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ON VARIOUS TRI-IDEALS IN TERNARY SEMIRINGS

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Abstract. In this paper we discuss the structure of tri quasi-ideals and bi-quasi ideals in ternary semiring and give some characterizations in terms of tri-ideals and quasi ideals in ternary semirings. m tri-ideals are the generalization of tri-ideals which are themselves the generalization of the left tri-ideals, right tri-ideals and lateral tri-ideals. The important properties of the m tri-ideals from the algebraic point of view have been described. The intersection of left, lateral and right ideals is a tri-ideal and product of left, lateral and right ideals is a tri-ideal. Also we discuss m-tri ideals towards some characterizations in terms of tri-ideals and its generators. Let I be a nonempty subset of R. Then the m tri-ideal generated by I is $< I >_m= \sum\limits_{i=1}^{m} N_0 I^i + (R^m I) + (R^m I R^m) + (I R^m)$. Some relevant counter examples are also indicated.

1. Introduction

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, combinatorics, functional analysis, graph theory, Euclidean geometry, probability theory, commutative and non commutative ring theory, optimization theory, discrete event dynamical systems, automata theory, form al language theory and the mathematical modeling of quantum physics and parallel computation systems and the like. Ternary semiring constitute a fairly natural generalization of semirings. There is a large literature dealing with ternary algebra and it appears more or less naturally in various domains of theoretical and mathematical physics.

D. H. Lehmer initiated the concept of ternary algebraic systems called triplexes in 1932 [**9**]. He investigated certain ternary algebraic systems called triplexes which

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turn out to be commutative ternary groups. The notion of semiring was introduced by Vandiver in 1934. In 1962, Hestenes [**6**] studied the notion of ternary algebra with application to matrices and linear transformation. In 1971, Lister characterized those additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. A great deal of research has been done and is being done in the area of ternary algebra The concept of bi ideals for associative rings were introduced by Lajos and Szasz [**8**]. Quasi-ideals are generalization of right ideals, lateral ideals, and left ideals whereas bi-ideals are generalization of quasi-ideals.

The results in ordinary semirings may be extended to n-ary semirings for arbitrary n but the transition from $n = 3$ to arbitrary n entails a great degree of complexity that makes it undesirable for exposition. The ring of integers Z which plays a role in the ring theory. The subset Z^+ of Z is an additive semigroup which is closed under the ring product, that is Z^+ is a semiring. Now, if we consider the subset Z^- of Z, then we see that Z^- is an additive semigroup which is closed under the triple ring product, that is Z^- forms a ternary semiring.

2. Preliminaries

From now onward, unless stated otherwise, R will denotes a simple Ternary Semiring.

Definition 2.1. ([**3**]) A non empty set R together with a binary operation called addition and ternary multiplication, denoted by juxtaposition ([]) is said to be a ternary semiring if R is an additive commutative semigroup and for all $a, b, c, d, e \in R$ satisfying

i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
ii) $[(a + b)cd] = [acd] + [bcd]$,
iii) $[a(b + c)d] = [abd] + [acd]$,
iv) $[ab(c + d)] = [abc] + [abd]$.

Definition 2.2. ([**3**]) An additive subsemigroup T of R is called a ternary subsemiring if $[t_1t_2t_3] \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.3. ([**4**]) An additive subsemigroup I of R is called a left (right, lateral) ideal of R if $s_1s_2i \in I$ ($is_1s_2 \in I$, $s_1is_2 \in I$ respectively) for all $s_1, s_2 \in R$ and $i \in I$. If I is a left, right and lateral ideal of R, then I is called an ideal of R.

Definition 2.4. ([**4**]) (i) An additive subsemigroup Q of R is called a quasi-ideal if $QRR \cap (RQR + RRQRR) \cap RRQ \subseteq Q$.

(ii) A ternary subsemiring B of R is called a bi-ideal if $BRBRB \subseteq B$.

Definition 2.5. ([**11**]) (i) A subsemiring L of semirings S is called an l-left ideal if $S^lL \subseteq L$.

