BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(1)(2021), 99-110
DOI: 10.7251/BIMVI2101099H

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SOME NEW FRACTIONAL INTEGRAL INEQUALITIES FOR HARMONICALLY h-CONVEX FUNCTIONS VIA CAPUTO k-FRACTIONAL DERIVATIVES

Rashida Hussain, Asghar Ali, Arzoo Ayub, and Asia Latif

Abstract. In present paper some Hermite-Hadamard type inequalities for harmonically h-convex functions via Caputo k-fractional derivatives are established. Moreover, some corresponding results for Caputo fractional derivatives are introduced. Also some inequalities involving Caputo k-fractional derivatives for convex functions are deduced.

1. Introduction

Convexity has been subject to comprehensive research for past few years due to its benefits in various branches of applied and pure mathematics. Several inequalities have been well-established by the researchers for convex functions, Hermite-Hadamard (HH) inequalities is the most celebrated one. It gives sufficient and necessary condition for a function to be convex.

It is stated as; let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function, then the subsequent inequalities hold

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. If f is concave the Inequalities (1.1) are reversed. Through the years various genelaizations and counter parts of HH inequalities have been developed for different classes of convex functions. In this paper HH inequalities are generalized for harmonically h-convex function. Harmonically h-convex functions

2010 *Mathematics Subject Classification.* 26A51, 26D10, 26A33.
Key words and phrases. Convex functions; Hermite-Hadamard inequalities; harmonically convex function; h-convex function; Caputo k-fractional derivatives.
Communicated by Daniel A. Romano.

were introduced by Noor et al. [**27**] as unification of harmonically convex functions [**13**] and h-convex functions [**28**]. For more work on harmonically convex functions and h-convex functions interested readers are referred to [**3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 18, 22, 24, 25, 26, 29**].

Fractional calculus deals with differentiation and integration of arbitrary order see [**2, 21, 23**]. In order to model real-life problems different fractional operators have been developed. The Caputo derivatives are often preferred for modelling initial value problems. Caputo derivatives are considered to be more natural as Caputo derivatives of constant is zero [**2**]. Many authors developed HH-type and Hermite-Hadamard-Fejer type inequalities using Caputo fractional derivatives see [**8, 19, 20, 29**] and references therein. The authors of this paper have developed said inequalities for k-fractional integrals [**1, 9, 10, 11, 12**], we are now motivated to formulate inequalities for Caputo k-fractional integrals. In the sequel some definition connected to our work are given.

Definition 1.1. ([**7**]) Let

$$\alpha > 0,\ k \geqslant 1,\ \alpha \notin \{1,2,3,...\},\ n = [\alpha]+1,\ f \in AC^n[a,b].$$

Then right and left Caputo k-fractional derivatives having order α are respectively given as follows

$$^{C}D^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt,\ x>a$$

and

$$^{C}D^{\alpha,k}_{b-}f(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt,\ x<b.$$

where $\Gamma_k(.)$ is the k-gamma function given by $\Gamma_k(\gamma)=\int_0^\infty t^{\gamma-1}e^{\frac{-t^k}{k}}dt$.

If $k=1$, the definition of Caputo k-fractional derivatives coincides with that of Caputo fractional derivatives. In addition if $\alpha \in \{1,2,3,...\}$, then

$$^{C}D^{\alpha,1}_{a+}f(x) = f^{(n)}(x) \text{ and } ^{C}D^{\alpha,1}_{b-}f(x) = (-1)^n f^{(n)}(x).$$

Further if $n=1$ and $\alpha=0$, then

$$^{C}D^{0,1}_{a+}f(x) = f(x) \text{ and } ^{C}D^{0,1}_{b-}f(x) = (-1)^n f(x).$$

G. Farid et al. [**8, 29**] introduced following notations.

Definition 1.2. Let $\alpha > 0$, $k \geqslant 1$, $\alpha \notin \{1,2,3,...\}$, $n=[\alpha]+1$, $f \in AC^n[a,b]$. Then right and left Caputo k-fractional derivatives of $f*g$ having order α are respectively given as follows

$$^{C}D^{\alpha,k}_{a+}(f*g)(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^x \frac{f^{(n)}(t)g^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt,\ x>a \tag{1.2}$$

and

$$^{C}D^{\alpha,k}_{b-}(f*g)(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})}\int_x^b \frac{f^{(n)}(t)g^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt,\ x<b. \tag{1.3}$$

In the sequel, I and $(0,1) \subseteq J$ are intervals in $\mathbb{R}$ and $[a,b]$ is in $\mathbb{R} \setminus \{0\}$.

In [**28**], Varošanec considered an important generalization of convex functions namely h-convex functions.

Definition 1.3. Let $h : J \longrightarrow \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. A function $f : I \longrightarrow \mathbb{R}$ is said to be h-convex if f is non-negative and

$$f\left(tx + (1-t)y\right) \leqslant h(t)f(y) + h(1-t)f(x), \quad \text{for all} \;\; x, y \in I \text{ and } t \in [0,1].$$

In [**13**], İşcan defined harmonically convex functions as follows.

