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THE DOUBLE GEO CHROMATIC NUMBER OF A GRAPH

S. Beulah Samli, J. John and S. Robinson Chellathurai

Abstract. A geodetic set S of G is said to be a double geo chromatic set S_{dg} if each $u - v$ geodesic, where $u, v \in S$ contains at least two entire color classes of G. The minimum cardinality of a double geo chromatic set of G is the double geo chromatic number of G and is denoted by $\chi'_{dg}(G)$. The double geo chromatic number of some certain standard graphs are determined and some general properties satisfied by this concept are studied. Connected graphs of order $p \geqslant 2$ with double geo chromatic number 2 are characterized. It is shown that for every positive integer x, y, z with $6 \leqslant x \leqslant y \leqslant z$, $y \leqslant x+1$ and $z \leqslant x+3$, there exists a connected graph G with $g(G) = x$, $\chi_{gc}(G) = y$ and $\chi'_{dg}(G) = z$. It is also shown that for every positive integer $a \geqslant 3$ with $\chi(G) = 2$, there exists a connected graph G such that $g(G) = a$, $\chi_{gc}(G) = \chi'_{dg}(G) = a + 1$.

1. Introduction

We consider finite simple connected graphs with at least two vertices. For any graph G the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to [**9**]. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. If the subgraph induced by its neighbors is complete, then a vertex v is called an *extreme vertex* of a graph G. A vertex x is said to lie on an $u - v$ geodesic P if x is an internal vertex of P. The closed interval $I[u, v]$ consists of u, v and all vertices lying on a $u - v$ geodesic of G, and

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for a non-empty set $S \subseteq V(G)$,$I[S] = \bigcup_{u,v \in S} I[u,v]$.If G is a connected graph, then a set S of vertices is a *geodetic set* if $I[S] = V(G)$. The *geodetic number* $g(G)$ of G is the minimum cardinality among all geodetic set of G. Geodetic number was introduced in [**3, 10**] and further studied in [**1, 5, 6, 7, 8, 12**]. The degree of a vertex u of G is the number of edges that are incident to the vertex. A vertex u is a *universal vertex* if $deg(u) = p - 1$. A *c-vertex coloring* of G is an assignment of c colors, $1, 2, \ldots, c$ to the vertices of G; the coloring is proper if no two distinct adjacent vertices have the same color. If $\chi(G) = c$,G is said to be *c- chromatic.* A set $C \subseteq V(G)$ is called *chromatic set* if C contains all c vertices of distinct colors in G. The *chromatic number* of G is the minimum cardinality among all chromatic sets of G. That is $\chi(G) = min\{|C|/C \textit{ is the chromatic set of } G\}$. For references on chromatic sets see [**11**].

The concept of geo chromatic number of a graph was introduced in [**2**]. A set $S_c \subseteq V(G)$ is said to be a *geo chromatic set* of G if S_c is both a geodetic set and a chromatic set of G. The minimum cardinality among all geo chromatic set of a graph G is its geo chromatic number $\chi_{gc}(G)$.

The concept of geo chromatic set has motivated us to introduce the new geo chromatic set conception of double geo chromatic set. We call the minimum cardinality of a double geo chromatic set of G, the double geo chromatic number of G.

In this paper we introduce the new concept as double geo chromatic number of a graph. In section 2, we introduce the definition of double geo chromatic number, we determine the double geo chromatic number of some standard graphs, also we characterize the graphs G which has double geo chromatic number $p - 1$ and 2. In section 3, we illustrate realization of the double geo chromatic number of G. The following theorems are used in sequel.

THEOREM 1.1. [**5**] *Every geodetic set of a graph contains its extreme vertices.*

THEOREM 1.2. [**9**] *Let G be a connected graph with cutvertices and let S be a geodetic set of G. If x is a cutvertex of G, then every component of $G - x$ contains an element of S.*

2. The double geo chromatic number of a graph

DEFINITION 2.1. A geodetic set S of G is said to be a double geo chromatic set S_{dg} if each $u - v$ geodesic contains at least two entire color classes of G, where $u, v \in S$. The minimum cardinality of a double geo chromatic set of G is the double geo chromatic number of G and is denoted by $\chi'_{dg}(G)$. A double geo chromatic set of minimum cardinality is called a χ'_{dg}-set of G. The minimum number of colors required for a double geo chromatic set is called double geo coloring.

EXAMPLE 2.1. For the graph G given in Figure 1, $S = \{a, b, g\}$ is a geodetic set. Hence $g(G) = 3$. By assigning a proper coloring of G, we get $\chi(G) = 2$. Clearly, by assigning two colors the geodesics $a - b$ and $a - g$ do not receive at least two color classes so that the set S is not a double geo chromatic set of G. Let

$S_{dg} = \{a, b, c, g\}$. It is clear that S_{dg} is a geodetic set and it receive every double geo colors. Hence S_{dg} is a double geo chromatic set of G and so $\chi'_{dg}(G) = 4$.

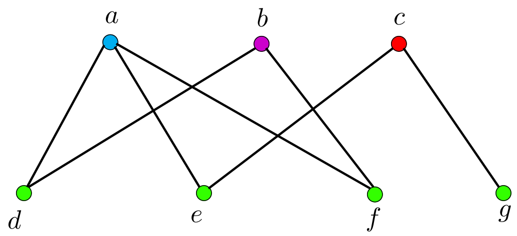

FIGURE 1. A graph G with $\chi'_{dg}(G) = 4$

REMARK 2.1. A minimum double geo chromatic set of G is not always unique. For the graph G given in Figure 2, $\{a, e, g, f, b\}$, $\{a, e, g, f, j\}$ and $\{a, e, g, f, h\}$ are three minimum double geo chromatic sets of cardinality 5.