(ii) A subsemiring N of semirings S is called an n-right ideal if $NS^n \subseteq N$.

where l, n are positive integers.

Remark 2.1. ([**11**]) For a semiring S and a positive integer m, $S^m = SS...S$ (m-times). Now $S^2 = SS \subseteq S$ and $S^3 = SSS \subseteq S^2 \subseteq S$. We conclude that

$S^l \subseteq S^m$ for all positive integers l and m, such that $l \geqslant m$. Consequently $S^m \subseteq S$ for all m.

Definition 2.6. ([**14**]) (i) A non-empty subset B of a semiring S is said to be right tri-ideal of S if B is a subsemiring of S and $BBSB \subseteq B$.

(ii) A non-empty subset B of a semiring S is said to be left tri-ideal of S if B is a subsemiring of S and $BSBB \subseteq B$.

(iii) A non-empty subset B of a semiring S is said to be tri-ideal of S if B is a right tri-ideal and left tri-ideal of S.

Definition 2.7. ([**14**]) A non empty subset B of a semiring S is called a
(i) left bi-quasi ideal of S if B is a subsemiring of S and $SB \cap BSB \subseteq B$.
(ii) right bi-quasi ideal of S if B is a subsemiring of S and $BS \cap BSB \subseteq B$.
(iii) bi-quasi ideal of S if B is a left bi-quasi ideal and right bi-quasi ideal of S.

3. Tri ideals in ternary Semirings

We introduce the concept of m-bi ideals and m-quasi ideals in ternary semirings.

Definition 3.1. A non empty subset B of a ternary semiring R is called a
(i) left tri-ideal if B is a sub ternary semiring of R and $BBRRBBB \subseteq B$.
(ii) lateral tri-ideal if B is a sub ternary semiring of R and $BBRBRBB \subseteq B$.
(iii) right tri-ideal if B is a sub ternary semiring of R and $BBBRRBB \subseteq B$.
(iv) tri-ideal if B is a left tri-ideal, lateral tri-ideal and right tri-ideal.

Theorem 3.1. *Every left (lateral, right) ideal is a left (lateral, right) tri-ideal.*

Converse of the Theorem 3.1 is need not be true which can be illustrated as follows.

Example 3.1. Consider the simple ternary semirings with binary operation usual addition and ternary usual multiplication.

$$R_1 = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & x_4 & x_5 \\ 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers, } i = 1, 2, ..., 6 \right\}$$

and

$$R_2 = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 & x_9 \\ 0 & 0 & 0 & x_{10} & x_{11} & x_{12} \\ 0 & 0 & 0 & 0 & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers, } i = 1, ..., 15 \right\}$$

Let

$$B_1 = \left\{ \begin{pmatrix} 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers } i = 1, 6 \right\}$$

and

$$B_2 = \left\{ \begin{pmatrix} 0 & x_1 & 0 & x_3 & 0 & x_5 \\ 0 & 0 & x_6 & 0 & 0 & x_9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers} \right\}.$$

Hence B_1 is a left and right tri-ideal of R_1 but not left and right ideal of R_1 and B_2 is a lateral tri-ideal of R_2, but not lateral ideal of R_2.

THEOREM 3.2. *Let R be a ternary semiring. Then the following are hold.*

(i) *If L is a left ideal , L_1 is a lateral ideal and L_2 is a right ideal of R, then $L \cap L_1 \cap L_2$ is a tri-ideal of R.*

(ii) *If L is a left ideal , L_1 is a lateral ideal and L_2 is a right ideal of R, then $L \cdot L_1 \cdot L_2$ is a tri-ideal of R.*

DEFINITION 3.2. A non empty subset B of R is called

(i) a left bi-quasi ideal of R if B is a sub ternary semiring of R and $RRB \cap BRBRB \subseteq B$.

(ii) a lateral bi-quasi ideal of R if B is a sub ternary semiring of R and $RBR \cap BRBRB \subseteq B$.