Definition 1.4. A function $f : [a,b] \longrightarrow \mathbb{R}$ is harmonically convex if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leqslant tf(y) + (1-t)f(x) \quad \text{for all} \;\; x, y \in [a,b] \text{ and } t \in [0,1].$$

In [**27**] Noor et al. defined harmonically h-convex functions.

Definition 1.5. Let $h : J \longrightarrow \mathbb{R}$ be a positive function. A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be harmonically h-convex if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leqslant h(t)f(y) + h(1-t)f(x), \quad \text{for all} \;\; x, y \in [a,b] \text{ and } t \in [0,1].$$

Latif et al.[**22**] defined harmonically symmetric function as follows.

Definition 1.6. A function $g : [a,b] \longrightarrow \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if

$$g(x) = g\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - \frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

2. Main Results

In this section some new HH-type inequalities for harmonically h-convex functions via Caputo k-fractional derivatives are given. In addition, corresponding results for Caputo fractional derivatives are also deduced. Moreover some analogous inequalities involving Caputo k-fractional derivatives for convex functions are retrieved.

Theorem 2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable function such that $f^{(n)}$ is harmonically h-convex and $f^{(n)} \in L[a,b]$. If $g : [a,b] \longrightarrow \mathbb{R}$ is a function such that $g^{(n)}$ is non-negative, integrable and harmonically symmetric w.r.t. $\frac{2ab}{a+b}$, then the subsequent inequalities for Caputo k-fractional derivatives hold*

$$\begin{aligned}
&\frac{(-1)^n}{h\left(\frac{1}{2}\right)}\, f^{(n)}\left(\frac{2ab}{a+b}\right)\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}(g \circ r)\left(\frac{1}{b}\right) \\
&\leqslant \left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}((f * g) \circ r)\left(\frac{1}{a}\right) + (-1)^n\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}((f * g) \circ r)\left(\frac{1}{b}\right)\right] \\
&\leqslant [f^{(n)}(a) + f^{(n)}(b)] \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{\left(g^{(n)} \circ r\right)(x)}{\left(x - \frac{1}{b}\right)^{\frac{\alpha}{k} - n + 1}} \overline{h}(x)dx,
\end{aligned} \tag{2.1}$$

where $r(x)=\frac{1}{x}$ *and* $\overline{h}(x)=h\left(\frac{ab}{b-a}\left(x-\frac{1}{b}\right)\right)+h\left(\frac{ab}{b-a}\left(\frac{1}{a}-x\right)\right)$ *for all* $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

PROOF. Since $f^{(n)}$ is harmonically h-convex function. Substituting $t=\frac{1}{2}$ and $x=\frac{ab}{ta+(1-t)b}$, $y=\frac{ab}{tb+(1-t)a}$ in Inequality (1.5), we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\leqslant h\left(\frac{1}{2}\right)\left[f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)+f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)\right]. \tag{2.2}$$

Multiplying Inequality (2.2) by $\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}$ and then integrating w.r.t. t over $[0,1]$, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\int_0^1\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt$$

$$\leqslant h\left(\frac{1}{2}\right)\left[\int_0^1\frac{f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt+\int_0^1\frac{f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt\right].$$

Let $\frac{ab}{tb+(1-t)a}=\frac{1}{x}$, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx$$

$$\leqslant h\left(\frac{1}{2}\right)\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{f^{(n)}\left(\frac{1}{x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{f^{(n)}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx\right] \tag{2.3}$$

Let $\frac{1}{a}+\frac{1}{b}-x=y$ in second integral at right hand side of Inequality (2.3) and utilizing the fact that $g^{(n)}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx$$

$$\leqslant h\left(\frac{1}{2}\right)\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{f^{(n)}\left(\frac{1}{x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{f^{(n)}\left(\frac{1}{y}\right)g^{(n)}\left(\frac{1}{y}\right)}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy\right]$$

Now utilizing the Definition 1.2, we have

$$\frac{(-1)^n}{h\left(\frac{1}{2}\right)}\ f^{(n)}\left(\frac{2ab}{a+b}\right)\,{}^{C}D^{\alpha,k}_{\frac{1}{a}-}(g\circ r)\left(\frac{1}{b}\right)$$

$$\leqslant\left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}((f*g)\circ r)\left(\frac{1}{a}\right)+(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{1}{a}-}((f*g)\circ r)\left(\frac{1}{b}\right)\right]. \tag{2.4}$$

where $r(x)$ is defined in the statement.

