BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(1)(2021), 39-45
DOI: 10.7251/BIMVI2101039U

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON HYPER PSEUDO BCI-ALGEBRAS

Didem Sürgevil Uzay and Alev Firat

Abstract. In this paper, we introduce the notion of hyper pseudo BCI-algebras and investigate some related properties. Also, we define some kinds of hyper pseudo subalgebras on hyper pseudo BCI-algebras.

1. Introduction

The study of BCK-algebras was initiated by K. Iseki [**4**] as a generalization of concept of set-theoretic difference and propositional calculus. In [**5**], the notion of BCI-algebra which is a generalization of a BCK-algebra was introduced by Iseki.

The hyperstructure theory (called also multialgebras) was introduced by F. Marty, [**9**], and hyper BCK-algebras were studied by many authors. Also the notion of hyper pseudo BCK-algebras which is a generalization of a hyper BCK-algebra and pseudo BCK-algebra was defined by R. A. Borzooei [**1**].

And then hyper BCI-algebras which is a generalization of a BCI-algebras was studied by X. L. Xin [**10**]. In [**11**], the notion of multiplier of hyper BCI-algebras was introduced and some properties of hyper BCI-algebras were investigated.

Also pseudo BCI-algebras was introduced by W. A. Dudek [**2**] and pseudo BCI-algebras were studied by many authors.[**3, 4, 7, 8, 12**]

In this paper, we introduce the notion of hyper pseudo BCI-algebras, which is a generalization of hyper BCI-algebras and pseudo BCI-algebras and discuss some related properties. Also we define some kinds of hyper pseudo subalgebras on hyper pseudo BCI-algebras.

2010 *Mathematics Subject Classification.* 06*F*35, 03*G*25.
Key words and phrases. hyper pseudo BCI-algebra, hyper pseudo subalgebra.
Communicated by Daniel A. Romano.

2. Preliminaries

DEFINITION 2.1. ([**5**]) An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a BCI-algebra if it satisfies: for all $x, y, z \in X$.

(BCI1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI2) $((x * (x * y)) * y = 0$,
(BCI3) $x * x = 0$,
(BCI4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

DEFINITION 2.2. ([**2**]) A pseudo BCI-algebra is a structure $\mathfrak{X} = \{X, \preceq, *, \circ, 0\}$ where $'' \preceq''$ is a binary relation on a set X, $''*''$ and $''\circ''$ are binary operations on X and 0 is an element of X, verifying the axioms, for all $x, y, z \in X$.

(a1) $(x * y) \circ (x * z) \preceq z * y$, $(x \circ y) * (x \circ z) \preceq z \circ y$,
(a2) $x * (x \circ y) \preceq y$, $x \circ (x * y) \preceq y$,
(a3) $x \preceq x$,
(a4) $x \preceq y$, $y \preceq x$ imply $x = y$,
(a5) $x \preceq y \Leftrightarrow x * y = 0 \Leftrightarrow x \circ y = 0$.

PROPOSITION 2.1 ([**2, 7, 8**]). *Let $\mathfrak{X}$ be a pseudo BCI-algebra. Then the following hold for all $x, y, z \in X$:*

(b_1) *if $x \preceq 0$, then $x = 0$,*
(b_2) *if $x \preceq y$, then $z * y \preceq z * x$ and $z \circ y \preceq z \circ x$,*
(b_3) *if $x \preceq y$ and $y \preceq z$, then $x \preceq z$,*
(b_4) $(x * y) \circ z = (x \circ z) * y$,
(b_5) *$x * y \preceq z$ iff $x \circ z \preceq y$,*
(b_6) $(x * y) * (z * y) \preceq x * z$, $(x \circ y) \circ (z \circ y) \preceq x \circ z$,
(b_7) *if $x \preceq y$, then $x * z \preceq y * z$ and $x \circ z \preceq y \circ z$,*
(b_8) $x * 0 = x = x \circ 0$,
(b_9) $x * (x \circ (x * y)) = x * y$, $x \circ (x * (x \circ y)) = x \circ y$,
(b_{10}) $0 * (x \circ y) \preceq y \circ x$,
(b_{11}) $0 \circ (x * y) \preceq y * x$,
(b_{12}) $0 * (x * y) = (0 \circ x) \circ (0 * y)$,
(b_{13}) $0 \circ (x \circ y) = (0 * x) * (0 \circ y)$,
(b_{14}) $0 * x = 0 \circ x$.