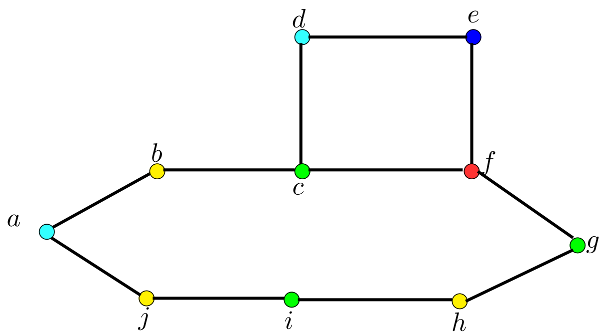

FIGURE 2. A graph G with $\chi'_{dg}(G) = 5$

OBSERVATION 2.1. Let G be a connected graph of order p. Then:

(1) Every double geo chromatic set of a connected graph G contains its extreme vertices. Also if the set of all extreme vertices of G, say $Ext(G)$ is a double geo chromatic set, then $Ext(G)$ is the unique minimum double geo chromatic set of G.

(2) Every double geo chromatic set of G contains a universal vertex if G has at least one universal vertex.

(3) Every double geo chromatic set of G contains at least one vertex from each component of $G - v$ if v is a cut vertex of G.

(4) Every double geo chromatic set of G is a geo chromatic set of G.

COROLLARY 2.1. *For the complete graph K_p $(p \geqslant 2)$, $\chi'_{dg}(K_p) = p$.*

PROOF. This follows from Observation 2.1(1). □

OBSERVATION 2.2. The double geo chromatic number of some standard graphs can be easily found and are given as follows:

(1) For the path P_p, $p \geqslant 2$, $\chi'_{dg}(P_p) = \begin{cases} 2 & \text{if} \quad p \text{ is even} \\ 3 & \text{if} \quad p \text{ is odd} \end{cases}$

(2) For the cycle C_p, $p \geqslant 6$, $\chi'_{dg}(C_p) = \begin{cases} 2 & \text{if} \quad p \equiv 2(mod4) \\ 3 & \text{if} \quad p \equiv 0(mod4) \\ 6 & \text{if} \quad p\text{is odd} \end{cases}$

(3) For the star $K_{1,p-1}$, $p \geqslant 2$, $\chi'_{dg}(K_{1,p-1}) = p$.

(4) For the wheel W_p, $p \geqslant 5$, $\chi'_{dg}(W_p) = \begin{cases} \frac{p+4}{2} & \text{if} \quad p \text{ is even} \\ \lfloor \frac{p}{2} \rfloor + 2 & \text{if} \quad p \text{ is odd} \end{cases}$

(5) For the complete bipartite graph $G = K_{m,n}$ $(m, n \in \mathbb{Z}^+)$,

(i) $\chi'_{dg}(G) = m + 1$ if $2 \leqslant m \leqslant 4$ and $n \geqslant 2$.

(ii) $\chi'_{dg}(G) = 6$ if $m, n \geqslant 5$.

THEOREM 2.1. *Let G be a connected graph of order $p \geqslant 2$. Then $2 \leqslant \chi_{gc}(G) \leqslant \chi'_{dg}(G) \leqslant p$.*

PROOF. Clearly $\chi_{gc}(G) \geqslant 2$. Since every double geo chromatic set S_{dg} of G is also a geo chromatic set S_c of G,$\chi_{gc}(G) \leqslant \chi'_{dg}(G)$. Also, $V(G)$ is a double geo chromatic set of G and so $\chi'_{dg}(G) \leqslant p$. Hence $2 \leqslant \chi_{gc}(G) \leqslant \chi'_{dg}(G) \leqslant p$. □

REMARK 2.2. The bounds in Theorem 2.1 are sharp. For the path P_{2p}, $\chi_{gc}(P_{2p}) = 2$. For the graph G given in Figure 3, $\chi_{gc}(G) = \chi'_{dg}(G) = 3$. For the complete graph K_p, $\chi'_{dg}(K_p) = p$. Also, all the inequalities in Theorem 2.1 are strict. For the graph G given in Figure 3, $\chi_{gc}(G) = 4$, $\chi'_{dg}(G) = 6$ and $p = 11$. Thus $2 < \chi_{gc}(G) < \chi'_{dg}(G) < p$.

COROLLARY 2.2. *Let G be a connected graph of order $p \geqslant 2$. If $\chi_{gc}(G) = p$, then $\chi'_{dg}(G) = p$.*

PROOF. This follows from Theorem 2.1. □

REMARK 2.3. The converse of the Corollary 2.2 need not be true. For the graph G of order $p = 5$ given in Figure 4, we have $\chi'_{dg}(G) = 5$ but $\chi_{gc}(G) = 4 \neq p$.

THEOREM 2.2. *Let G be a connected graph of order p. If $deg(v) = p - 1$, then v belongs to every double geo chromatic set of G.*

PROOF. Let S_c be a geo chromatic set and S_{dg} be a double geo chromatic set of G. Since $deg(v) = p - 1$, v receive distinct color by a proper coloring of G so that $v \in S_c$. By Observation 2.1(4), $S_c \subseteq S_{dg}$. Hence $v \in S_{dg}$. □

REMARK 2.4. The converse of the Theorem 2.2 need not be true. For the graph G given in Figure 3, the vertex i belongs to all double geo chromatic set of G. But $deg(i) \neq p - 1$.