Again employing the definition of harmonically convexity of the $f^{(n)}$ such that we have

$$f^{(n)}\left(\frac{ab}{tl+(1-t)e}\right)+f^{(n)}\left(\frac{ab}{te+(1-t)b}\right)\leqslant[f^{(n)}(a)+f^{(n)}(b)][h(t)+h(1-t)] \tag{2.5}$$

Multiplying Inequality (2.5) by $\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}$ and then integrating w.r.t. t over $[0,1]$, we obtain

$$\int_0^1 \frac{f^{(n)}\Big(\frac{ab}{tb+(1-t)a}\Big)\, g^{(n)}\Big(\frac{ab}{tb+(1-t)a}\Big)}{t^{\frac{\alpha}{k}-n+1}}dt + \int_0^1 \frac{f^{(n)}\Big(\frac{ab}{ta+(1-t)b}\Big)\, g^{(n)}\Big(\frac{ab}{tb+(1-t)a}\Big)}{t^{\frac{\alpha}{k}-n+1}}dt$$

$$\leqslant [f^{(n)}(a)+f^{(n)}(b)] \int_0^1 \frac{g^{(n)}\Big(\frac{ab}{tb+(1-t)a}\Big)}{t^{\frac{\alpha}{k}-n+1}}[h(t)+h(1-t)]dt$$

Let $\frac{ab}{tb+(1-t)a} = \frac{1}{x}$, we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \frac{f^{(n)}\left(\frac{1}{x}\right) g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx + \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{f^{(n)}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right) g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx$$

$$\leqslant [f^{(n)}(a)+f^{(n)}(b)] \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}} \left[h\Big(\frac{ab}{b-a}\Big(x-\frac{1}{b}\Big)\Big) + h\Big(\frac{ab}{b-a}\Big(\frac{1}{a}-x\Big)\Big)\right]dx \tag{2.6}$$

Let $\frac{1}{a}+\frac{1}{b}-x=y$ in second integral at left hand side of inequality (2.6) and utilizing the fact that $g^{(n)}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$, we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \frac{f^{(n)}\left(\frac{1}{x}\right) g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx + \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{f^{(n)}\left(\frac{1}{y}\right) g^{(n)}\left(\frac{1}{y}\right)}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy$$

$$\leqslant [f^{(n)}(a)+f^{(n)}(b)] \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx$$

where $\overline{h}(x)$ is given in the statement. Finally on employing the Definition 1.2, we have

$$\left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}\left((f*g)\circ r\right)\left(\frac{1}{a}\right) + (-1)^n\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}\left((f*g)\circ r\right)\left(\frac{1}{b}\right)\right]$$

$$\leqslant [f^{(n)}(a)+f^{(n)}(b)] \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{\left(g^{(n)}\circ r\right)(x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx \tag{2.7}$$

where $r(x)$ and $\overline{h}(x)$ are given in the statement.

On combining inequality (2.4) and inequality (2.7) we obtain the Inequalities (2.1). Hence proved. □

COROLLARY 2.1. *In Theorem 2.1, if we take $k=1$ then the subsequent inequalities hold*

$$\frac{(-1)^n}{h\left(\frac{1}{2}\right)}\, f^{(n)}\left(\frac{2ab}{a+b}\right)\, {}^{C}D^{\alpha}_{\frac{1}{a}-}\left(g\circ r\right)\left(\frac{1}{b}\right)$$

$$\leqslant \left[{}^{C}D^{\alpha}_{\frac{1}{b}+}\left((f*g)\circ r\right)\left(\frac{1}{a}\right) + (-1)^n\, {}^{C}D^{\alpha}_{\frac{1}{a}-}\left((f*g)\circ r\right)\left(\frac{1}{b}\right)\right]$$

$$\leqslant [f^{(n)}(a) + f^{(n)}(b)] \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{\left(g^{(n)} \circ r\right)(x)}{\left(x - \frac{1}{b}\right)^{\alpha - n + 1}} \overline{h}(x) dx.$$

Corollary 2.2. *In Theorem 2.1, if we take $h(x)$ to be the identity function i.e. $f^{(n)}(x)$ to be convex function, then* [Theorem 2.1, [**29**]] *is retrieved.*

Theorem 2.2. *Let $f : I \subset (0, \infty) \longrightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $e < l$. If $f^{(n)}$ is harmonically h-convex function on $[a, b]$, then the subsequent inequalities for Caputo k-fractional integrals hold*

$$\frac{1}{h\left(\frac{1}{2}\right)\left(n - \frac{\alpha}{k}\right)} \left(\frac{b-a}{ab}\right)^{n - \frac{\alpha}{k}} f^{(n)}\left(\frac{2ab}{a+b}\right)$$

$$\leqslant k\Gamma_k\left(n - \frac{\alpha}{k}\right)\left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}(f \circ r)\left(\frac{1}{a}\right) + (-1)^n\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}(f \circ r)\left(\frac{1}{b}\right)\right]$$

$$\leqslant [f^{(n)}(a) + f^{(n)}(b)] \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{1}{\left(x - \frac{1}{b}\right)^{\frac{\alpha}{k} - n + 1}} \overline{h}(x) dx, \tag{2.8}$$

where $r(x) = \frac{1}{x}$ and $\overline{h}(x) = h\left(\frac{ab}{b-a}\left(x - \frac{1}{b}\right)\right) + h\left(\frac{ab}{b-a}\left(\frac{1}{a} - x\right)\right)$ for all $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

