BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(1)(2021), 47-53
DOI: 10.7251/BIMVI2101047R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

HOMOMORPHISMS OF PSEUDO-UP ALGEBRAS

Daniel A. Romano

Abstract. As a generalization of UP-algebra, the notion of pseudo-UP algebras was introduced, and some of their properties were explored by the author in his article (Pseudo-UP algebra. An Introduction. *Bul. Int. Math. Virtual Inst.*, 10(2)(2020), 349-355). In addition, this author presented the concepts of pseudo-UP ideals and pseudo-UP filters in such algebraic structures. In this article, as a continuation of the author's works, the concept of homomorphisms between pseudo-UP algebras are introduced and discussed.

1. Introduction

Iampan [**1**] introduced a new algebraic structure which is called UP-algebras as a generalization of KU-alebras. He studied ideals and congruences in UP-algebras. He also introduced the concept of homomorphism of UP-algebras and investigated some related properties. Moreover, he derived some straightforward consequences of the relations between quotient UP-algebras and isomorphism. In the study of this algebraic structure, this author took part also ([**5, 6, 7, 8, 9**]).

The concept of pseudo-UP algebra was introduced in [**10**] and some of its characteristic properties were proved. In his article [**11**], the author introduced the concepts of pseudo-UP ideals and pseudo-UP filters in pseudo-UP algebras. The concept of homomorphisms between pseudo-UP algebras is introduced and discussed in this article, as a continuation of the papers mentioned above. The notion of homomorphisms between pseudo-UP algebras is designed in the same way as it was done in articles [**2, 3, 4**] when analyzing pseudo-BCK and pseudo-BCI algebras.

2010 *Mathematics Subject Classification.* 03G25, 06F99, 03G25 .
Key words and phrases. UP-algebra, Pseudo-UP algebra, Pseudo-UP ideal, Homomorphism between pseudo-UP algebras.

2. Preliminaries

In this section we will describe some elements of UP-algebras from the literature [**1, 10, 11**] necessary for our intentions in this text.

2.1. UP-algebras.

DEFINITION 2.1. ([**1**]) An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra where A is a nonempty set, $'\cdot'$ is a binary operation on A, and 0 is a fixed element of A (i.e. a nullary operation) if it satisfies the following axioms:
(UP-1) $(\forall x, y \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

2.2. Pseudo-UP algebras.

DEFINITION 2.2. ([**10**]) A pseudo-UP algebra is a structure $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$, where $' \leqslant '$ is a binary relation on a set A, $'\cdot'$ and $'*'$ are internal binary operations on A and $'0'$ is an element of A, verifying the following axioms:
(pUP-1) $(\forall x, yz \in A)(y \cdot z \leqslant (x \cdot y) * (x \cdot z) \ \wedge \ y * z \leqslant (x * y) \cdot (x * z))$;
(pUP-4) $(\forall x, y \in A)((x \leqslant y \ \wedge \ y \leqslant x) \implies x = y)$;
(pUP-5) $(\forall x, y \in A)((y \cdot 0) * x = x \ \wedge \ (y * 0) \cdot x = x)$ and
(pUP-6) $(\forall x, y \in A)((x \leqslant y \iff x \cdot y = 0) \ \wedge \ (x \leqslant y \iff x * y = 0))$.

Each UP-algebra $(A, \cdot, 0)$ can be viewed as a pseudo-UP algebra $((A, \leqslant), \cdot, *, 0)$ with $* = \cdot$.

From the previous definition, it immediately follows

LEMMA 2.1 ([**10**]). *In a pseudo-UP algebra* $\mathfrak{A}$ *the following holds:*
(8) $(\forall x \in A)(x \cdot 0 = 0 \ \wedge \ x * 0 = 0)$*;*
(9) $(\forall x \in A)(0 \cdot x = x \ \wedge \ 0 * x = x)$*; and*
(10) $(\forall x \in A)(x \cdot x = 0 \ \wedge \ x * x = 0)$.

In the following definitions, we introduce the concept of pseudo-UP ideals and pseudo-UP filters in pseudo-UP algebras

DEFINITION 2.3. ([**11**]) A nonempty subset J of a pseudo-UP algebra $\mathfrak{A}$ is called a pseudo-UP ideal of $\mathfrak{A}$ if it satisfies
(pJ1) $0 \in J$;
(pJ2) $(\forall x, y, z \in A)((x \cdot (y * z) \in J \ \wedge \ y \in J) \implies x \cdot z \in J)$ and
(pJ3) $(\forall x, y, z \in A)((x * (y \cdot z) \in J \ \wedge \ y \in J) \implies x * z \in J)$.