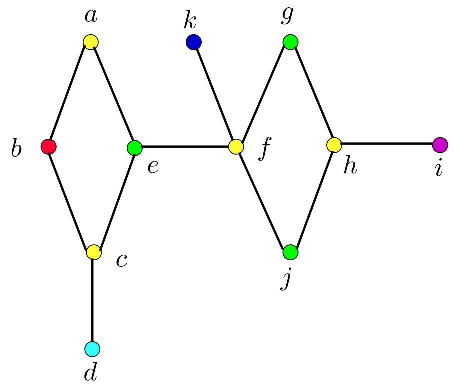

FIGURE 3. A graph G with $2 < \chi_{gc}(G) < \chi'_{dg}(G) < p$

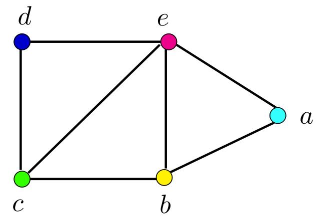

FIGURE 4. A graph G with $\chi'_{dg}(G) = 5$ and $\chi_{gc}(G) = 4$

The following theorem chracterizes graphs for which the double geo chromatic number is $p - 1$ if $g(G) = 2$.

THEOREM 2.3. *Let G be a conncected graph of order $p \geqslant 6$ with $g(G) = 2$ and z be the only vertex which has degree 2. Then every vertex of G except the vertex z has degree $p - 2$ if and only if $\chi'_{dg}(G) = p - 1$.*

Proof. Let G be a connected graph such that every vertex of degree $p-2$ except the vertex z, then there exists a connected graph $G \setminus \{z\}$ of order $n = p-1$. We claim that $\chi'_{dg}(G \setminus \{z\}) = n$. Since $n-2$ vertices of a connected graph $G \setminus \{z\}$ has degree $n-1$ and two vertices has degree $n-2$, $diam(G \setminus \{z\}) = 2$. Clearly, every vertex of $G \setminus \{z\}$ lies in a $x-y$ geodesic such that $d(x,y) = 2$. The set of vertices $\{x, y\}$ is a unique minimum geodetic set of $G \setminus \{z\}$. Consider a proper coloring of $G \setminus \{z\}$ such that the vertices which has degree $n-1$ receive distinct colors and the vertices x, y receive same color and so every vertices of $G \setminus \{z\}$ is a unique minimum double geo chromatic set of $G \setminus \{z\}$ so that $\chi'_{dg}(G \setminus \{z\}) = n$. Now, claim that $\chi'_{dg}(G \setminus \{z\}) = \chi'_{dg}(G)$. Since the vertex z has degree 2 and $G \setminus \{z\}$ has degree $p-2$ and so $diam(G) = 2$. The vertices x, y are not adjacent in $G \setminus \{z\}$ and so the vertices x, y are not adjacent in G. Let the vertex z be adjacent with the vertices x, y. Cleary, every vertices of G lies in a $x-y$ geodesic. Consider a proper coloring of G such that the vertices of a minimum geodetic set S receive same color, the vertex z was repeated by a color which was assigned in the vertex which has degree $p-2$. Let $w \in G$ be a vertex which has degree $p-2$, the vertices w, z receive same color and the vertices of $G \setminus S \cup \{w, z\}$ receive distinct colors. Now, a $x-y$ geodesic has more than two color classes. Hence $V(G) \setminus \{z\}$ is a minimum double geo chromatic set of G which is also a unique minimum geo chromatic set of G. Hence it follows that $\chi'_{dg}(G \setminus \{z\}) = n = p-1$.

Conversely, let $\chi'_{dg}(G) = p-1$. Assume to the contrary that there exists a vertex which does not have degree $p-2$ except the vertex z. Then either there exists a vertex which has degree $p-1$ or there exists a vertex which has degree less than $p-2$. Let us consider two cases.

Case 1. Suppose there exists a vertex which has degree $p-1$ and all the other vertices of G has degree $p-2$ except the vertex z. Since the vertices x and y are adjacent with the vertex z, we consider two subcases.

Sub case 1.1. Suppose there exists a vertex which has degree $p-1$ in $G \setminus \{x, y, z\}$. Clearly the vertex z has degree 3. This contradicts to our assumption that the vertex z has degree 2.

Sub case 1.2. Suppose, either the vertex x or y has degree $p-1$. Let us assume that the vertex x has degree $p-1$. Now we claim that the vertex y has degree $p-1$. Suppose to the contrary that the vertex y does not have degree $p-1$, Clearly the vertex x does not have degree $p-1$, which contradicts to our assumption that x has degree $p-1$. Hence $deg(y) = p-1$. which implies the vertex $G \setminus \{x, y\}$ is a unique minimum geodetic set S of G, $g(G) = p-2$, which contradicts to our fact that $g(G) = 2$.

Case 2. Suppose there exists a vertex which has degree less than $p-2$. Let the vertex be x and all the other vertices of G has degree $p-2$ except the vertex z. We consider three subcases.

Sub case 2.1. Let the vertex x has degree 2. Clearly the vertices of G except the vertices of $N[x]$, $N[z]$ has degree $p-3$ and the vertices of $N(x) \setminus \{z\}$, $N(z) \setminus \{x\}$ has degree $p-2$. Let the vertex of $N(x) \setminus \{z\}$ be x'. Clearly the vertex set of $G \setminus \left\{x, y, x'\right\}$ is a minimum geodetic set of G and so $g(G) > 2$, which is impossible.