Proof. Considering Inequality (2.2) Multiplying it by $\frac{1}{t^{\frac{\alpha}{k} - n + 1}}$ and integrating w.r.t. t over $[0, 1]$, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right) \int_0^1 t^{n - \left(1 + \frac{\alpha}{k}\right)} dt \leqslant h\left(\frac{1}{2}\right)\left[\int_0^1 \frac{f^{(n)}\left(\frac{ab}{tb + (1-t)a}\right)}{t^{\frac{\alpha}{k} - n + 1}} dt + \int_0^1 \frac{f^{(n)}\left(\frac{ab}{ta + (1-t)b}\right)}{t^{\frac{\alpha}{k} - n + 1}} dt\right].$$

Let $\frac{ab}{tb + (1-t)a} = \frac{1}{x}$ and $\frac{ab}{ta + (1-t)b} = \frac{1}{y}$, we have

$$f^{(n)}\left(\frac{2ab}{e+l}\right) \cdot \frac{1}{\left(n - \frac{\alpha}{k}\right)} \leqslant h\left(\frac{1}{2}\right)\left(\frac{ab}{l-e}\right)^{n - \frac{\alpha}{k}}\left[\int_{\frac{1}{b}}^{\frac{1}{a}} \frac{f^{(n)}\left(\frac{1}{x}\right)}{\left(x - \frac{1}{b}\right)^{\frac{\alpha}{k} - n + 1}} dx + \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{f^{(n)}\left(\frac{1}{y}\right)}{\left(\frac{1}{a} - y\right)^{\frac{\alpha}{k} - n + 1}} dy\right]$$

On employing the Definition 1.1, we have

$$\frac{1}{h\left(\frac{1}{2}\right)\left(n - \frac{\alpha}{k}\right)} \left(\frac{b-a}{ab}\right)^{n - \frac{\alpha}{k}} f^{(n)}\left(\frac{2ab}{a+b}\right)$$

$$\leqslant k\Gamma_k\left(n - \frac{\alpha}{k}\right)\left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}(f \circ r)\left(\frac{1}{a}\right) + (-1)^n\, {}^{C}D^{\alpha,k}_{\frac{1}{a}-}(f \circ r)\left(\frac{1}{b}\right)\right] \tag{2.9}$$

where $r(x)$ is given in the statement.

Considering Inequality (2.5) and multiplying it by $\frac{1}{t^{\frac{\alpha}{k} - n + 1}}$ and then integrating w.r.t. t over $[0, 1]$, we have

$$\int_0^1 \frac{f^{(n)}\left(\frac{ab}{tb + (1-t)a}\right)}{t^{\frac{\alpha}{k} - n + 1}} dt + \int_0^1 \frac{f^{(n)}\left(\frac{ab}{ta + (1-t)b}\right)}{t^{\frac{\alpha}{k} - n + 1}} dt \leqslant [f^{(n)}(a) + f^{(n)}(b)] \int_0^1 \frac{1}{t^{\frac{\alpha}{k} - n + 1}} [h(t) + h(1-t)] dt$$

Using change of variable $\frac{ab}{tb+(1-t)a}=\frac{1}{x}$ and $\frac{ab}{ta+(1-t)b}=\frac{1}{y}$, we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{f^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{f^{(n)}\left(\frac{1}{y}\right)}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_{0}^{1}\frac{1}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}\left[h\left(\frac{ab}{l-e}\left(x-\frac{1}{b}\right)\right)+h\left(\frac{ab}{l-e}\left(\frac{1}{a}-x\right)\right)\right]dt \tag{2.10}$$

Now employing Definition 1.1, we have

$$k\Gamma_k\left(n-\frac{\alpha}{k}\right)\left[{}^{C}D^{\alpha,k}_{\frac{1}{b}+}\ (f\circ r)\left(\frac{1}{a}\right)+(-1)^n\ {}^{C}D^{\alpha,k}_{\frac{1}{a}-}\ (f\circ r)\left(\frac{1}{b}\right)\right]$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{1}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx. \tag{2.11}$$

where $r(x)$ and $\overline{h}(x)$ are given in the statement.