DEFINITION 2.3 ([**2**]). By a pseudo BCI-subalgebra of a pseudo BCI-algebra $\mathfrak{X}$, we mean subset S of X which satisfies $x * y \in S$ and $x \circ y \in S$ for all $x, y \in S$.

DEFINITION 2.4 ([**7**]). Let $\mathfrak{X} = (X, \preceq, *, \circ, 0)$ be a pseudo BCI-algebra. For any nonempty subset J of X and any element y of X, we denote

$$*(y, J) := \{x \in X | x * y \in J\} \text{ and } \circ(y, J) := \{x \in X | x \circ y \in J\}$$

Note that $*(y, J) \cap \circ(y, J) = \{x \in X | x * y \in J, x \circ y \in J\}$.

A nonempty subset J of a pseudo BCI-algebra $\mathfrak{X}$ is called a pseudo BCI-ideal of $\mathfrak{X}$ if it satisfies

(I_1) $0 \in I$,
(I_2) $(\forall y \in J)$ $*(y, J) \subseteq J$ and $\circ(y, J) \subseteq J$.

Note that if $\mathfrak{X}$ is a pseudo BCI-algebra satisfying $x * y = x \circ y$ for all $x, y \in X$, then the notion of a pseudo BCI-ideal and a BCI-ideal coincide.

DEFINITION 2.5. ([**3**]) A pseudo BCI-algebra $\mathfrak{X}$ is said to be p-semisimple if it satisfies for all $x \in X$,

$$\text{if } 0 \preceq x, \text{ then } x = 0.$$

PROPOSITION 2.2 ([**3**]). *Let $\mathfrak{X}$ be a pseudo BCI-algebra. Then the following are equivalent for $x, y \in X$,*

(i) *X is p-semisimple,*
(ii) *if $x \preceq y$, then $x = y$,*
(iii) $x * (x \circ y) = y = x \circ (x * y)$,
(iv) $0 * (0 \circ x) = x = 0 \circ (0 * x)$,
(v) $x * (0 \circ y) = y \circ (0 * x)$.

3. ON HYPER PSEUDO BCI-ALGEBRAS

DEFINITION 3.1. Let H be a nonempty set and $''\circ, *''$ be two hyper operations on H and $''0''$ is a constant element. $(H, \circ, *, 0)$ is called to be a hyper pseudo BCI-algebra, if it satisfies the following conditions:

(HPI1) $(x \circ z) \circ (y \circ z) \ll x \circ y$, $(x * z) * (y * z) \ll x * y$,
(HPI2) $(x \circ y) * z = (x * z) \circ y$,
(HPI3) $x \ll x$,
(HPI4) $x \ll y$, $y \ll x \Rightarrow x = y$,
(HPI5) $0 \circ (0 \circ x) \ll x$, $0 * (0 * x) \ll x$,
(HPI6) $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Moreover for every $A, B \subseteq H$, $A \ll B$ is defined by there exists a $b \in B$ such that $a \ll b$ for all $a \in A$.

EXAMPLE 3.1. Let $H = \{0, 1, 2\}$ and $''\circ, *''$ be a hyper operation on H with Cayley table as follows:

$\circ$	0	1	2
0	{0,1}	{0,1}	{0,1}
1	{1}	{0,1}	{0,1}
2	{2}	{2}	{0,1}

$*$	0	1	2
0	{0,1}	{0,1}	{0,1}
1	{1}	{0,1}	{0,1}
2	{2}	{2}	{0,1,2}

Then it is easily checked that $(H, \circ, *, 0)$ is a hyper pseudo BCI-algebra.