Sub case 2.2. Let the vertex x has degree $p-3$. Clearly the vertex x is not adjacent with exactly two vertices of $G \setminus \{z\}$. Let the vertices which are not adjacent to the vertex x be a, y such that $d(a, y) = 1$, so that $d(x, a) = d(x, y) = 2$. Therefore the set of vertices $\{x, a, y\}$ of G is a minimum geodetic set of G so that $g(G) = 3$. This contradicts the fact that $g(G) = 2$.

Sub case 2.3. If $2 < deg(x) < p-3$, then there exist vertices which have degree at most $p-2$. Choose a vertex w such that $d(x, w) = 2$. Clearly, there are some vertices which does not lie in a $x-w$ geodesic. Hence $g(G) > 2$, which contradicts to our assumption. It follows that every vertices of G except the vertex z has degree $p-2$. □

THEOREM 2.4. *Let G be a connected graph of order $p \geqslant 2$. Then $\chi_{gc}(G) = 2$ if and only if $\chi'_{dg}(G) = 2$.*

PROOF. Suppose that $\chi_{gc}(G) = 2$. Let $S = \{x, y\}$ be a minimum geodetic set of G. Since x, y belongs to distinct color classes, every vertices of G lies in a $x-y$ geodesic, say P. Hence P contains exactly two color classes of G. Hence $\chi'_{dg}(G) = 2$. Conversely, let $\chi'_{dg}(G) = 2$. By Theorem 2.1, $\chi_{gc}(G) = 2$. □

OBSERVATION 2.3. Let G be a connected graph of order $p \leqslant 4$. Then $\chi_{gc}(G) = \chi'_{dg}(G)$

3. Realization result

In this section we give realization result concerning the double geo chromatic number.

THEOREM 3.1. *For every positive integer x, y, z with $6 \leqslant x \leqslant y \leqslant z$, $y \leqslant x+1$ and $z \leqslant x+3$, there exists a connected graph G with $g(G) = x$, $\chi_{gc}(G) = y$ and $\chi'_{dg}(G) = z$.*

PROOF. We consider the following cases.

Case 1. If $x = y = z$, consider the complete graph K_x. Then by Corollary 2.9 and Observation 2.4(4), $g(G) = \chi_{gc}(G) = \chi'_{dg}(G) = x$.

Case 2. $x = y < z$: Let us consider three subcases.

Subcase 2.1. $z = x+1$: Let P_3 be a path with vertex set $\{x_1, x_2, x_3\}$ such that $d(x_1, x_3) = 2$. Let $K_{1,x-1}$ be a star with vertex set $\{y_0, y_1, y_2, ..., y_{x-1}\}$, where y_0 is a centre vertex. Let G be the graph obtained from this by identifying y_0 with a pendent vertex of P_3, say x_3. It is easily seen that the set of all extreme vertices of G is a minimum geodetic set S so that $g(G) = x$. Since $\chi(G) = 2$, S has the vertices of two color classes. So that $\chi_{gc}(G) = y = x$. If G has two color classes, then there exist a geodesic which does not has at least two color classes. It is clear that, a geodetic set is not a double geo chromatic set of G. Define a coloring by, $y_1, y_2, ..., y_{x-1}, x_2$ belongs to distinct color classes and x_1, x_3 belongs to one color class. Since $\{y_1, y_2, ..., y_{x-1}, x_1\}$ is a geodetic set S of G, S not have any vertex from one color class. Let $S \cup \{x_2\}$. Then $S \cup \{x_2\}$ is a double geo chromatic set of G. Hence $\chi^{'}_{dg}(G) = z = x + 1$.

Subcase 2.2. $z = x + 2$: Let H be the graph obtained as follows:

(1) Take two copies of P_5 with the vertex set $\{v_{i_1} : 1 \leqslant i \leqslant 5\}$ and $\{u_{i_1} : 1 \leqslant i \leqslant 5\}$.
(2) Take a copy of P_4 with the vertex set $\{x_{j_1} : 1 \leqslant j \leqslant 4\}$ and a copy of P_3 with the vertex set $\{w_{k_1} : 1 \leqslant k \leqslant 3\}$ such that $deg(w_{11}) = deg(w_{31}) = deg(x_{11}) = deg(x_{41}) = 1$ and $d(x_{11}, x_{31}) = 2$.
(3) Take one copy of P_2 with the vertex set $\{y_1, y_2\}$.
(4) Add two new vertices z_1, z_2 and join to both w_{11}, x_{11}. Also join w_{31} to x_{31}.
(5) Join one pendant vertex from each copy of P_5, say v_{11}, u_{11} to w_{11} and again join the remaining pendant vertex, say v_{51}, u_{51} of P_5 to x_{41}.
(6) Take a copy of $K_{1,x-6}$ with the vertex set $\{a_0, a_1, ..., a_{x-6}\}$, where a_0 is the centre vertex of $K_{1,x-6}$.

Let G be the graph obtained from H by identifying a_0 with x_{31}. The graph G is shown in Figure 5.

