, $\mathcal{T}_1 z$ is a fixed point of $\mathcal{T}_1^p$. Similarly, $\mathcal{T}_2(\mathcal{T}_2^q z)=\mathcal{T}_2 z$. Now, using equation (3.8) and Lemma 2.1, we obtain

$$\begin{aligned} S(z,z,\mathcal{T}_1 z) &= S(\mathcal{T}_1^p z,\mathcal{T}_1^p z,\mathcal{T}_1^p(\mathcal{T}_1 z)) \\ &\leqslant \phi\Big(S(z,z,\mathcal{T}_1 z),S(\mathcal{T}_1 z,\mathcal{T}_1 z,\mathcal{T}_1^p(\mathcal{T}_1 z)),S(z,z,\mathcal{T}_1^p z), \\ &\quad \frac{1}{3}\big[S(z,z,\mathcal{T}_1^p z)+S(\mathcal{T}_1 z,\mathcal{T}_1 z,\mathcal{T}_1^p z)+S(z,z,\mathcal{T}_1^p z)\big]\Big) \\ &= \phi\Big(S(z,z,\mathcal{T}_1 z),S(\mathcal{T}_1 z,\mathcal{T}_1 z,\mathcal{T}_1 z),S(z,z,z), \\ &\quad \frac{1}{3}\big[S(z,z,z)+S(\mathcal{T}_1 z,\mathcal{T}_1 z,z)+S(z,z,z)\big]\Big) \\ &= \phi\Big(S(z,z,\mathcal{T}_1 z),0,0,\frac{1}{3}S(z,z,\mathcal{T}_1 z)\Big). \end{aligned}$$

Since ϕ satisfies the condition $(R3)$, then we get

$$\begin{aligned} S(z,z,\mathcal{T}_1 z) &\leqslant k\,S(z,z,\mathcal{T}_1 z) \\ &\Rightarrow S(z,z,\mathcal{T}_1 z)=0, \text{ since } 0<k<1. \end{aligned}$$

Thus, we have $z=\mathcal{T}_1 z$ for all $z\in X$. Similarly, we can show that $z=\mathcal{T}_2 z$. This shows that z is a common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$. For uniqueness of z, let $z'\neq z$ be another common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$. Then clearly z' is also a common fixed point of $\mathcal{T}_1^p$ and $\mathcal{T}_2^q$ which implies $z'=z$. Hence $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique common fixed point. This completes the proof. □

THEOREM 3.4. *Let $\{\mathcal{G}_\alpha\}$ be a family of continuous self mappings on a complete S-metric space (X, S) satisfying*

$$\begin{aligned} S(\mathcal{G}_\alpha x, \mathcal{G}_\alpha y, \mathcal{G}_\beta z) \;\leqslant\; & \phi\Big(S(x,y,z), S(z,z,\mathcal{G}_\beta z), S(y,y,\mathcal{G}_\alpha y), \\ & \frac{1}{3}\big[S(x,x,\mathcal{G}_\alpha y) + S(z,z,\mathcal{G}_\alpha y) \\ & + S(y,y,\mathcal{G}_\alpha x)\big]\Big) \end{aligned} \tag{3.9}$$

for $\alpha, \beta \in \Phi$ with $\alpha \neq \beta$ and $x, y, z \in X$. Then there exists a unique $q \in X$ satisfying $\mathcal{G}_\alpha q = q$ for all $\alpha \in \Phi$.

PROOF. For $x_0 \in X$, we define a sequence as follows:

$$x_{2n+1} = \mathcal{G}_\alpha x_{2n},\ x_{2n+2} = \mathcal{G}_\beta x_{2n+1},\ n = 0, 1, 2, \ldots.$$