(iii) a right bi-quasi ideal of R if B is a sub ternary semiring of R and $BRR \cap BRBRB \subseteq B$.

(iv) a bi-quasi ideal of R if B is a left bi-quasi ideal, lateral bi-quasi ideal and right bi-quasi ideal.

EXAMPLE 3.2. Consider the simple ternary semirings R_2 in Example 3.1 with binary operation usual addition and ternary usual multiplication. Let

$$B = \left\{ \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| a_i^{'s} \text{ are non positive real numbers} \right\}.$$

Hence B is a bi-quasi ideal of R_2.

DEFINITION 3.3. A non empty subset B of R is called

(i) a left tri quasi ideal of R if B is a sub ternary semiring of R and $RRB \cap BBRRBBB \subseteq B$.

(ii) a lateral tri quasi ideal of R if B is a sub ternary semiring of R and $RBR \cap BBRBRBB \subseteq B$.

(iii) a right tri quasi ideal of R if B is a sub ternary semiring of R and $BRR \cap BBBRRBB \subseteq B$.

(iv) a tri quasi ideal of R if B is a left tri quasi ideal, lateral tri quasi ideal and right tri quasi ideal.

EXAMPLE 3.3. Consider the simple ternary semirings R_2 in Example 3.1 with binary operation usual addition and ternary usual multiplication. Let

$$B = \left\{ \begin{pmatrix} 0 & b_1 & 0 & b_2 & b_3 & b_4 \\ 0 & 0 & 0 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & b_8 & 0 & b_9 \\ 0 & 0 & 0 & 0 & b_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \text{ are non positive real numbers} \right\}.$$

Hence B is a tri quasi-ideal of R_2.

THEOREM 3.3. *The following holds*

(i) *Every left bi-quasi ideal is a left tri quasi ideal.*

(ii) *Every lateral bi-quasi ideal is a lateral tri quasi ideal.*

(iii) *Every right bi-quasi ideal is a right tri quasi ideal.*

Proof. (i) Suppose that B is a left bi-quasi ideal of R, $RRB \cap BRBRB \subseteq B$. Now, $RRB \cap BBRRRBBB \subseteq RRB \cap BRBRB \subseteq B$. Thus, B is a left tri quasi ideal of R.

Similarly we can to prove (ii) and (iii). $\square$

Converse of the Theorem 3.3 is need not be true which can be illustrated as follows.

Example 3.4. Consider the simple ternary semirings R_2 in Example 3.1 with binary operation usual addition and ternary usual multiplication. Let

$$B = \left\{ \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers} \right\}.$$

Now, $R_2R_2B \cap BBR_2R_2BBB \subseteq B$. But

$$R_2R_2B \cap BR_2BR_2B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x \text{ is a non positive real number} \right\} \nsubseteq B.$$

Hence B is a left (lateral, right) tri quasi-ideal of R_2, but not left (not lateral, not right) bi quasi-ideal of R_2 respectively.

Theorem 3.4. *The following holds*
(i) *If B is a left bi-quasi ideal of R, then B is a tri ideal of R.*
(ii) *If B is a lateral bi-quasi ideal of R, then B is a tri ideal of R.*
(iii) *If B is a right bi-quasi ideal of R, then B is a tri ideal of R.*

Proof. (iii) Suppose that B is a right bi-quasi ideal of R. Then $BRR \cap BRBRB \subseteq B$. Now, $BBBRRBB \subseteq BRR \cap BRBRB \subseteq B$ and $BBRRBBB \subseteq BRR \cap BRBRB \subseteq B$ and $BBRBRBB \subseteq BRR \cap BRBRB \subseteq B$. Thus, B is a tri-ideal of R.

Similarly to prove (i) and (ii). $\square$

Corollary 3.1. *If B is a bi-quasi ideal of R, then B is a tri-ideal of R.*

Theorem 3.5. *Every bi-ideal is a left (lateral, right) tri-ideal of R.*

Converse of the Theorem 3.5 is need not be true which can be illustrated as follows.