Clearly, $I[a_i, a_j] \cup I[y_2, a_i] \cup I[y_2, z_1] \cup I[z_1, z_2] \cup I[a_i, w_{21}] \cup I[u_{31}, v_{31}] = V(G)$,

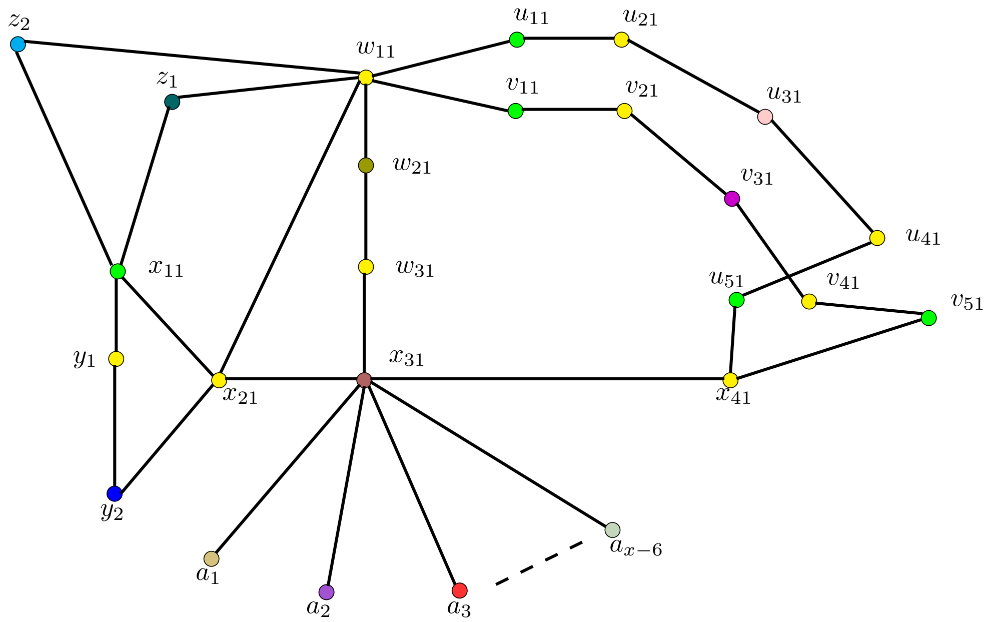

FIGURE 5. A graph G with $g(G) = \chi_{gc}(G) = x$ and $\chi^{'}_{dg}(G) = x + 2$

where $1 \leqslant i, j \leqslant x - 6\,(i \neq j)$. Then $S = \{a_1, a_2, ..., a_{x-6}, y_2, z_1, z_2, w_{21}, u_{31}, v_{31}\}$ is a geodetic set of G. Also, the removal of at least one vertex from S is not a geodetic set of G and so $g(G) = x$. It is clear that $\chi(G) = 3$ and S is a chromatic set of G. Therefore S is a geo chromatic set of G and $\chi_{gc}(G) = x$. By using three colors, we can't say that at least two color classes of G lie on a geodesic. Hence $z > x$. Define a coloring of G such that different vertices of S receive distinct colors, say color 1, color 2,...,color x and some vertices of $G \setminus S$ receive color $x + 1$ and also another some vertices of $G \setminus S$ receive color $x + 2$. It is clear that at least two color classes lie on each geodesic of G and also G receive $x + 2$ double geo colors. Let the vertices of $G \setminus S$ belong to the color classes either $C_{l_{(x+1)}}$ or $C_{l_{(x+2)}}$. Since S receive x colors, S does not receive any vertex from each $C_{l_{(x+1)}}$ and $C_{l_{(x+2)}}$. For obtaining S as a double geo chromatic set, choose atleast one vertex from each $C_{l_{(x+1)}}$ and $C_{l_{(x+2)}}$. Let $v_{11} \in C_{l_1}$ and $v_{21} \in C_{l_2}$. If $v_{11}, v_{21} \in S$, then $S_{dg} = S \cup \{v_{11}, v_{21}\}$ is a double geo chromatic set of G. Therefore $\chi^{'}_{dg}(G) \leqslant x + 2$. But, the removal of at least one vertex from $S_{dg} = S \cup \{v_{11}, v_{21}\}$ is not a double geo chromatic set of G. Hence $\chi^{'}_{dg}(G) = x + 2$.

Subcase 2.3. $z = x + 3$: Let $P_4 : x_1, x_2, x_3, x_4$ be a path of length 4. For

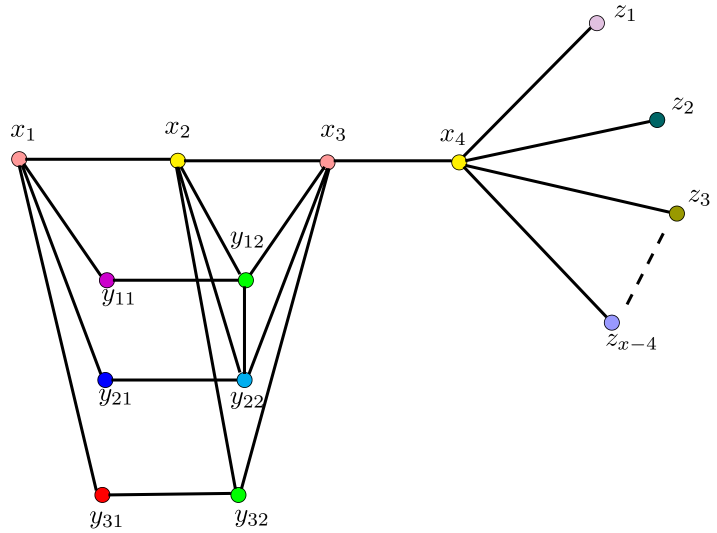

FIGURE 6. A graph G with $g(G) = \chi_{gc}(G) = x$ and $\chi^{'}_{dg}(G) = x + 3$

