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GRADINGS ON QUATERNION BLOCKS WITH THREE SIMPLE MODULES

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Abstract. In this paper we show that tame blocks of group algebras with quaternion defect groups and three isomorphism classes of simple modules can be non-trivially graded. We prove this by using the transfer of gradings via derived equivalences.

1. Introduction and preliminaries

Let A be an algebra over a field k. We say that A is a graded algebra if A is the direct sum of subspaces $A = \bigoplus_{i\in\mathbb{Z}} A_i$, such that $A_iA_j \subset A_{i+j}$, $i, j \in \mathbb{Z}$. The subspace A_i is said to be the homogeneous subspace of degree i. It is obvious that we can always trivially grade A by setting $A_0 = A$. A given grading is positive if $A_i = 0$ for all $i < 0$. In this paper we study the problem of existence of non-trivial gradings on blocks of group algebras with quaternion defect groups and three simple modules. We refer the reader to [**1**] for details on defect groups of blocks of group algebras.

Tame blocks of group algebras appear only for group algebras over fields of characteristic 2, so for the remainder of this paper we will assume that the field k has characteristic 2. All algebras in this paper are finite dimensional algebras over the field k, and all modules will be left modules (unless otherwise stated). The category of finite dimensional A–modules is denoted by A–mod and the full subcategory of finite dimensional projective A–modules is denoted by P_A. The derived category of bounded complexes over the category A–mod is denoted by $D^b(A)$, and the homotopy category of bounded complexes over P_A will be denoted by $K^b(P_A)$.

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For some algebras, such as group algebras, it is not obvious how one can construct non-trivial gradings on these algebras. More complex methods, such as transfer of gradings via derived and stable equivalences [**10, 5, 6**], had to be developed in order to introduce non-trivial gradings on certain blocks of group algebras. The aim of this paper is to show how one can use transfer of gradings via derived equivalences to grade tame blocks with quaternion defect groups and three simple modules.

Let A and B be two symmetric algebras over a field k and let us assume that A is a graded algebra. The following theorem is due to Rouquier.

THEOREM 1.1 ([**10**, Theorem 6.3]). *Let A and B be two symmetric algebras and assume that A is graded. Let T be a tilting complex of A-modules that induces a derived equivalence between A and B. Then there exists a grading on B and a structure of a graded complex T' on T, such that T' induces an equivalence between the derived categories of graded A-modules and graded B-modules.*

This theorem tells us that derived equivalences are compatible with gradings, that is, gradings can be transferred between symmetric algebras via derived equivalences.

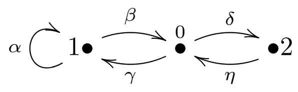
For a given tilting complex T of A-modules, which is a bounded complex of finitely generated projective A-modules, there exists a structure of a complex of graded A-modules T' on T. If T is a tilting complex that tilts from A to B, then $\mathrm{End}_{K^b(P_A)}(T) \cong B^{op}$. Viewing T as a graded complex T', and by computing its endomorphism ring as a graded object, we get a graded algebra which is isomorphic to the opposite algebra of the algebra B. We notice here that the choice of a grading on T' is unique up to shifting the grading of each indecomposable summand of T'. This follows from the fact that if we have two different gradings on an indecomposable module (bounded complex), then they differ only by a shift (see [3, Lemma 2.5.3]). We refer the reader to [**9, 5, 6**] for details on categories of graded modules and their derived categories, and [**11**] for basics of homological algebra.

If B is a block with a quaternion defect group of order 2^n and with three isomorphism classes of simple modules, then B is Morita equivalent to one of the algebras $Q(3A)_2^{2^{n-2}}$, $Q(3B)^{2^{n-2}}$, and $Q(3K)^{2^{n-2}}$ (cf. [**7**]):

(1) For any integer $r \geqslant 2$ let $Q(3A)_2 := Q(3A)_2^r$ be the algebra defined by the quiver and relations