DEFINITION 2.4. ([**11**]) A nonempty subset F of a pseudo-UP algebra $\mathfrak{A}$ is called a pseudo-UP filter of $\mathfrak{A}$, if it satisfies the following properties:
(pF1) $0 \in F$;
(pF2) $(\forall x, y \in A)((x \in F \ \wedge \ x \cdot y \in F \implies y \in F)$; and
(pF3) $(\forall x, y \in A)((x \in F \ \wedge \ x * y \in F \implies y \in F)$.

3. The main results

3.1. Lattices of pseudo-UP ideals and pseudo-UP filters.

THEOREM 3.1. *The family $\mathfrak{J}(A)$ of all pseudo-UP ideals in a pseudo-UP algebra $\mathfrak{A}$ forms a complete lattice.*

PROOF. (1) Let $\{J_k\}_{k\in K}$ be a family of pseudo-UP ideals in a pseudo-UP algebra $\mathfrak{A}$. It is obvious that $0 \in \bigcap_{k\in K} J_k$ is valid.

Let $x, y, z \in A$ be arbitrary elements such that $x \cdot (y * z) \in \bigcap_{k\in K} J_k$ and $y \in \bigcap_{k\in K} J_k$. Then for every index $k \in K$, it is valid $x \cdot (y * z) \in J_k$ and $y \in J_k$. Thus $x \cdot z \in J_k \subseteq \bigcap_{j\in K} J_j$. The implication (pJ3) can be proved by analogy with the previous proof. So, the intersection $\bigcap_{j\in K} J_j$ is a pseudo-UP ideal in $\mathfrak{A}$.

(2) Let $\mathfrak{X}$ be the family of all pseudo-UP ideals of $\mathfrak{A}$ that contain the union $\bigcup_{k\in K} J_k$. Then, according to the first part of this proof, $\cap\mathfrak{X}$ is the minimal pseudo-UP ideal in $\mathfrak{A}$ that contains $\bigcup_{k\in K} J_k$.

(3) If we put $\sqcup_{k\in K} J_l = \cap\mathfrak{X}$ and $\sqcap_{k\in K} J_k = \bigcap_{k\in K} J_k$, then $(\mathfrak{J}(A), \sqcup, \sqcap)$ is a complete lattice. □

COROLLARY 3.1. *Let B be an arbitrary subset in a pseudo-UP algebra $\mathfrak{A}$. Then there exists the minimal pseudo-UP ideal J_B in $\mathfrak{A}$ that contains B.*

PROOF. The proof of this corollary follows directly from the second part of the proof of the previous theorem. □

COROLLARY 3.2. *Let a be an arbitrary element in a pseudo-UP algebra $\mathfrak{A}$. Then there exists the minimal pseudo-UP ideal J_a in $\mathfrak{A}$ that contains a.*

PROOF. The proof of this corollary follows directly from the the previous Corollary if we put $B = \{a\}$. □

The proof of the following theorem can be deducted in a similar way as the proof of the preceding theorem, so we shall leave it out.

THEOREM 3.2. *The family $\mathfrak{F}(A)$ of all pseudo-UP filters in a pseudo-UP algebra $\mathfrak{A}$ forms a complete lattice and holds $\mathfrak{F}(A) \subseteq \mathfrak{J}(A)$.*

3.2. The concept of pseudo-UP homomorphisms. The pseudo homomorphisms between pseudo-BCK algebras were studied by Y. B. Jun, M. Kondo and K. H. Kum in [**2**] and K. J. Lee and C. H. Park in [**4**]. The pseudo homomorphisms between pseudo-BCI algebras were studied by Y. B. Jun, H. S. Kim and J. Neggers in [**3**].

We will transfer the idea of determining homomorphisms on these algebras to pseudo-UP algebras in the following definition.

DEFINITION 3.1. Let $A = ((A, \leqslant_A), \cdot_A, *_A, 0_A)$ and $B = ((B, \leqslant_B), \cdot_B, *_B, 0_A)$ be pseudo-UP algebras. A mapping $f : A \longrightarrow B$ is called a pseudo-UP homomorphism if

$(\forall x, y \in A)(f(x \cdot_A y) = f(x) \cdot_B f(y))$ and

$(\forall x, y \in A)(f(x *_A y) = f(x) *_B f(y))$.