It follows from (3.9), $(SM2)$ and Lemma 2.1 that

$$\begin{aligned} S(x_{2n+1}, x_{2n+1}, x_{2n}) &= S(\mathcal{G}_\alpha x_{2n}, \mathcal{G}_\alpha x_{2n}, \mathcal{G}_\beta x_{2n-1}) \\ &\leqslant \phi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n-1}, x_{2n-1}, \mathcal{G}_\beta x_{2n-1}), S(x_{2n}, x_{2n}, \mathcal{G}_\alpha x_{2n}), \\ &\qquad \frac{1}{3}\big[S(x_{2n}, x_{2n}, \mathcal{G}_\alpha x_{2n}) + S(x_{2n-1}, x_{2n-1}, \mathcal{G}_\alpha x_{2n}) \\ &\qquad + S(x_{2n}, x_{2n}, \mathcal{G}_\alpha x_{2n})\big]\Big) \\ &= \phi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n-1}, x_{2n-1}, x_{2n}), S(x_{2n}, x_{2n}, x_{2n+1}), \\ &\qquad \frac{1}{3}\big[S(x_{2n}, x_{2n}, x_{2n+1}) + S(x_{2n-1}, x_{2n-1}, x_{2n+1}) \\ &\qquad + S(x_{2n}, x_{2n}, x_{2n+1})\big]\Big) \\ &= \phi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n+1}, x_{2n+1}, x_{2n}), \\ &\qquad \frac{1}{3}\big[S(x_{2n+1}, x_{2n+1}, x_{2n}) + S(x_{2n+1}, x_{2n+1}, x_{2n-1}) \\ &\qquad + S(x_{2n+1}, x_{2n+1}, x_{2n})\big]\Big) \\ &= \phi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n+1}, x_{2n+1}, x_{2n}), \\ &\qquad \frac{1}{3}\big[2S(x_{2n+1}, x_{2n+1}, x_{2n}) + 2S(x_{2n+1}, x_{2n+1}, x_{2n}) \\ &\qquad + S(x_{2n}, x_{2n}, x_{2n-1})\big]\Big) \\ &= \phi\Big(S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n}, x_{2n}, x_{2n-1}), S(x_{2n+1}, x_{2n+1}, x_{2n}), \\ &\qquad \frac{1}{3}\big[4S(x_{2n+1}, x_{2n+1}, x_{2n}) + S(x_{2n}, x_{2n}, x_{2n-1})\big]\Big). \end{aligned} \tag{3.10}$$

Since ϕ satisfies the condition $(R1)$, there exists $k \in [0, 1)$ such that

$$S(x_{2n+1}, x_{2n+1}, x_{2n}) \leqslant kS(x_{2n}, x_{2n}, x_{2n-1}) \leqslant k^{2n} S(x_1, x_1, x_0). \tag{3.11}$$

Thus for all $n < m$, by using $(SM2)$, Lemma 2.1 and equation (3.11), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leqslant 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\dots \\ &\leqslant 2[k^n + \cdots + k^{m-1}]S(x_0, x_0, x_1) \\ &\leqslant \Big(\frac{2k^n}{1-k}\Big)S(x_0, x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to \infty$, we get $S(x_n, x_n, x_m) \to 0$ as $\Big(\frac{2k^n}{1-k}\Big) \to 0$ since $0 < k < 1$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). By the completeness of the space, there exists $q \in X$ such that $x_n \to q \in X$ as $n \to \infty$. By the continuity of $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$, it is clear that $\mathcal{G}_\alpha q = \mathcal{G}_\beta q = q$. Therefore q is a common fixed point of $\mathcal{G}_\alpha$ for all $\alpha \in \Phi$.

In order to prove the uniqueness, let us take another common fixed point q' of $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ where $q' \neq q$. Then using equation (3.9) and Lemma 2.1, we obtain

$$\begin{aligned} S(q, q, q') &= S(\mathcal{G}_\alpha q, \mathcal{G}_\alpha q, \mathcal{G}_\beta q') \\ &\leqslant \phi\Big(S(q, q, q'), S(q', q', \mathcal{G}_\beta q'), S(q, q, \mathcal{G}_\alpha q), \\ &\qquad \frac{1}{3}\big[S(q, q, \mathcal{G}_\alpha q) + S(q', q', \mathcal{G}_\alpha q) \\ &\qquad\quad + S(q, q, \mathcal{G}_\alpha q)\big]\Big) \\ &= \phi\Big(S(q, q, q'), S(q', q', q'), S(q, q, q), \\ &\qquad \frac{1}{3}\big[S(q, q, q) + S(q', q', q) + S(q, q, q)\big]\Big) \\ &= \phi\Big(S(q, q, q'), 0, 0, \frac{1}{3}S(q, q, q')\Big). \end{aligned}$$

Since ϕ satisfies the condition $(R3)$, then we get

$$\begin{aligned} S(q, q, q') &\leqslant k\, S(q, q, q') \\ &\Rightarrow S(q, q, q') = 0, \ \text{ since } 0 < k < 1. \end{aligned}$$

Thus, we have $q = q'$ for all $q \in X$. This shows that q is a unique common fixed point of $\mathcal{G}_\alpha$ for all $\alpha \in \Phi$. This completes the proof. □

Next, we give an analogue of fixed point theorem in metric spaces for S-metric spaces by combining Theorem 3.1 with $\phi \in \Phi$ and ϕ satisfies the conditions $(R1)$, $(R2)$ and $(R3)$. The following corollary is an analogue of Banach's contraction principle.