On combining inequality (2.9) and inequality (2.11), we get the required inequalities. Hence proved. □

Corollary 2.3. *In Theorem 2.2, if we take $k=1$, then the subsequent inequalities hold*

$$\frac{1}{h\left(\frac{1}{2}\right)(n-\alpha)}\left(\frac{b-a}{ab}\right)^{n-\alpha}f^{(n)}\left(\frac{2ab}{a+b}\right)$$

$$\leqslant\Gamma(n-\alpha)\left[{}^{C}D^{\alpha}_{\frac{1}{b}+}\ (f\circ r)\left(\frac{1}{a}\right)+(-1)^n\ {}^{C}D^{\alpha}_{\frac{1}{a}-}\ (f\circ r)\left(\frac{1}{b}\right)\right]$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{1}{\left(x-\frac{1}{b}\right)^{\alpha-n+1}}\overline{h}(x)dx. \tag{2.12}$$

Corollary 2.4. *In Theorem 2.2, if we take $h(x)$ to be the identity function i.e. $f^{(n)}(x)$ to be convex function, then* [Theorem 2.2, [**29**]] *is retrieved.*

Theorem 2.3. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a function such that $f^{(n)}$ is a harmonically h-convex, where $e<l$ and $f^{(n)}\in L[a,b]$. If $g:[a,b]\longrightarrow\mathbb{R}$ is a function such that $g^{(n)}$ is a non-negative, integrable and harmonically symmetric w.r.t. $\frac{2ab}{a+b}$, then the subsequent inequalities for Caputo k-fractional integrals hold*

$$\frac{(-1)^n}{h\left(\frac{1}{2}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right){}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}(g\circ r)\left(\frac{1}{b}\right)$$

$$\leqslant\left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}\ ((f*g)\circ r)\left(\frac{1}{a}\right)+(-1)^n\ {}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}\ ((f*g)\circ r)\left(\frac{1}{b}\right)\right]$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{\left(g^{(n)}\circ r\right)(x)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx, \tag{2.13}$$

where $r(x)=\frac{1}{x}$ and $\overline{h}(x)=h\left(\frac{ab}{b-a}\left(x-\frac{1}{b}\right)\right)+h\left(\frac{ab}{b-a}\left(\frac{1}{a}-x\right)\right)$ for all $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

PROOF. Since $f^{(n)}$ is harmonically h-convex therefore again considering inequality (2.2). Multiplying (2.2) by $\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}$ and integrating w.r.t. t over $\left[0,\frac{1}{2}\right]$, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\int_0^{\frac{1}{2}}\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt$$

$$\leqslant h\Big(\frac{1}{2}\Big)\Bigg[\int_0^{\frac{1}{2}}\frac{f^{(n)}\Big(\frac{ab}{tb+(1-t)a}\Big)\,g^{(n)}\Big(\frac{ab}{tb+(1-t)a}\Big)}{t^{\frac{\alpha}{k}-n+1}}dt+\int_0^{\frac{1}{2}}\frac{f^{(n)}\Big(\frac{ab}{ta+(1-t)b}\Big)\,g^{(n)}\Big(\frac{ab}{tb+(1-t)a}\Big)}{t^{\frac{\alpha}{k}-n+1}}dt\Bigg]$$

Let $\frac{ab}{tb+(1-t)a}=\frac{1}{x}$ in above inequality, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx$$

$$\leqslant h\left(\frac{1}{2}\right)\left[\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(\frac{1}{x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx\right]. \tag{2.14}$$

On right hand side of Inequality (2.14) putting $\frac{1}{a}+\frac{1}{b}-x=y$ in second integral and utilizing the fact that $g^{(n)}$ is harmonically symmetric w.r.t. $\frac{2ab}{a+b}$, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx$$

$$\leqslant h\left(\frac{1}{2}\right)\left[\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(\frac{1}{x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(\frac{1}{y}\right)g^{(n)}\left(\frac{1}{y}\right)}{\left(\frac{1}{a}-y\right)^{\frac{\alpha}{k}-n+1}}dy\right]$$

By using the Definition 1.2, we have

$$\frac{(-1)^n}{h\left(\frac{1}{2}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right){}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}(g\circ r)\left(\frac{1}{b}\right)$$

$$\leqslant\left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}((f*g)\circ r)\left(\frac{1}{a}\right)+(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}((f*g)\circ r)\left(\frac{1}{b}\right)\right]. \tag{2.15}$$

Now multiplying inequality (2.5) by $\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}$ and integrating w.r.t. t over $\left[0,\frac{1}{2}\right]$, we have

$$\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt+\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_0^{\frac{1}{2}}\frac{g^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}[h(t)+h(1-t)]dt.$$

Let $\frac{ab}{tb+(1-t)a}=\frac{1}{x}$, we have

$$\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(\frac{1}{x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g^{(n)}\left(\frac{1}{x}\right)}{\left(x-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dx$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{g^{(n)}(\frac{1}{x})}{(x-\frac{1}{b})^{\frac{\alpha}{k}-n+1}}\left[h\left(\frac{ab}{b-a}\left(x-\frac{1}{b}\right)\right)+h\left(\frac{ab}{b-a}\left(\frac{1}{a}-x\right)\right)\right]dx \tag{2.16}$$