Example 3.5. Consider the simple ternary semirings R_2 in Example 3.1 with binary operation usual addition and ternary usual multiplication.

$$R_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ x_2 & x_3 & 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_6 & 0 & 0 & 0 \\ x_7 & x_8 & x_9 & x_{10} & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers, } i = 1, 2, ..., 15 \right\}.$$

Let

$$B_1 = \left\{ \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers, } i = 1, 10, 15 \right\}.$$

and

$$B_2 = \left\{ \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{10} & 0 & x_{12} \\ 0 & 0 & 0 & 0 & x_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers} \right\},$$

$$B_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{15} & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers, } i = 1, 6, 15 \right\}.$$

Hence B_1 and B_2 are left tri-ideal and lateral tri-ideal of R_2 respectively, but not bi ideal of R_2 and B_3 is a right tri-ideal of R_3, but not bi ideal of R_3.

THEOREM 3.6. *Every interior ideal is a left(lateral, right) tri-ideal of R.*

PROOF. Suppose that I is a interior ideal of R. Then $RIRIR \subseteq I$. Now, $IIRRIII \subseteq RIRIR \subseteq I$. Thus, I is a left tri-ideal of R. Similarly other parts. □

THEOREM 3.7. *Let B be a ternary subsemiring of R. If U_1 is a right ideal, U_2 is a lateral ideal and U_3 is a left ideal of R such that $U_1U_2U_3 \subseteq B \subseteq U_1 \cap U_2 \cap U_3$, then B is a tri-ideal of R.*

PROOF. Suppose that U_1 is a right ideal, U_2 is a lateral ideal and U_3 is a left ideal of R such that $U_1U_2U_3 \subseteq B \subseteq U_1 \cap U_2 \cap U_3$. Then $BBRRBBB \subseteq (U_1 \cap U_2 \cap U_3)(U_1 \cap U_2 \cap U_3)RR(U_1 \cap U_2 \cap U_3)(U_1 \cap U_2 \cap U_3)(U_1 \cap U_2 \cap U_3) \subseteq U_1U_2RRU_3U_3U_3 \subseteq U_1U_2RRU_3 \subseteq U_1U_2U_3 \subseteq B$. Thus B is a left tri-ideal of R. Similarly, B is a right (lateral) tri-ideal of R. Hence B is a tri-ideal of R. □

THEOREM 3.8. *The following holds*

(i) *The intersection of a left tri-ideal B of R and an ideal A of R is a left tri-ideal of R.*

(ii) *The intersection of a lateral tri-ideal B of R and an ideal A of R is a lateral tri-ideal of R.*

(iii) *The intersection of a right tri-ideal B of R and an ideal A of R is a right tri-ideal of R.*

PROOF. (i) Suppose $C = B \cap A$. Then $CCRRCCC \subseteq BBRRBBB \subseteq B$. Since A is an ideal, $CCRRCCC \subseteq AARRAAA \subseteq ARR \subseteq A$ and $CCRRCCC \subseteq AARRAAA \subseteq RAR \subseteq A$ and $CCRRCCC \subseteq AARRAAA \subseteq RRA \subseteq A$. Thus, $CCRRCCC \subseteq B \cap A = C$. Hence C is a left tri-ideal of R.

Similar we can to prove (ii) and (iii). □

COROLLARY 3.2. *(i)The intersection of a tri-ideal and ideal is a tri-ideal of R.*
(ii) The intersection of a tri-ideals is a tri-ideal of R.

THEOREM 3.9. *The intersection of a tri-ideal and interior (bi quasi, tri quasi) ideal of R is a tri-ideal of R.*

PROOF. Suppose that I is a tri-ideal of R and B is a bi quasi ideal of R. To prove that $B \cap I$ is the tri-ideal of R. Now, $(B \cap I)(B \cap I)RR(B \cap I)(B \cap I)(B \cap I) \subseteq BRBRB$ and $(B \cap I)(B \cap I)RR(B \cap I)(B \cap I)(B \cap I) \subseteq RRB$.