Corollary 3.1. *Let (X, S) be a complete S-metric space. Suppose that the mapping $\mathcal{T}\colon X \to X$ satisfies the following condition:*

$$S(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leqslant \gamma\, S(x, y, z)$$

for all $x, y, z \in X$, *where* $\gamma \in [0,1)$ *is a constant. Then* $\mathcal{T}$ *has a unique fixed point in* X. *Moreover,* $\mathcal{T}$ *is continuous at the fixed point.*

Proof. The assertion follows using Theorem 3.1 with $\phi(a,b,c,d) = \gamma\, a$ for some $\gamma \in [0,1)$ and all $a,b,c,d \in \mathbb{R}_+$. □

Again, we give an analogue of fixed point theorem in metric spaces for S-metric spaces by combining Theorem 3.2 with $\phi \in \Phi$ and ϕ satisfies the conditions $(R1)$, $(R2)$ and $(R3)$. Then we have the following corollary.

Corollary 3.2. *Let* (X,S) *be a complete* S*-metric space. Suppose that the mappings* $\mathcal{T}_1, \mathcal{T}_2 \colon X \to X$ *satisfies the following condition:*

$$S(\mathcal{T}_1 x, \mathcal{T}_1 y, \mathcal{T}_2 z) \leqslant \delta\, S(x,y,z)$$

for all $x, y, z \in X$, *where* $\delta \in [0,1)$ *is a constant. Then* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *have a unique common fixed point in* X.

Proof. The assertion follows using Theorem 3.2 with $\phi(a,b,c,d) = \delta\, a$ for some $\delta \in [0,1)$ and all $a,b,c,d \in \mathbb{R}_+$. □

Example 3.1. Let $X = \mathbb{R}$ be the usual S-metric space as in Example 2.4. Now, we consider the mapping $\mathcal{T} \colon X \to X$ by $\mathcal{T}(x) = \frac{x}{7}$ for all $x \in [0,1]$. Then

$$\begin{aligned} S(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) &= |\mathcal{T}x - \mathcal{T}z| + |\mathcal{T}y - \mathcal{T}z| \\ &= \left|\left(\frac{x}{7}\right) - \left(\frac{z}{7}\right)\right| + \left|\left(\frac{y}{7}\right) - \left(\frac{z}{7}\right)\right| \\ &= \frac{1}{7}\big[|x-z| + |y-z|\big] \\ &\leqslant \frac{2}{7}\big[|x-z| + |y-z|\big] \\ &= \frac{2}{7}\, S(x,y,z) \\ &= \gamma\, S(x,y,z) \end{aligned}$$

where $\gamma = \frac{2}{7} < 1$. Thus $\mathcal{T}$ satisfies all the conditions of Corollary 3.1 and clearly $0 \in X$ is the unique fixed point of $\mathcal{T}$.

Example 3.2. Let $X = \mathbb{R}$ be the usual S-metric space as in Example 2.4. Now, we consider the mapping $\mathcal{T}_1, \mathcal{T}_2 \colon X \to X$ by $\mathcal{T}_1(x) = \frac{x}{2}$ and $\mathcal{T}_2(x) = 0$ for all $x \in [0,1]$. Then

$$\begin{aligned} S(\mathcal{T}_1 x, \mathcal{T}_1 y, \mathcal{T}_2 z) &= |\mathcal{T}_1 x - \mathcal{T}_2 z| + |\mathcal{T}_1 y - \mathcal{T}_2 z| \\ &= \left|\left(\frac{x}{2}\right) - 0\right| + \left|\left(\frac{y}{2}\right) - 0\right| \\ &= \frac{1}{2}\big[|x| + |y|\big] \end{aligned}$$

and

$$\begin{aligned} &\leqslant \frac{3}{4}\big[|x-z|+|y-z|\big] \\ &= \frac{3}{4}\,S(x,y,z) \\ &= \delta\,S(x,y,z) \end{aligned}$$

where $\delta = \frac{3}{4} < 1$. Thus $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy all the conditions of Corollary 3.2 and clearly $0 \in X$ is the unique common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

EXAMPLE 3.3. Let $X = [0,1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x=y=z, \\ \max\{x,y,z\} & \text{if otherwise.} \end{cases}$$

for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. Let $\mathcal{T}\colon X \to X$ be a mapping defined as $\mathcal{T}(x) = \frac{x}{3}$ for all $x \in X$.