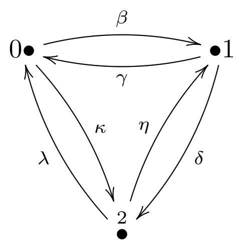
$$\begin{tikzcd} 1\bullet \arrow[r, bend left, "\beta"] & \overset{0}{\bullet} \arrow[l, bend left, "\gamma"] \arrow[r, bend left, "\delta"] & \bullet 2 \arrow[l, bend left, "\eta"] \end{tikzcd} \qquad \begin{aligned} \beta\gamma\beta &= (\beta\delta\eta\gamma)^{r-1}\beta\delta\eta, \\ \gamma\beta\gamma &= (\delta\eta\gamma\beta)^{r-1}\delta\eta\gamma, \\ \eta\delta\eta &= (\eta\gamma\beta\delta)^{r-1}\eta\gamma\beta,\ \beta\gamma\beta\delta = 0, \\ \delta\eta\delta &= (\gamma\beta\delta\eta)^{r-1}\gamma\beta\delta,\ \eta\delta\eta\gamma = 0. \end{aligned}$$

(2) For any integer $r \geqslant 3$ let $Q(3B) := Q(3B)^r$ be the algebra defined by the quiver and relations



$\beta\gamma = \alpha^{r-1}$, $\alpha\beta = \beta\delta\eta\gamma\beta\delta\eta$,
$\gamma\alpha = \delta\eta\gamma\beta\delta\eta\gamma$, $\eta\delta\eta = \eta\gamma\beta\delta\eta\gamma\beta$,
$\delta\eta\delta = \gamma\beta\delta\eta\gamma\beta\delta$, $\alpha^2\beta = 0$,
$\beta\delta\eta\delta = 0$.

(3) For any integer $r \geqslant 2$ let $Q(3K) := Q(3K)^r$ be the algebra defined by the quiver and relations



$\beta\delta = \kappa\lambda\kappa$, $\eta\gamma = \lambda\kappa\lambda$,
$\delta\lambda = \gamma\beta\gamma$,
$\lambda\beta = (\eta\delta)^{r-1}\eta$, $\gamma\kappa = (\delta\eta)^{r-1}\delta$,
$\kappa\eta = \beta\gamma\beta$, $\gamma\beta\delta = \delta\eta\gamma\lambda\kappa\eta = 0$.

We recommend [**2**] and [**4**] as a good introduction to path algebras of quivers.

Holm proved in [**8**] that the algebras $Q(3A)_2^{2^{n-2}}$, $Q(3B)^{2^{n-2}}$ and $Q(3K)^{2^{n-2}}$ are derived equivalent. In other words, it was proven that all blocks with common quaternion defect group and three isomorphism classes of simple modules are derived equivalent. We will use tilting complexes given in [**8**] to transfer gradings via derived equivalences from $Q(3A)_2$ to $Q(3B)$ and $Q(3K)$.

2. Transfer of gradings via derived equivalences

The rest of the paper is devoted to proving the following theorem.

THEOREM 2.1. *Let A be a tame block of a group algebra with quaternion defect groups and three isomorphism classes of simple modules. There exists a non-trivial grading on A.*

For the remainder of this paper, if we say that an algebra given by a quiver and relations is graded, we will assume that it is graded in such a way that the arrows and the vertices of its quiver are homogeneous. Also, right multiplication by a given path ρ will be denoted by the same letter ρ.

Let us assume that $Q(3A)_2$ is graded and that $\deg(\beta) = d_2$, $\deg(\gamma) = d_3$, $\deg(\delta) = d_4$ and $\deg(\eta) = d_5$ in the quiver of $Q(3A)_2$. We write Σ for $d_2+d_3+d_4+d_5$. Since the relations of $Q(3A)_2$ are homogeneous the following equations hold

$$\begin{aligned} 2d_2+2d_3 &= r\Sigma,\\ 2d_4+2d_5 &= r\Sigma. \end{aligned}$$

If we add these equations, then we get that $\Sigma = 0$, which implies that $d_3 = -d_2$ and $d_5 = -d_4$. This additional information will make our calculations easier.

REMARK 2.1. From $d_3 = -d_2$ and $d_5 = -d_4$ it follows that d_3 and d_2, and d_4 and d_5 cannot be positive simultaneously. Hence, there does not exist a non-trivial positive grading on $Q(3A)$ such that the arrows of its quiver are homogeneous elements.