Using the change of variable $\frac{1}{a}+\frac{1}{b}-x=y$ and harmonically symmetry of $g^{(n)}$ and definition in the second integral term on left side of inequality (2.16). Then applying Definition 1.2, we have

$$\left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}\left((f*g)\circ r\right)\left(\frac{1}{a}\right)+(-1)^{n}\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}\left((f*g)\circ r\right)\left(\frac{1}{b}\right)\right]$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{\left(g^{(n)}\circ r\right)(x)}{(x-\frac{1}{b})^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx. \tag{2.17}$$

where $r(x)$ and $\overline{h}(x)$ are given in the statement. On combining inequality (2.15) and inequality (2.17) we get the required Inequalities (2.13). □

COROLLARY 2.5. *In Theorem 2.1, if we take $k=1$ then the subsequent inequalities hold*

$$\frac{1}{h\left(\frac{1}{2}\right)}(-1)^{n}f^{(n)}\left(\frac{2ab}{a+b}\right){}^{C}D^{\alpha}_{\frac{a+b}{2ab}-}(g\circ r)\left(\frac{1}{b}\right)$$

$$\leqslant\left[{}^{C}D^{\alpha}_{\frac{a+b}{2ab}+}\left((f*g)\circ r\right)\left(\frac{1}{a}\right)+(-1)^{n}\,{}^{C}D^{\alpha}_{\frac{a+b}{2ab}-}\left((f*g)\circ r\right)\left(\frac{1}{b}\right)\right]$$

$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{\left(g^{(n)}\circ r\right)(x)}{\left(x-\frac{1}{b}\right)^{\alpha-n+1}}\overline{h}(x)dx. \tag{2.18}$$

COROLLARY 2.6. *In Theorem 2.3, if we take $h(x)$ to be the identity function i.e. $f^{(n)}(x)$ to be convex function, then* [Theorem 2.3, [**29**]] *is retrieved.*

THEOREM 2.4. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a function such that $f^{(n)}\in L[a,b]$, $a,b\in I$. If $f^{(n)}$ is harmonically h-convex function on $[a,b]$, then the subsequent inequalities for Caputo k-fractional integrals hold*

$$\frac{2^{\frac{\alpha}{k}-n}\left(\frac{b-a}{ab}\right)^{n-\frac{\alpha}{k}}}{h\left(\frac{1}{2}\right)\left(n-\frac{\alpha}{k}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right)$$

$$\leqslant k\Gamma_{k}\left(n-\frac{\alpha}{k}\right)\left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}(f\circ r)\left(\frac{1}{a}\right)+(-1)^{n}\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}(f\circ r)\left(\frac{1}{b}\right)\right]$$

$$\leqslant\left[f^{(n)}(a)+f^{(n)}(b)\right]\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\frac{1}{\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx, \tag{2.19}$$

where $r(x)=\frac{1}{x}$ *and* $\overline{h}(x)=h\left(\frac{ab}{b-a}\left(x-\frac{1}{b}\right)\right)+h\left(\frac{ab}{b-a}\left(\frac{1}{a}-x\right)\right)$ *for all* $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

PROOF. Multiplying Inequality (2.2) by $\frac{1}{t^{\frac{\alpha}{k}-n+1}}$ and integrating w.r.t. t over $\left[0,\frac{1}{2}\right]$, we have

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\int_0^{\frac{1}{2}}t^{n-\left(1+\frac{\alpha}{k}\right)}dt\leqslant h\left(\frac{1}{2}\right)\left[\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt+\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)}{t^{\frac{\alpha}{k}-n+1}}dt\right]$$

$$f^{(n)}\left(\frac{2ab}{a+b}\right)\cdot\frac{2^{\frac{\alpha}{k}-n}}{\left(n-\frac{\alpha}{k}\right)}\leqslant h\left(\frac{1}{2}\right)\left[\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt+\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)}{t^{\frac{\alpha}{k}-n+1}}dt\right]$$

Let $\frac{ab}{tb+(1-t)a}=\frac{1}{x}$ and $\frac{ab}{ta+(1-t)b}=\frac{1}{y}$, we have

$$\frac{2^{\frac{\alpha}{k}-n}}{\left(n-\frac{\alpha}{k}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right)$$
$$\leqslant h\left(\frac{1}{2}\right)\left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}}\left[\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\frac{f^{(n)}\left(\frac{1}{x}\right)}{\left(\frac{1}{a}-x\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}\left(\frac{1}{y}\right)}{\left(y-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dy\right]$$

$$\frac{2^{\frac{\alpha}{k}-n}}{\left(n-\frac{\alpha}{k}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right)$$
$$\leqslant h\left(\frac{1}{2}\right)\left(\frac{ab}{b-a}\right)^{n-\frac{\alpha}{k}}\left[\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\frac{f^{(n)}(r(x))}{\left(\frac{1}{a}-x\right)^{\frac{\alpha}{k}-n+1}}dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\frac{f^{(n)}(r(y))}{\left(y-\frac{1}{b}\right)^{\frac{\alpha}{k}-n+1}}dy\right]$$