each integer i with $1 \leqslant i \leqslant 3$, let $P_2 : y_{i_1}, y_{i_2}$ be a path of order 2. Join each y_{i_1} to x_1 and join each y_{i_2} to x_2, x_3 and also join y_{12} to y_{22}. Let G be the graph obtained from this by adding $x - 4$ new vertices $z_1, z_2, ..., z_{x-4}$ to P_4 and join each $z_j\,(1 \leqslant j \leqslant x-4)$ to x_4. The graph G is shown in Figure 6. It is easily to seen that $I[z_r, z_s] \cup I[z_r, x_1] \cup I[x_1, y_{12}] \cup I[x_1, y_{22}] \cup I[x_1, y_{32}] = V(G)$, where $1 \leqslant r, s \leqslant x-4\,(r \neq s)$. Then $S = \{x_1, y_{11}, y_{21}, y_{31}, z_1, z_2, ..., z_{x-4}\}$ is a geodetic set of G and so $|S| \leqslant x$. If $|S| \leqslant x-1$, then S is not a geodetic set of G. Therefore $g(G) = x$. Clearly, S is a chromatic set of G and so S is a geo chromatic set S_c of G. Hence $\chi_{gc}(G) = x = y$. It is clear that S_c is not a double geo chromatic set of G. Define a coloring of G such that different vertices of S (except x_1) receive distinct colors, say color 1, color 2,...,color $x-1$ and the vertices x_1, x_3 receive color x, the vertices x_2, x_4 receive color $x+1$, the vertices y_{12}, y_{32} receive color $x+2$ and also the vertex y_{22} receive color $x+3$. Clearly, each geodesic of G have at least two color classes. But no vertex of S receive color $x+1$, color $x+2$, color $x+3$. Let the vertices which receive color $x+1$, color $x+2$ and color $x+3$ belong to the color classes $C_{l(x+1)}, C_{l(x+2)}$ and $C_{l(x+3)}$. For obtaining S as a double geo chromatic set, choose at least one vertex from each $C_{l(x+1)}, C_{l(x+2)}$ and $C_{l(x+3)}$. Let $x_2 \in C_{l(x+1)}$, $y_{12} \in C_{l(x+2)}$ and $y_{22} \in C_{l(x+3)}$. If $x_2, y_{22}, y_{12} \in S$, then $S \cup \{x_2, y_{12}, y_{22}\}$, which is a double geo chromatic set S_{dg} of G. Therefore $\chi'_{dg}(G) \leqslant x+3$. But, the removal of at least one vertex from $S \cup \{x_2, y_{12}, y_{22}\}$ is not a double geo chromatic set of G. Hence $\chi'_{dg}(G) = x+3$.

Case 3. $x < x+1 \leqslant z$: We consider three subcases.

Subcase 3.1. $z = x+1$: Let $C_6 : x_1, x_2, x_3, x_4, x_5, x_6, x_1$ be a cycle of order 6 with $d(x_3, x_1) = 2$ and $d(x_4, x_1) = 3$. Now add $x-2$ new vertices $y_1, y_2, ..., y_{x-2}$. Let G be the graph obtained from this by joining x_3 and x_5 to $y_1, y_2, ..., y_{x-2}$ and also join x_3 to x_5. The graph G is shown in Figure 7. It is easily to seen that

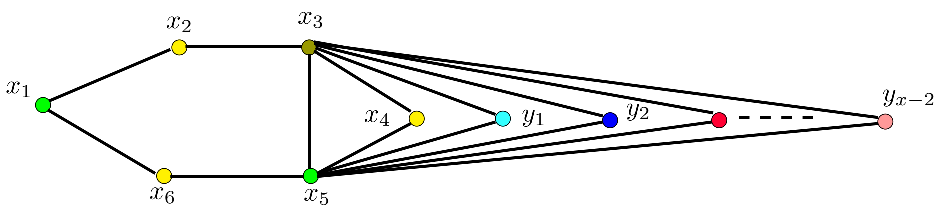

FIGURE 7. A graph G with $g(G) = x$, $\chi_{gc}(G) = \chi'_{dg}(G) = x+1$

$S = \{x_1, x_4, y_1, y_2, ..., y_{x-2}\}$ is a minimum geodetic set of G. Therefore $g(G) = x$. Since $\chi(G) = 3$, S does not have any vertex from one color class of G. Let a vertex from that color class be x_3. Now, the set S becomes $S \cup \{x_3\}$, which is a chromatic set of G. Clearly $S \cup \{x_3\}$ is a minimum geo chromatic set S_c of G so that $\chi_{gc}(G) = x+1$. It is clear that using three color classes are not enough for double geo chromatic set of G. Let us increase the color classes by assigning the colors $1, 2, 3, ..., x-2, x-1$ to $y_1, y_2, ..., y_{x-2}, x_3$ and two colors, say $x, x+1$ are required to color the vertices of $G \setminus \{y_1, y_2, ..., y_{x-2}, x_3\}$. Clearly, each geodesic contains at least two color classes of G. But S does not receive every colors which have been assigned. Let the color be $x-1$. Since x_3 receive a color $x-1$, $S \cup \{x_3\}$ is the set which has vertices from each defined color classes. Hence $S \cup \{x_3\}$ is a minimum double geo chromatic set S_{dg} of G, $\chi'_{dg}(G) = x+1$.

Subcase 3.2. $z = x+2$: Let $C_6 : x_1, x_2, x_3, x_4, x_5, x_6, x_1$ be a cycle of order 6. Join x_1 to x_3, x_5 and x_4 to x_2, x_6. Add $x-2$ new vertices $y_1, y_2, ..., y_{x-2}$. Let G be the graph obtained from this by joining each $y_i\,(1 \leqslant i \leqslant x-2)$ to x_4 and x_5 and x_3 to x_5. The graph G is shown in Figure 8. Let $S = \{x_1, x_4, y_1, y_2, ..., y_{x-2}\}$.