EXAMPLE 3.1. Let $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$ be a pseudo-UP algebra and $a \in A$. Then the mappings $f_a : A \longrightarrow A$ and $g_a : A \longrightarrow A$, defined by $f_a(x) = a \cdot x$ and $g_a(x) = a * x$, are not pseudo-UP homomorphisms.

LEMMA 3.1. *If $f : A \longrightarrow B$ is a pseudo-UP homomorphism, then $f(0_A) = 0_B$ holds.*

PROOF. The assertion of this lemma follows directly from the Definition 3.1 with respect to equality (10) in the article [**10**]. □

COROLLARY 3.3. *If $f : A \longrightarrow B$ is a pseudo-UP homomorphism, then*
(a) $(\forall x, y \in A)(x \leqslant_A y \implies f(x) \leqslant_B f(y))$.

PROOF. Let $x, y \in A$ such that $x \leqslant_A y$. Then $x \cdot y = 0_A$ and $x * y =_A 0$ by (pUP-6) in [**10**]. Thus $f(x) \cdot_B g(y) = f(x \cdot_A y) = 0_B = f(x *_A y) = f(x) *_N f(y)$ by Corollary 3.3. So, it follows $f(x) \leqslant_B f(y)$ in accordance with (pUP-6) again. □

PROPOSITION 3.1. *Let A, B and C be pseudo-UP algebras and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be pseudo-UP homomorphisms. Then $g \circ f : A \longrightarrow C$, defined by*

$$(\forall x \in A)((g \circ f)(x) = g(f(x)))$$

is a pseudo-UP homomorphism.

PROOF. Straightforward. □

THEOREM 3.3. *Let $f : A \longrightarrow B$ be a pseudo-UP homomorphism between pseudo-UP algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then*

(i) *If J is a pseudo-UP ideal of $\mathfrak{B}$, then $f^{-1}(J)$ is a pseudo-UP ideal of $\mathfrak{A}$. Association $J \longmapsto f^{-1}(J)$, induced by the homomorphism f, realizes the correspondence between family $\mathfrak{J}(B)$ and the family $\mathfrak{J}(A)$.*

(ii) *If f is surjective and I is a pseudo-UP ideal of $\mathfrak{A}$, then $f(I)$ is a pseudo-UP ideal of B. Association $I \longmapsto f(I)$, induced by the homomorphism f, realizes the correspondence between family $\mathfrak{J}(A)$ and the family $\mathfrak{J}(B)$.*

PROOF. (i) Assume that J is a pseudo-UP ideal of $\mathfrak{A}$. Obviously $0_A \in f^{-1}(J)$.

Let $x, y, z \in A$ be such that $x \cdot_A (y *_A z) \in f^{-1}(J)$ and $y \in f^{-1}(J)$. Then $f(x) \cdot_B (f(y) *_B f(z)) \in J$. Since J is a pseudo-UP ideal in B, we have $f(x) \cdot_B f(z) \in J$. Thus $f(x \cdot_A z) \in J$ and $x \cdot_A z \in f^{-1}(J)$.

The implication $x *_A (y \cdot_A z) \in f^{-1}(J) \wedge y \in f^{-1}(J) \implies x *_A z \in f^{-1}(J)$ can be proved in an analogous way.

(ii) Assume that f is surjective and let I be a pseudo-UP ideal of A. Obviously, $0_B = f(0_A) \in f(I)$.

Let $a, b, c \in B$ be such that $a \cdot_B (b *_B c) \in f(I)$ and $b \in f(I)$. Then there exist elements $x, z \in B$ and $y, u \in I$ such that $a = f(x)$, $b = f(y)$, $c = f(z)$ and $a \cdot_B (b *_B c) = f(u)$. Thus, from $f(x \cdot_A (y *_A z)) = f(x) \cdot_B (f(y) *_B f(z))) = a \cdot_A (b *_A c)) = f(u) \in f(I) \subseteq B$, it follows $x \cdot_A (y *_A z) \in I$. Now, from $x \cdot_A (y *_A z) \in I$ and $y \in I$ it follows $x \cdot_A z \in I$ because I is a pseudo-UP ideal in $\mathfrak{A}$. Hence $f(x) \cdot_B f(z) = f(x \cdot_A z) \in f(I)$. So, $f(I)$ is a pseudo-UP ideal in $\mathfrak{B}$.