Appling the definition of Caputo k-fractional derivative in above inequality, we have

$$\frac{2^{\frac{\alpha}{k}-n}\left(\frac{b-a}{ab}\right)^{n-\frac{\alpha}{k}}}{h\left(\frac{1}{2}\right)\left(n-\frac{\alpha}{k}\right)}f^{(n)}\left(\frac{2ab}{a+b}\right)$$

$$\leqslant k\Gamma_k\left(n-\frac{\alpha}{k}\right)\left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}(f\circ r)\left(\frac{1}{a}\right)+(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}(f\circ r)\left(\frac{1}{b}\right)\right]\tag{2.20}$$

where $r(x)$ is given in the statement.

Now multiplying Inequality (2.5) with $\frac{1}{t^{\frac{\alpha}{k}-n+1}}$ and integrating w.r.t. t over $\left[0,\frac{1}{2}\right]$, we have

$$\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{tb+(1-t)a}\right)}{t^{\frac{\alpha}{k}-n+1}}dt+\int_0^{\frac{1}{2}}\frac{f^{(n)}\left(\frac{ab}{ta+(1-t)b}\right)}{t^{\frac{\alpha}{k}-n+1}}dt$$
$$\leqslant[f^{(n)}(a)+f^{(n)}(b)]\int_0^{\frac{1}{2}}\frac{1}{t^{\frac{\alpha}{k}-n+1}}[h(t)+h(1-t)]dt.$$

Using change of variables $\frac{ab}{tb+(1-t)a}=\frac{1}{x}$ and $\frac{ab}{tb+(1-t)a}=\frac{1}{y}$. Then the Definition 1.1, we have

$$k\Gamma_k\left(n-\frac{\alpha}{k}\right)\left[{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}+}(f\circ r)\left(\frac{1}{a}\right)+(-1)^n\,{}^{C}D^{\alpha,k}_{\frac{a+b}{2ab}-}(f\circ r)\left(\frac{1}{b}\right)\right]$$

$$\leqslant \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\frac{1}{\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx. \tag{2.21}$$

where $r(x)$ and $\overline{h}(x)$ are given in the statement.

On combining inequality (2.20) and inequality (2.21) we get Inequalities(2.19). □

COROLLARY 2.7. *In Theorem 2.1, if we take $k=1$ then the subsequent inequalities hold for Caputo fractional derivatives*

$$\frac{2^{\alpha-n}\left(\frac{b-a}{ab}\right)^{n-\alpha}}{h\left(\frac{1}{2}\right)(n-\alpha)}f^{(n)}\left(\frac{2ab}{a+b}\right)$$
$$\leqslant \Gamma(n-\alpha)\left[{}^{C}D^{\alpha}_{\frac{a+b}{2ab}+}(f\circ r)\left(\frac{1}{a}\right)+(-1)^{n}\,{}^{C}D^{\alpha}_{\frac{a+b}{2ab}-}(f\circ r)\left(\frac{1}{b}\right)\right]$$
$$\leqslant \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\frac{1}{\left(\frac{1}{b}-x\right)^{\alpha-n+1}}\overline{h}(x)dx.$$

COROLLARY 2.8. *In Theorem 2.4, if we take $h(x)$ to be the identity function i.e. $f^{(n)}(x)$ to be convex function, then* [Theorem 2.4, [**29**]] *is retrieved.*

References

[1] A. Ali, G. Ghulshan, R. Hussain, A. Latif and M. Muddassar. Generalized inequalities of the type of Hermite-Hadamard-Fejer with quasi convex functions by way of k-fractional derivative. *J. Comput. Anal. Appl.*, **22**(7)(2017), 1208–1219.

[2] R. Almeida, D. Tavares and D. F. M. Torres. *The Variable-Order Fractional Calculus of Variations.* SpringerBriefs Appl. Sci. Technol., Springer, Cham, 2019.

[3] M. Bombardelli and S. Varošanec. Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.*, **58**(9)(2009), 1869–1877.

[4] P. Burai and A. Hazy. On approximately h-convex functions. *J. Convex Anal.*, **18**(2)(2011), 447–454.

[5] G. Cristescu, M. A. Noor and M. U. Awan. Bounds of the second degree cumulative frontier gaps of functions with generalized convexity. *Carpathian J. Math.*, **31**(2)(2015), 173–180.

[6] S. S. Dragomir. Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces. *Proyecciones J. Math.*, **34**(4)(2015), 323–341.

[7] G. Farid, A. Javed, A. Ur. Rehman and M. I. Qureshi. On Hadamard-type inequalities for differentiable functions via Caputo k-fractional derivatives. *Cogent Math.*, **4**(2017), 135–429.