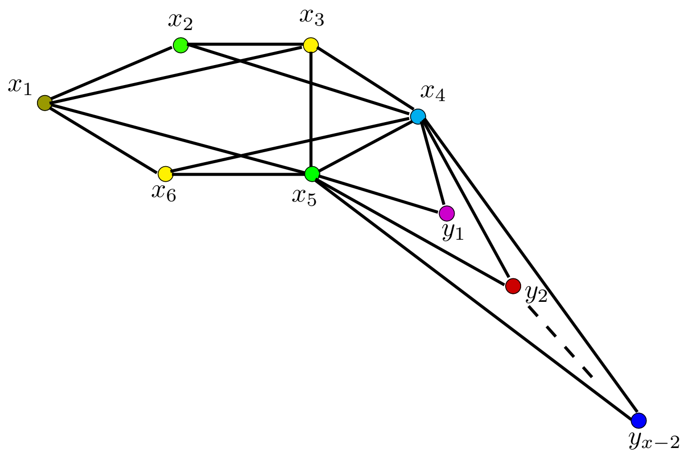

FIGURE 8. A graph G with $g(G) = x$, $\chi_{gc}(G) = x+1$ and $\chi'_{dg}(G) = x+2$

It is clear that S is a minimum geodetic set of G and $g(G) = x$. It is clear that $\chi(G) = 3$ and so G have three color classes, say C_{l_1}, C_{l_2} and C_{l_3}. But S does not receive any vertex from C_{l_3} so that S is not a chromatic set of G. Let $x_5 \in C_{l_3}$. If $x_5 \in S$, then $S \cup \{x_5\}$ is a geo chromatic set S_c of G. Therefore $\chi_{gc}(G) = x+1$. Define a double geo coloring of G such that different vertices of S receive distinct colors, say color 1, color 2,...,color x and some vertices of $G \setminus S$ receive color $x+1$ and also another some vertices of $G \setminus S$ receive color $x+2$. Let the vertices which receive color 1, color 2,...,color x, color $x+1$, color $x+2$ belong to the color classes, namely C_{l_1}, C_{l_2},...,C_{l_x}, $C_{l_{(x+1)}}$, $C_{l_{(x+2)}}$. It is clear that each geodesic have at least two color classes of G, but S do not receive any vertex from each $C_{l_{(x+1)}}$ and $C_{l_{(x+2)}}$. Let $x_3 \in C_{l_{(x+1)}}$ and $x_5 \in C_{l_{(x+2)}}$. If $x_3, x_5 \in S$, then $S \cup \{x_3, x_5\}$ is a double geo chromatic set of G. The removal of at least one vertex from $S \cup \{x_3, x_5\}$ is not a double geo chromatic set of G. Hence $\chi'_{dg}(G) = x + 2$.

Subcase 3.3. $z = x + 3$: Let $C_8 : a_1, a_2, ..., a_8, a_1$ be a cycle of order 8. Join a_2 to a_7, a_8, a_3 to a_8, a_7 and a_4 to a_6. Add $x - 2$ new vertices $b_1, b_2, ..., b_{x-2}$. Let G be the graph obtained from this by joining each b_j $(1 \leqslant j \leqslant x - 2)$ to a_4 and a_6. The graph G is shown in Figure 9.

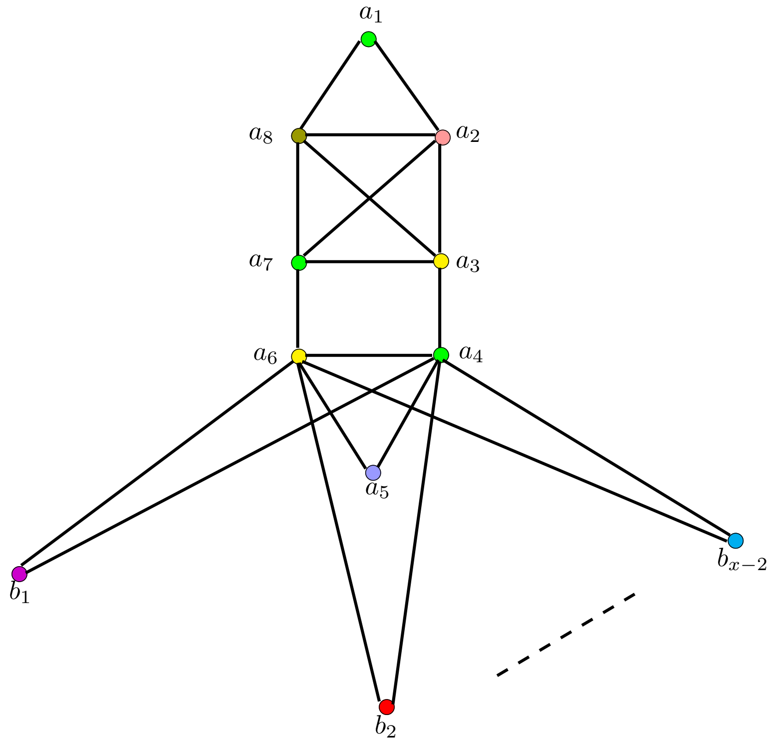

FIGURE 9. A graph G with $g(G) = x$, $\chi_{gc}(G) = x + 1$ and $\chi'_{dg}(G) = x + 3$