Thus, $(B \cap I)(B \cap I)RR(B \cap I)(B \cap I)(B \cap I) \subseteq RRB \cap BRBRB \subseteq B$. Now, $(B \cap I)(B \cap I)RR(B \cap I)(B \cap I)(B \cap I) \subseteq IIRRIII \subseteq I$. Hence $(B \cap I)(B \cap I)RR(B \cap I)(B \cap I)(B \cap I) \subseteq B \cap I$. Hence $B \cap I$ is a left tri ideal of R. Similarly, $B \cap I$ is a lateral and right tri ideal of R. Thus, $B \cap I$ is a tri ideal of R. Similarly other cases. □

4. m Tri-ideals in ternary Semirings

Definition 4.1. (i) A left m tri-ideal B of R is a sub ternary semiring of R such that $BBR^mBBB \subseteq B$.

(ii) A lateral m tri-ideal B of R is a sub ternary semiring of R such that $BBR^mBR^mBB \subseteq B$.

(iii) A right m tri-ideal B of R is a sub ternary semiring of R such that $BBBR^mBB \subseteq B$.

(iv) A m tri-ideal B of R if B is a left m tri-ideal, lateral m tri-ideal, right m tri-ideal of R , where m is a positive integer.

Theorem 4.1. *For $m \geqslant 1$,*

(i) *Every left tri-ideal is an m left tri-ideal of R.*

(ii) *Every lateral tri-ideal is an m lateral tri-ideal of R.*

(iii) *Every right tri-ideal is an m right tri-ideal of R.*

Converse of the Theorem 4.1 is need not be true which can be illustrated as follows.

Example 4.1. Consider the simple ternary semirings with binary operation usual addition and ternary usual multiplication.

$$R_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_7 & x_8 & x_9 & x_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 0 & 0 & 0 & 0 & 0 \\ x_{16} & x_{17} & x_{18} & x_{19} & x_{20} & x_{21} & 0 & 0 & 0 & 0 \\ x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & 0 & 0 & 0 \\ x_{29} & x_{30} & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & 0 & 0 \\ x_{37} & x_{38} & x_{39} & x_{40} & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & 0 \end{pmatrix} \middle| \, x_i^{'s} \text{ are non positive real numbers} \right\},$$

$$R_2 = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ 0 & 0 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \\ 0 & 0 & 0 & x_{18} & x_{19} & x_{20} & x_{21} & x_{22} & x_{23} & x_{24} \\ 0 & 0 & 0 & 0 & x_{25} & x_{26} & x_{27} & x_{28} & x_{29} & x_{30} \\ 0 & 0 & 0 & 0 & 0 & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{36} & x_{37} & x_{38} & x_{39} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{40} & x_{41} & x_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{43} & x_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x_i^{'s} \text{ are non positive real numbers} \right\},$$

$$B_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_8 & 0 & x_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{11} & 0 & x_{13} & 0 & x_{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{17} & 0 & x_{19} & 0 & x_{21} & 0 & 0 & 0 & 0 \\ x_{22} & 0 & x_{24} & 0 & x_{26} & 0 & x_{28} & 0 & 0 & 0 \\ 0 & x_{30} & 0 & x_{32} & 0 & x_{34} & 0 & x_{36} & 0 & 0 \\ x_{37} & 0 & x_{39} & 0 & x_{41} & 0 & x_{43} & 0 & x_{45} & 0 \end{pmatrix} \middle| \, x_i^{'s} \text{ are non positive real numbers} \right\},$$

$$B_2 = \left\{ \begin{pmatrix} 0 & x_1 & 0 & x_3 & 0 & x_5 & 0 & x_7 & 0 & x_9 \\ 0 & 0 & x_{10} & 0 & x_{12} & 0 & x_{14} & 0 & x_{16} & 0 \\ 0 & 0 & 0 & x_{18} & 0 & x_{20} & 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & 0 & 0 & x_{25} & 0 & x_{27} & 0 & x_{29} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{31} & 0 & x_{33} & 0 & x_{35} \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{36} & 0 & x_{38} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{40} & 0 & x_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{43} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x_i^{'s} \text{ are non positive real numbers} \right\}.$$

Hence B_1 is a m left tri-ideal but not left tri-ideal of R_1 and B_2 is a m right tri-ideal but not right tri-ideal of R_2.