[8] G. Farid and A. Javed. On Hadamard and Fejér-Hadamard inequalities for Caputo k-fractional derivatives. *Int. J. Nonlinear Anal. Appl.*, **9**(1)(2018), 69–81.

[9] R. Hussain, A. Ali, G. Ghulshan and A. Latif. Hermite-Hadamard type inequalities for k-Riemann-Liouville fractional integrals via two kinds of convexity. *Aust. J. Math. Anal. Appl. (AJMAA)*, **13**(1)(2016), 1–12.

[10] R. Hussain, A. Ali, A. Latif and G. Ghulshan. Some k-fractional associates of hermite-hadamard's inequality for quasi-convex functions and applications to special means. *Fractional Differ. Calc.*, **7**(2)(2017), 301–309.

[11] R. Hussain, A. Ali, K. Rauf, Bareera Kehkashan and A. Latif. Fractional Inequalities of hermite-Hadamard type for h-convex, co-ordinated h-convex and s-convex functons. *Ilirias J. Math.*, **5**(1)(2016), 12–20.

[12] R. Hussain, B. Kehkashan, A. Ali, A. Latif and K. Rauf. Some generalized k-fractional companions of Hadamard's inequality. *Nig. J. Math. Appl.*, **25**(1)(2016), 8–18.

[13] I. İşcan. Hermite-Hadamard's type inequalities for harmonic convex functions. *Hacet. J. Math. Stat.*, **43**(6)(2014), 935–942.
[14] I. İ şcan, and Wu. Shanhe. Hermite-Hadamard type inequalities for harmonic convex functions via fractional integrals. *J. Appl. Math. Comput.*, **238**(2014), 237–244.
[15] I. İşcan. Hermite-Hadamard type inequalities for harmonic convex functions. *Hacet. J. Math. Stat.*, **43**(2014), 935–942.
[16] I. İşcan. *Hermite-Hadamard type inequalities for harmonic* $(\alpha; m)$*-convex functions.* 17 Apr 2015. (http://arxiv.org/abs/1307.5402).
[17] I. İşcan, M. Kunt and N. Yazici. Hermite-Hadamard-Fejer type inequalities for harmonic convex functions via fractional integrals. *New Trends in Math. Sci.*, **4**(3)(2016), 239–253.
[18] I. İşcan. Some new general integral inequalities for h-convex and h-concave functions, *Adv. Pure. Appl. Math.*, **5**(1)(2014), 21–29.
[19] C. Li, D. Qian and Y. Q. Chen. On Riemann-Liouville and Caputo Derivatives. *Hindawi Publishing Corporation, Discrete Dyn. Nat. Soc.*, **2011** 15 pages.Article ID 562494, (doi:10.1155/2011/562494)
[20] S. M. Kang, G. Farid, W. Nazeer and S. Naqvi. Aversion of the Hadamard inequality for Caputo fractional derivatives and related results. *J. Comput. Anal. Appl.*, **27**(6)(2019), 962–972.
[21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo. *Theory and applications of fractional differential equations.* North-Holland Mathematics Studies, Elsevier, New York-London, **204**(2006).
[22] M. A. Latif, S. S. Dragomir and E. Momoniant, Some Fejér type inequalities for harmonic-convex functions with applications to special means. *Int. J. Anal. Appl.*, **13**(1)(2017), 1–14.
[23] S. Miller and B. Ross. *An introduction to fractional calculus and fractional differential equations.* John Wiley & Sons, New York, NY, USA, **2**, (1993).
[24] N. P. N. Ngoc, N. V. Vinh and P. T. T. Hien. Integral Inequalities of Hadamard Type for r-Convex Functions. *Int. Math. Forum*, **4**(35)(2009), 1723–1728.
[25] C. P. Niculescu. Convexity according to the geometric mean. *Math. Inequal. Appl.*, **3**(2)(2000), 155–167.
[26] C. P. Niculescu. Convexity according to means. *Math. Inequal. Appl.*, **4**(6)(2003), 571–579.
[27] M. A. Noor, K. I. Noor, M. U. Awan and S. Costache. Some integral inequalities for harmonic h-convex functions. *U.P.B. Sci. Bull. Series A*, **77**(1)(2015), 5–16.
[28] S. Varošanec. On h-convexity. *J. Math. Anal. Appl.*, **326**(2007), 303–311.
[29] A. Waheed, G. Farid, A. U. Rehman and W. Ayub. k-Fractional Integral Inequalities for harmonic convex functions via Caputo k-fractional Integral Derivatives. *Bull. Math. Anal. Appl.*, **10**(1)(2018), 55–67.

Received by editors 28.10.2019; Revised version 11.07.2020; Available online 20.07.2020.

Department of Mathematics, Mirpur University Of Science And Technology, Mirpur, Pakistan
E-mail address: drrashida@must.edu.pk
E-mail address: drrali.maths@must.edu.pk
E-mail address: arzooay7@gmail.com
E-mail address: asialatif.maths@must.edu.pk