Proposition 2.1. *The gradings on $Q(3A)$, such that the arrows of the quiver of $Q(3A)$ are homogeneous, are parameterized by $\mathbb{Z} \times \mathbb{Z}$.*

Proof. If we choose two integers d_2 and d_4 arbitrarily, then by setting $d_3 = -d_2$ and $d_5 = -d_4$ we obtain a non-trivial grading on $Q(3A)$. These gradings are parameterized by $\mathbb{Z} \times \mathbb{Z}$. $\square$

The graded radical layers of the projective indecomposable $Q(3A)_2$-modules are:

$$
\begin{array}{ccccc}
 & & S_0 & & \\
d_2 & S_1 & & S_2 & d_5 \\
0 & S_0 & & S_0 & 0 \\
d_5 & S_2 & & S_1 & d_2 \\
0 & S_0 & & S_0 & 0 \\
 & \vdots & & \vdots & \\
d_2 & S_1 & & S_2 & d_5 \\
0 & S_0 & & S_0 & 0 \\
d_5 & S_2 & & S_1 & d_2 \\
0 & & S_0 & &
\end{array}
,\qquad
\begin{array}{ccccc}
 & & S_1 & & \\
d_3 & & S_0 & & \\
d_3+d_5 & S_2 & & & \\
d_3 & S_0 & & & \\
0 & S_1 & & & \\
 & \vdots & & S_1 & 0 \\
d_3 & S_0 & & & \\
d_3+d_5 & S_2 & & & \\
d_3 & & S_0 & & \\
0 & & S_1 & &
\end{array}
,
$$

$$
\begin{array}{ccccc}
 & & S_2 & & \\
d_4 & & S_0 & & \\
d_4+d_2 & S_1 & & & \\
d_4 & S_0 & & & \\
0 & S_2 & & & \\
 & \vdots & & S_2 & 0 \\
d_4 & S_0 & & & \\
d_4+d_2 & S_1 & & & \\
d_4 & & S_0 & & \\
0 & & S_2 & &
\end{array}
.
$$

The integers that appear to the left and right of simple modules in the above radical layers denote homogeneous degrees of these simple modules.

With respect to this grading, the graded Cartan matrix of $Q(3A)_2$ is

$$
\begin{pmatrix}
4r & 2r\,q^{d_3} & 2r\,q^{d_4} \\
2r\,q^{d_2} & r+2 & r\,q^{d_2+d_4} \\
2r\,q^{d_5} & r\,q^{d_3+d_5} & r+2
\end{pmatrix}
$$

and its graded Cartan determinant is $16r$.

Let us now transfer gradings from $Q(3A)_2$ to $Q(3B)$. A tilting complex T, defined in [**8**, Section 4], that tilts from $Q(3A)_2$ to $Q(3B)$ (see [**8**, Section 4] for the proof that $\mathrm{End}_{K^b(P_{Q(3A)_2})}(T) \cong Q(3B)^{op}$) is given by $T := T_0 \oplus T_1 \oplus T_2$, where T_1

is the stalk complex with P_1 in degree 0, T_2 is given by

$$0 \longrightarrow P_1 \xrightarrow{\beta\delta} P_2\langle d_2 + d_4\rangle \longrightarrow 0\ ,$$

with non-zero terms in degrees 0 and 1, and T_0 is given by

$$0 \longrightarrow P_1\langle -d_2\rangle \oplus P_1\langle -d_2\rangle \xrightarrow{(\beta,\beta\delta\eta)} P_0 \longrightarrow 0\ ,$$

with non-zero terms in degrees 0 and 1.

The only summand of $\mathrm{Endgr}_{K^b(P_{Q(3A)_2})}(T_1)$ is $k\langle 0\rangle$. It follows that $\deg(\alpha) = 0$ in the quiver of $Q(3B)$. From this and the relations of $Q(3B)$ we conclude that $\deg(\gamma) = -\deg(\beta)$ and $\deg(\eta) = -\deg(\delta)$. This means that we are left to determine the degrees of β and δ in the quiver of $Q(3B)$.

It is obvious that

$$\mathrm{Homgr}_{K^b(P_{Q(3A)_2})}(T_1, T_2) \cong \mathrm{Homgr}_{Q(3A)_2}(P_1, \ker(\beta\delta)) \cong k\langle 0\rangle \oplus k\langle 0\rangle.$$

It follows that all paths in the quiver of $Q(3B)$ starting at vertex 1 and ending at vertex 2 are in degree 0. Because $\beta\gamma$ is one of these two paths it follows that $\deg(\beta) = -\deg(\delta)$. If $d = \deg(\beta)$, then $\deg(\gamma) = -d$, $\deg(\delta) = -d$ and $\deg(\eta) = d$.

Note that $\mathrm{Homgr}_{K^b(P_{Q(3A)_2})}(T_1, T_0) \cong \mathrm{Homgr}_{Q(3A)_2}(P_1, \ker(\beta, \beta\delta\eta))$. The latter space is isomorphic to $k\langle -d_2\rangle \oplus k\langle -d_2\rangle \oplus k\langle -d_2\rangle \oplus k\langle -d_2\rangle$. Hence, $d = \deg(\beta) = d_2$.