Definition 4.2. A m bi-ideal B of R is a sub ternary semiring of R such that $BR^mBR^mB \subseteq B$.

Theorem 4.2. *For $m \geqslant 1$, Every m bi-ideal is an m tri-ideal of R.*

Theorem 4.3. *The product of atleast three m tri-ideals is also m tri-ideals of R.*

Theorem 4.4. *If B is a m tri-ideal of R and T_1, T_2 are two ternary subsemirings with identity element e, then BT_1T_2, T_1BT_2 and T_1T_2B are m tri-ideals of R.*

Proof. Let B be a m tri-ideal of R , T_1 and T_2 are two ternary subsemirings with identity element e. Clearly, BT_1T_2 is closed under addition. Now, $(BT_1T_2)(BT_1T_2)R^mR^m(BT_1T_2)(BT_1T_2)(BT_1T_2) \subseteq BBR^mR^mBBBT_1T_2 \subseteq BT_1T_2$ Thus BT_1T_2 is an m-left tri ideal of R.
Similarly BT_1T_2 are m-lateral tri ideal and m-right tri-ideal of R.
Similarly, T_1BT_2 and T_1T_2B is an m-tri-ideal of R. □

Theorem 4.5. *If B is a m tri-ideal of R and T is a ternary subsemiring of R, then $B \cap T$ is a m tri-ideal of T.*

Proof. Since $B \cap T \subseteq B$ and $B \cap T \subseteq T$, $(B \cap T)(B \cap T)(B \cap T) \subseteq BBB \subseteq B$. $(B \cap T)(B \cap T)T^mT^m(B \cap T)(B \cap T)(B \cap T) \subseteq (B \cap T)(B \cap T)R^mR^m(B \cap T)(B \cap T)(B \cap T) \subseteq BBR^mR^mBBB \subseteq B$. Therefore $B \cap T$ is a m left tri-ideal of T. similarly, $B \cap T$ is a m lateral tri-ideal and m right tri-ideal of T. □

Definition 4.3. An additive ternary subsemigroup Q of R is called a m quasi ideal if $QR^m \cap R^mQR^m \cap R^mQ \subseteq Q$.

Theorem 4.6. *Every m quasi ideal is a m tri-ideal of R.*

Proof. Suppose Q is a m quasi-ideal of R. Then $QR^m \cap R^mQR^m \cap R^mQ \subseteq Q$. Now, $QQR^mQQQ \subseteq QR^m$, $QQR^mQQQ \subseteq R^mQR^m$ and $QQR^mQQQ \subseteq R^mQ$. Hence $QQR^mQQQ \subseteq Q$. Thus, Q is m left tri-ideal of R. similarly, Q is m lateral tri-ideal and right tri-ideal of R. □

Converse of the Theorem 4.6 need not be true by the following Example.

EXAMPLE 4.2. Consider the simple ternary semirings R_2 in Example 3.1 with binary operation usual addition and ternary usual multiplication. Let

$$B = \left\{ \begin{pmatrix} 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| x, y, z \text{ are non positive real numbers} \right\}.$$

Hence B is a m tri-ideal but not 2-quasi ideal of R_2.

DEFINITION 4.4. (i) A ternary subsemiring L of R is called an l left tri-ideal if

$$LLR^lLLL \subseteq L.$$

(ii) A ternary subsemiring X of R is called an m lateral tri-ideal if

$$XXR^mXR^mXX \subseteq X.$$

(iii) A ternary subsemiring N of R is called an n right tri-ideal if

$$NNNR^nNN \subseteq N,$$

where l, m, n are positive integers.