The implication $a *_B (y \cdot_B z) \in f(I) \wedge b \in f(I) \Longrightarrow a *_B z \in f(I)$ can be proved in an analogous way. □

COROLLARY 3.4. *Let $f : A \longrightarrow B$ be a pseudo-UP homomorphism of pseudo-UP algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then the kernel $Ker(f) = \{x \in A : f(x) = 0_B\}$ of f is a pseudo-UP ideal of A.*

PROOF. Obviously, $0_A \in Ker(f)$.
Let $x, y, z \in A$ be such that $x \cdot_A (y *_A z) \in Ker(f)$ and $y \in Ker(f)$. Then $f(y) = 0_B$ and $0_B = f(x \cdot_A (y *_A z)) = f(x) \cdot_B (f(y) *_B f(z)) = f(x) \cdot_B (0_B *_B f(z)) = f(x) \cdot_B f(z) = f(x \cdot_A z)$. Thus $x \cdot_A z \in Ker(f)$.
The second implication can be proven analogous to the previous one. □

THEOREM 3.4. *Let $f : A \longrightarrow B$ be a pseudo-UP homomorphism between pseudo-UP algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then:*
(iii) *If G is a pseudo-UP filter of $\mathfrak{B}$, then $f^{-1}(G)$ is a pseudo-UP filter of $\mathfrak{A}$.*
(iv) *If f is surjective and F is a pseudo-UP filter of $\mathfrak{A}$, then $f(F)$ is a pseudo-UP filter of B.*

PROOF. The proof of this theorem can be deducted by direct verification, similar to the proof of the Theorem 3.3. □

3.3. The concept of congruences of pseudo-UP algebras. We define the notion of congruence relations on pseudo-UP algebras.

DEFINITION 3.2. Let $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$ be a pseudo-UP algebra and θ be an equality relation on the set A. θ is called:
(a) Left congruence relation on $\mathfrak{A}$ if
$(\forall x, y, z \in A)((x, y) \in \theta \Longrightarrow ((z \cdot x, z \cdot y) \in \theta \wedge (z * x, z * y) \in \theta))$.
(b) Right congruence relation on $\mathfrak{A}$ if
$(\forall x, y, z \in A)((x, y) \in \theta \Longrightarrow ((x \cdot z, y \cdot z) \in \theta \wedge (x * z, y * z) \in \theta))$.
(c) Congruence relation on $\mathfrak{A}$ if it is a left and right congruence.

EXAMPLE 3.2. Let us suppose that the following formula

$$(\forall x, y, z \in A)((x \cdot (y \cdot z) = (x \cdot y) \cdot z) \wedge (x * (y * z) = (x * y) * z)$$

is a valid formula in a pseudo-UP algebra A. For any $a \in A$, let us define

$$R_a = \{(x, y) \in A \times A : a \cdot x = a \cdot y \wedge a * x = a * y\}.$$

Then R_a is a right congruence on $\mathfrak{A}$. Obviously, the following is valid $R_a = Ker(f_a) \cap Ker(g_a)$.

LEMMA 3.2. *The condition* (c) *is equivalent to the following implication*
$(\forall x, y, u, v \in A)(((x, y) \in \theta \wedge (u, v) \in \theta) \Longrightarrow ((x \cdot u, y \cdot v) \in \theta \wedge (x * u, y * v) \in \theta))$

PROPOSITION 3.2. *Let θ a congruence relation on a pseudo-UP algebra $\mathfrak{A}$. Then the set $C_0 = \{x \in A : (x, 0) \in \theta\}$ is a pseudo-UP ideal in A.*

PROOF. Since $(0,0) \in \theta$, it is obvious that $0 \in C_0$ holds.

Let $x, y, z \in A$ be such that $x \cdot (y * z) \in C_0$ and $y \in C_0$. Then $(x \cdot (y * z), 0) \in \theta$ and $(y, 0) \in \theta$. From $(y, 0) \in \theta$ it follows $(y * z, 0 * z) \in \theta$ and $(y * z, z) \in \theta$ by (9). Thus $(x \cdot (y * z), x \cdot z) \in \theta$. Now, from $(x \cdot (y * z), 0) \in \theta$ and $(x \cdot (y * z), z \cdot z)$ it follows $0, x \cdot z) \in \theta$ because θ is a transitive relation. So, $x \cdot z \in C_0$.