Without loss of generality we may assume that $x > y > z$, then we have

$$S(\mathcal{T}x,\mathcal{T}y,\mathcal{T}z) = \max\Big\{\frac{x}{3},\frac{y}{3},\frac{z}{3}\Big\} = \frac{x}{3},$$

$$S(x,y,z) = \max\big\{x,y,z\big\} = x,$$

$$S(z,z,\mathcal{T}z) = \max\Big\{z,z,\frac{z}{3}\Big\} = z,$$

$$S(y,y,\mathcal{T}y) = \max\Big\{y,y,\frac{y}{3}\Big\} = y,$$

$$S(x,x,\mathcal{T}y) = \max\Big\{x,x,\frac{y}{3}\Big\} = x,$$

$$S(z,z,\mathcal{T}y) = \max\Big\{z,z,\frac{y}{3}\Big\} = z,$$

$$S(y,y,\mathcal{T}x) = \max\Big\{y,y,\frac{x}{3}\Big\} = y.$$

Now, we consider inequality (3.1), we have

$$S(\mathcal{T}x,\mathcal{T}y,\mathcal{T}z) = \frac{x}{3} \leqslant \phi\Big\{x,z,y,\Big(\frac{x+z+y}{3}\Big)\Big\}.$$

Since ϕ satisfies the condition $(R1)$, there exists $k \in (0,1)$ such that

$$\frac{x}{3} \leqslant k\,x,$$

or $k \geqslant \frac{1}{3}$. If we take $0 \leqslant k < 1$, then $\mathcal{T}$ satisfies all the conditions of Theorem 3.1. Hence, applying Theorem 3.1, $\mathcal{T}$ has a unique fixed point. Here it is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

EXAMPLE 3.4. Let $X = [0,1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x=y=z, \\ \max\{x,y,z\} & \text{if otherwise.} \end{cases}$$

for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. Let $\mathcal{T}_1, \mathcal{T}_2\colon X \to X$ be two mappings defined as $\mathcal{T}_1(x) = \frac{x}{4}$ and $\mathcal{T}_2(x) = \frac{x}{5}$ for all $x \in X$.

Without loss of generality we may assume that $x > y > z$, then we have

$$S(\mathcal{T}_1x, \mathcal{T}_1y, \mathcal{T}_2z) = \max\left\{\frac{x}{4}, \frac{y}{4}, \frac{z}{5}\right\} = \frac{x}{4},$$
$$S(x, y, z) = \max\{x, y, z\} = x,$$
$$S(z, z, \mathcal{T}_2z) = \max\left\{z, z, \frac{z}{5}\right\} = z,$$
$$S(y, y, \mathcal{T}_1y) = \max\left\{y, y, \frac{y}{4}\right\} = y,$$
$$S(x, x, \mathcal{T}_1y) = \max\left\{x, x, \frac{y}{4}\right\} = x,$$
$$S(z, z, \mathcal{T}_1y) = \max\left\{z, z, \frac{y}{4}\right\} = z,$$
$$S(y, y, \mathcal{T}_1x) = \max\left\{y, y, \frac{x}{4}\right\} = y.$$

Now, we consider inequality (3.4), we have

$$S(\mathcal{T}_1x, \mathcal{T}_1y, \mathcal{T}_2z) = \frac{x}{4} \leqslant \phi\Big\{x, z, y, \Big(\frac{x+z+y}{3}\Big)\Big\}.$$

Since ϕ satisfies the condition $(R1)$, there exists $k \in (0, 1)$ such that $\frac{x}{4} \leqslant k\,x$, or $k \geqslant \frac{1}{4}$. If we take $0 \leqslant k < 1$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy all the conditions of Theorem 3.2. Hence, applying Theorem 3.2, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique common fixed point. Here it is seen that $0 \in X$ is the unique common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

4. Conclusion

In this paper, we establish some fixed point and common fixed point theorems under an implicit relation in the framework of S-metric spaces. We support our results by some examples. Our results extend, unify and generalize several results from the existing literature.

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