THEOREM 4.7. *Every l-left ideal ,m-lateral ideal and n-right ideal of R with e is an l tri-ideal, m tri-ideal,n tri-deal of R respectively.*

PROOF. Let L be the m-lateral ideal of R, then $R^mLR^m \subseteq L$. Now, $LLR^mLLL \subseteq LLR^mLee...eLL \subseteq LLR^mLR^mLL \subseteq L$ and $LLR^mLR^mLL \subseteq LLLLL \subseteq L$ and $LLLR^mLL \subseteq LLLR^mLL \subseteq LLeee.eLR^mLL \subseteq LLR^mLR^mLL \subseteq L$. Therefore L is a m tri-ideal of R. Similarly, L is a l tri-ideal and n tri-ideal of R. $\square$

THEOREM 4.8. *The intersection of l-left ideal, m-lateral ideal and n-right ideal is an l-left ideal, m-lateral ideal and n-right ideal of R respectively.*

THEOREM 4.9. *Let U, V, W be an l left tri-ideal, m lateral tri-ideal and n right tri-ideal of R respectively. Then $U \cap V \cap W$ is an t tri-ideal , where $t = \max(l, m, n)$.*

PROOF. Clearly, $X = U \cap V \cap W$ is a ternary subsemiring of R. By Theorem 4.7, U, V and W are l tri-ideals, m tri-ideals and n tri-ideals respectively. The intersection of U, V and W becomes $\max(l, m, n)$ tri-ideals.

$$\begin{aligned} XXR^tXXX &\subseteq UUR^tUUU \subseteq RRR^tRRU \\ &= R^{t+2}R^2U \subseteq R^tU \subseteq U. \end{aligned}$$

Similarly $XXR^tXXX \subseteq V$ and $XXR^tXXX \subseteq W$. Hence $XXR^tXXX \subseteq X$. Similarly, $XXXR^tXX \subseteq X$ and $XXR^tXR^tXX \subseteq X$. $\square$

THEOREM 4.10. *Let U be a m left(lateral, right) tri-ideal of R and V be a m left(lateral, right) tri-ideal of U such that $V^3 = V$. Then V is a m left(lateral, right) tri-ideal of R.*

Proof. Since U is a m left tri-ideal of R, $UUR^mUUU \subseteq U$ and V is a m left tri-deal of U, $VVU^mVVV \subseteq V$.

$$\begin{aligned} VVR^mVVV &= (VVV)(VVV)R^mV(VVV)(VVV) \\ &= VVVV(VVR^mVVV)VVVV \\ &\subseteq VVVV(UUR^mUUU)VVVV \\ &\subseteq VVVVUVVVV \\ &= VVVVUVVV(VVV) \\ &= VVU^3U^3VVV \\ &\subseteq VVU^3VVV \\ &. \\ &= VVU^mVVV \\ &\subseteq V \end{aligned}$$

Thus V is a m left tri-ideal of R. Similar to prove other cases. □

Corollary 4.1. *Let U be a m tri-ideal of R and V be a m tri-ideal of U such that $V^3 = V$. Then V is a m tri-ideal of R.*

Theorem 4.11. *The following holds*

(i) *Let U, V and W be three ternary subsemirings of R and $B = UVW$. Then B is a m tri- ideal if at least one of U, V, W is m right ideal or m left ideal of R.*

(ii) *Let U, V and W be three ternary subsemirings of R and $B = UVW$. Then B is a m lateral tri-ideal if at least one of U, V, W is a m lateral tri-ideal of R.*

5. m tri-ideal generators of ternary semirings

Theorem 5.1. *Let I be a nonempty subset of R. Then the m left tri-ideal generated by I is $< I >_m = \sum\limits_{i=1}^{m} N_0 I^i + (R^m I)$.*