Proof of the Second Implication $(x * (y \cdot z) \in C_0 \wedge y \in C_0) \Longrightarrow x * z \in C_0$ can be demonstrated similarly to the proof of a previous implication. □

THEOREM 3.5. *Let $f : A \longrightarrow B$ be a pseudo-UP homomorphism and $\theta = \{(x, y) \in A \times A : f(x) =_B f(y)\}$. Then θ is a congruence relation on $\mathfrak{A}$,*

PROOF. The proof of this theorem is obtained by direct verification. Indeed:

Since $f(x) =_B f(x)$ for any $x \in A$ because f is a mapping, we conclude that θ is reflexive. Of course, it is obvious that θ is a symmetric and transitive relation.

Let $x, y, z \in A$ such that $(x, y) \in \theta$. Then $f(x) =_B f(x)$. Thus

$f(z \cdot_A x) =_B f(z) \cdot_B f(x) =_B f(z) \cdot_B f(y) =_B f(z \cdot_A y)$, and

$f(z *_A x) =_B f(z) *_B f(x) =_B f(z) *_B f(y) =_B f(z *_A y)$.

So, hold $(z \cdot_A x, z \cdot_A y) \in \theta$ and $(z *_A x, z *_A y) \in \theta$. Therefore, we concluded by part (a) of Definition 3.2 that θ is a left congruence on the pseudo-UP algebra $\mathfrak{A}$.

Analogously to this one can prove that θ is a right congruence at $\mathfrak{A}$. Finally, θ is a congruence at $\mathfrak{A}$ according to (c) in the definition 3.2. □

4. Final observation

The concept of pseudo-UP homomorphism between pseudo-UP algebras is introduced and discussed in this text. The obtained results, in further research, can serve as base to construct isomorphism theorems for pseudo-UP algebras. The problem we encounter with respect to any congruence of θ on a pseudo-UP algebra $\mathfrak{A}$ is that, generally speaking, the factor structure of $\mathfrak{A}/\theta$ need not be a pseudo-UP algebra. With little effort it can be verified that the factor structure $\mathfrak{A}/\theta$ satisfies the axioms (pUP-1), (pUP-5) and (pUP-6) in the Definition 2.3 except the axiom (pUP- 4).

This conclusion predicts to us that further exploration of the properties of homomorphisms on pseudo-UP algebras would have difficulty with designing isomorphism theorems with these algebras if we proceeded into the usual framework of algebraic considerations.

References

[1] A. Iampan. A new branch of the logical algebra: UP-algebras. *J. Algebra Relat Top.*, **5**(1)(2017), 35-54.

[2] Y. B. Jun, M. Kondo and K. H. Kim. Pseudo-ideals of pseudo-BCK algebras. *Sci. Math. Japonicae Online*, **8**(2003), 87–91.

[3] Y. B. Jun, H. S. Kim and J. Neggers. On pseudo-BCI ideals of pseudo-BCI algebras. *Mat. Vesnik*, **58**(1)(2006), 39-46.

[4] K. J. Lee and C. H. Park. Some ideals of pseudo-BCI-algebras. *J. Appl. Math. Inform.*, **27**(1-2)(2009), 217-231.

[5] D. A. Romano. Proper UP-filters in UP-algebra. *Universal Journal of Mathematics and Applications*, **1**(2)(2018), 98–100.

[6] D. A. Romano. Notes on UP-ideals in UP-algebras. *Communications in Advanced Mathematical Sciences*, **1**(1)(2018), 35–38.
[7] D. A. Romano. Some properties of proper UP-filters of UP-algebras. *Fundamental Journal of Mathematics and Applications*, **1**(2)(2018), 109–111.
[8] D. A. Romano. Some decompositions of UP-ideals and proper UP-filters. *Mathematical Advances in Pure and Applied Sciences*, **2**(1)(2019), 16–18.
[9] D. A. Romano. Pseudo-valuations on UP-algebras. *Universal Journal of Mathematics and Applications*. **2**(3)(2019), 138–140.
[10] D. A. Romano. Pseudo-UP algebras, An introduction. *Bull. Int. Math. Virtual Inst.*, **10**(2)(2020), 349–355.
[11] D. A. Romano. Pseudo-UP ideals and pseudo-UP filters in pseudo-UP algebras. *Mathematical Sciences and Applications E-Notes*, **8**(1)(2020), 155–158.

Received by editors 23.10.2019; Revised version 25.06.2020; Available online 06.07.2020.

International Mathematical Virtual Institute,
6, Kordunaška Street, 78000 Banja Luka, Bosnia and Herzegovina
E-mail address: bato49@hotmail.com