It is easily seen that $\{a_1, a_5, b_1, b_2, ..., b_{x-2}\}$ is a minimum geodetic set of G and $g(G) = x$. Since $\chi(G) = 4$, there exist four color classes (say) C_{l_1}, C_{l_2}, C_{l_3} and C_{l_4}. No vertex from one color class, say C_{l_4} belong to S so that S is not a chromatic set of G. To obtain S as a chromatic set, choose at least one vertex from C_{l_4}. Let $a_4 \in C_{l_4}$. If $a_4 \in S$, then $S_c = S \cup \{a_4\}$ is a minimum geo chromatic set of G. Therefore $\chi_{gc}(G) = x + 1$. Now, define a double geo coloring of G such that $b_1, b_2, ..., b_{x-2}, a_5, a_8, a_2$ receive distinct colors, say color 1, color 2,...,color x, color

$x+1$ and a_1, a_4, a_7 receive color $x+2$ and also a_3, a_6 receive color $x+3$. Let the vertices which receive color 1, color 2,..., color $x+2$, color $x+3$ belong to the color classes, namely C_{l_1}, C_{l_2},...,$C_{l_{(x+2)}}$, $C_{l_{(x+3)}}$. It is clear that each geodesic have at least two color classes of G. But no vertex from C_{l_x}, $C_{l_{(x+1)}}$, $C_{l_{(x+3)}}$ belong to S so that S do not receive every double geo colors. If $a_2, a_3, a_8 \in S$, then $S \cup \{a_2, a_3, a_8\}$ is a double geo chromatic set of G. Therefore $\chi'_{dg}(G) \leqslant x+3$. Also, the removal of at least one vertex from $S \cup \{a_2, a_3, a_8\}$ is not a double geo chromatic set of G. Hence $\chi'_{dg}(G) = x+3$. □

Theorem 3.2. *For an integer $a > 3$ with $\chi(G) = 2$, there exists a connected graph G such that $g(G) = a$ and $\chi_{gc}(G) = \chi'_{dg}(G) = a+1$.*

Proof. Let $P_3 : x_1, x_2, x_3$ be a path of order 3. Add $z_1, z_2, ..., z_a$ new vertices and join each z_i $(1 \leqslant i \leqslant a)$ to x_1 and x_3. Let $P_2 : w_1, w_2$ be a path of order 2 and join w_2 to x_3 of P_3 and join w_1 to z_a. Let $K_{1,a-1}$ be a star with the vertex set $\{y_0, y_1, y_2, ..., y_{a-1}\}$, where y_0 is the centre vertex. Let G be the graph obtained from this by identifying y_0 with w_2. The graph G is shown in Figure 10. It is easily

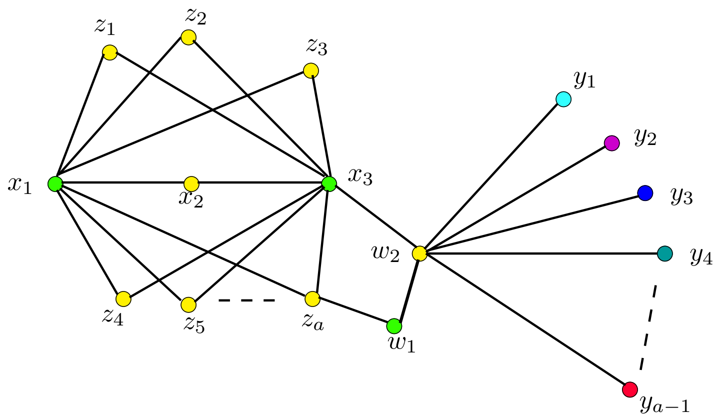

Figure 10. A graph G with $g(G) = a$ and $\chi_{gc}(G) = \chi'_{dg}(G) = a+1$

seen that $S = \{x_1, y_1, y_2, ..., y_{a-1}\}$ is a minimum geodetic set of G and $g(G) = a$. Since $\chi(G) = 2$, there exist two color classes, say C_{l_1} and C_{l_2}. By assigning a proper coloring of G, the vertices of S receive same color, say color 1 so that S is not a chromatic set of G. It is clear that S does not receive any vertex from C_{l_2} (say) and so choose at least one vertex from C_{l_2} to obtain S as a geo chromatic set of G. Let $x_2 \in C_{l_2}$. If $x_2 \in S$, then $S_c = S \cup \{x_2\}$ is a geo chromatic set of G and so $\chi_{gc}(G) = a+1$. Also, there exists a geodesic which do not contain at least two color classes of G so that two colors are not enough for double geo coloring.

Define a coloring of G such that $y_1, y_2, ..., y_{a-1}$ receive distinct colors, say color 1, color 2,...,color $a-1$ and some vertices of $G \setminus \{y_j\}\,(1 \leqslant j \leqslant a-1)$ receive color a and also another some vertices of $G \setminus \{y_j\}\,(1 \leqslant j \leqslant a-1)$ receive color $a+1$. It is clear that each geodesic of G contain at least two color classes of G. Therefore G receives $a+1$ double geo colors. It follows that $S_{dg} = S_c$ is a double geo chormatic set of G and also the removal of at least one vertex from $S_{dg} = S \cup \{x_2\}$ is not a double geo chromatic set of G. Hence $\chi^{'}_{dg}(G) = a+1$. □

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Department of Mathematics, Scott Christian College (Autonomous), Nagercoil - 629 003, India

Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamil Nadu, India

E-mail address: beulahsamlisam1991@gmail.com

Department of Mathematics, Government College of Engineering, Tirunelveli - 627 007, India

E-mail address: john@gcetly.ac.in

Department of Mathematics, Scott Christian College (Autonomous), Nagercoil - 629 003, India

E-mail address: robinchel@rediffmail.com