Proof. We show that $< I >_m = \sum\limits_{i=1}^{m} N_0 I^i + (R^m I)$ is the smallest m left tri-ideal of R containing I. Let $a, b, c \in < I >_m$. Then

$$a = \sum_{finite} (n_j x_{j1} x_{j2} ... x_{jm}) + \sum_{finite} (a_{j1} a_{j2} ... a_{jm} r_j),$$

$$b = \sum_{finite} (n_j^{'} x_{j1}^{'} x_{j2}^{'} ... x_{jm}^{'}) + \sum_{finite} (a_{j1}^{'} a_{j2}^{'} ... a_{jm}^{'} r_j^{'}),$$

$$c = \sum_{finite} (n_j^{'} x_{j1}^{''} x_{j2}^{''} ... x_{jm}^{''}) + \sum_{finite} (a_{j1}^{''} a_{j2}^{''} ... a_{jm}^{''} r_j^{''})$$

where $n_j, n_j^{'}, n_j^{''} \in N_0$, $x_{j1} x_{j2} ... x_{jm}, x_{j1}^{'} x_{j2}^{'} ... x_{jm}^{'}, x_{j1}^{''} x_{j2}^{''} ... x_{jm}^{''}, r_j, r_j^{'}, r_j^{''} \in I$ and $a_{j1} a_{j2} ... a_{jm}, a_{j1}^{'} a_{j2}^{'} ... a_{jm}^{'}, a_{j1}^{''} a_{j2}^{''} ... a_{jm}^{''} \in R$. Put

$$x = n_j x_{j1} x_{j2} ... x_{jm}, \quad x^{'} = n_j^{'} x_{j1}^{'} x_{j2}^{'} ... x_{jm}^{'}, \quad x^{''} = n_j^{'} x_{j1}^{''} x_{j2}^{''} ... x_{jm}^{''}$$

and $y = a_{j1}a_{j2}...a_{jm}r_j$, $y^{'} = a^{'}_{j1}a^{'}_{j2}...a^{'}_{jm}r^{'}_j$, $y^{''} = a^{''}_{j1}a^{''}_{j2}...a^{''}_{jm}r^{''}_j$. Now,

$$a+b+c = \Big[\sum_{finite} x + \sum_{finite} y\Big] + \Big[\sum_{finite} x^{'} + \sum_{finite} y^{'}\Big] + \Big[\sum_{finite} x^{''} + \sum_{finite} y^{''}\Big] =$$

$$\Big[\sum_{finite} x + \sum_{finite} x^{'} + \sum_{finite} x^{''}\Big] + \Big[\sum_{finite} y + \sum_{finite} y^{'} + \sum_{finite} y^{''}\Big]$$

$$= \sum_{finite}\Big[x + x^{'} + x^{''}\Big] + \sum_{finite}\Big[y + y^{'} + y^{''}\Big] \in \sum_{i=1}^{m} N_0 I^i + (R^m I).$$

Put $p = \sum_{i=1}^{m} N_0 I^i$ and $q = R^m I$. Thus, $R^m < I >_m \subseteq < I >_m$. Let B' be any other m left tri-ideal of R containing I. Then $N_0 I^i \subseteq B'$ for all $i \in N$ and $R^m I \subseteq R^m B' \subseteq B'$. Therefore $< I >_m = \sum_{i=1}^{m} N_0 I^i + (R^m I) \subseteq B'$. Hence, $< I >_m$ is the smallest m left tri-ideal of R containing I. $\square$

THEOREM 5.2. *Let I be a nonempty subset of R. Then the mlateral tri-ideal generated by I is* $< I >_m = \sum_{i=1}^{m} N_0 I^i + (R^m I R^m)$.

THEOREM 5.3. *Let I be a nonempty subset of R. Then the m right tri-ideal generated by I is* $< I >_m = \sum_{i=1}^{m} N_0 I^i + (I R^m)$.

COROLLARY 5.1. *Let I be a nonempty subset of R. Then the m tri-ideal generated by I is* $< I >_m = \sum_{i=1}^{m} N_0 I^i + (R^m I) + (R^m I R^m) + (I R^m)$.

Conclusion. In this paper mainly we start the tri-ideals in ternary semirings. By using tri-ideals and its generators, we characterized them and results in this paper may apply to many algebraic structures of partial ternary semirings and gamma ternary semirings.

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