With respect to this grading, the graded quiver of $Q(3B)$ is given by

$$\begin{tikzcd} 1\bullet \arrow[loop left, "0"] \arrow[r, bend left, "d_2"] & \overset{0}{\bullet} \arrow[l, bend left, "-d_2"] \arrow[r, bend left, "-d_2"] & \bullet 2 \arrow[l, bend left, "d_2"] \end{tikzcd}$$

and we see that this grading only depends on our choice of d_2. The graded Cartan matrix of $Q(3B)$ with respect to this grading is

$$\begin{pmatrix} 8 & 4\,q^{-d_2} & 4\,q^{-d_2} \\ 4\,q^{d_2} & r+2 & 2 \\ 4\,q^{d_2} & 2 & 4 \end{pmatrix}$$

and its graded Cartan determinant is $16r$.

We proceed by transferring gradings from $Q(3A)_2$ to $Q(3K)$. A tilting complex T that tilts from $Q(3A)_2$ to $Q(3K)$ is given by $T := T_0 \oplus T_1 \oplus T_2$, where T_1 is the stalk complex with P_2 in degree 0, T_2 is the stalk complex with P_1 in degree 0, and T_0 is given by

$$0 \longrightarrow P_1\langle -d_2\rangle \oplus P_2\langle -d_5\rangle \xrightarrow{(\beta,\eta)} P_0 \longrightarrow 0\ ,$$

with non-zero terms in degrees 0 and 1.

Clearly,

$$\mathrm{Homgr}_{K^b(P_{Q(3A)_2})}(T_1, T_0) \cong \mathrm{Homgr}_{Q(3A)_2}(P_2, \ker(\beta, \eta)) \cong k\langle -d_5\rangle \oplus k\langle -d_5\rangle.$$

This means that $\deg(\gamma) = d_5$. Similarly, $\mathrm{Homgr}_{K^b(P_{Q(3A)_2})}(T_2, T_0) \cong k\langle -d_2\rangle$, and $\deg(\lambda) = d_2$ in the quiver of $Q(3K)$.

Since $\mathrm{Endgr}_{K^b(P_{Q(3A)_2})}(T_1)$ is isomorphic to the sum of $r+2$ copies of $k\langle 0\rangle$, the paths starting and ending at vertex 1 are in degree 0. Similarly, the graded vector space $\mathrm{Endgr}_{K^b(P_{Q(3A)_2})}(T_2)$ is isomorphic to the sum of $r+2$ copies of $k\langle 0\rangle$. The paths starting and ending at vertex 2 are in degree 0. Hence, $\deg(\beta) = -\deg(\gamma) = -d_5$, and $\deg(\kappa) = -\deg(\lambda) = -d_2$. Also, $\deg(\delta) = -\deg(\eta)$. From the relations of the algebra $Q(3K)$ we conclude that $\deg(\delta) = \deg(\kappa) - \deg(\beta) = -d_2 + d_5$. It follows that $\deg(\eta) = d_2 - d_5$.

With respect to this grading, the graded quiver of $Q(3K)$ is given by

$$\begin{tikzcd}
0\bullet \arrow[rr, bend left, "-d_5"] \arrow[dr, bend left, "-d_2"] & & \bullet 1 \arrow[ll, bend left, "d_5"] \arrow[dl, bend left, "-d_2+d_5"] \\
& \overset{2}{\bullet} \arrow[ul, bend left, "d_2"] \arrow[ur, bend left, "d_2-d_5"] &
\end{tikzcd}$$

This grading only depends on our choice of d_2 and d_5.
With respect to this grading, the graded Cartan matrix of $Q(3K)$ is given by

$$\begin{pmatrix} 4 & 2\,q^{-d_5} & 2\,q^{-d_2} \\ 2\,q^{d_5} & r+2 & r\,q^{-d_2+d_5} \\ 2\,q^{d_2} & r\,q^{d_2-d_5} & r+2 \end{pmatrix}$$

and its graded Cartan determinant is $16r$.

We see that for each quaternion block the resulting grading obtained by transfer of gradings via derived equivalences is a non-trivial grading for an appropriate choice of the homogeneous degrees of the arrows of an appropriate quiver. Thus, we have proved Theorem 2.1.

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