THEOREM 3.1. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra. If $itsatisfiesx *$ $y = x \circ y$, for all $x, y \in H$, then H is a hyper BCI algebra; and if $'' * ''$, $'' \circ ''$ are singleton, then H is a pseudo BCI-algebra.*

PROOF. Let H be a hyper pseudo BCI-algebra. If $x * y = x \circ y$, for all $x, y \in H$, then H is a hyper BCI-algebra and if $'' * ''$, $'' \circ ''$ are singleton, then H is a pseudo BCI-algebra.

By $(a4)$, $(a5)$ we have $x \ll y \Leftrightarrow x * y = \{0\} \Leftrightarrow x \circ y = \{0\}$. Using $(HPI1)$, we get $(x \circ z) \circ (y \circ z) \ll (x \circ y)$. Therefore we have $((x \circ z) \circ (y \circ z)) * (x \circ y) = 0$. Using $(HPI2)$, we get $((x \circ z) * (x \circ y)) \circ (y \circ z) = 0$. Then we have $((x \circ z) * (x \circ y)) \ll (y \circ z)$.

Similarly we can see that; $((x * z) \circ (x * y)) \ll (y * z)$. Hence $(a1)$ holds. $(HPI3)$ $x \ll x$. Then $(a3)$ holds. Using $(HPI2)$ and $(a3)$, we get $(x * (x \circ y)) \circ y = (x \circ y) * (x \circ y)$. Therefore $0 \in (x \circ y) * (x \circ y)$. So that $(x * (x \circ y)) \ll y$.

Similarly, it can be proved that $(x \circ (x * y)) \ll y$. Hence $(a2)$ holds. Hence, H is a hyper pseudo BCI-algebra. □

PROPOSITION 3.1. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra. Then it satisfies the following conditions for all $x, y, z \in H$:*

(i) $x \ll 0$ *implies* $x = 0$.
(ii) $0 \in x \circ (x * 0)$, $0 \in x * (x \circ 0)$,
(iii) $x \ll x \circ 0$, $x \ll x * 0$,
(iv) $0 \circ (x \circ y) \ll y \circ x$, $0 * (x * y) \ll y * x$,
(v) $A \ll A$,
(vi) $A \subseteq B$ *implies* $A \ll B$,
(vii) $A \ll \{0\}$ *implies* $A = \{0\}$,
(viii) $x \circ 0 \ll \{y\}$ *implies* $x \ll y$ *and* $x * 0 \ll \{y\}$ *implies* $x \ll y$.
(ix) $y \ll z$ *implies* $x \circ z \ll x \circ y$ *and* $x * z \ll x * y$.
(x) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$ *and* $x \circ z \ll y \circ z$.
$x * y = \{0\}$ *implies* $(x * z) * (y * z) = \{0\}$ *ve* $x * z \ll y * z$.
(xi) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$.

PROOF. (i) Let $x \ll 0$. Then $0 \in x \circ 0$ and $0 \in x * 0$ and so using $(HPI1)$, we have

$$0 \in 0 \circ (x \circ 0) \subseteq (0 \circ 0) \circ (x \circ 0) \ll 0 \circ x,$$
$$0 \in 0 * (x * 0) \subseteq (0 * 0) * (x * 0) \ll 0 * x.$$

Therefore $0 \ll 0 \circ x$, $0 \ll 0 * x$. On the other hand using $(HPI5)$, we have $0 \in 0 \circ (0 \circ x) \ll x$, $0 \in 0 * (0 * x) \ll x$. Therefore we get $0 \ll x$. By $(HPI4)$, we find $x = 0$.

(ii) Using $(HPI2)$, we find $0 \in (x \circ 0) * (x \circ 0) = (x * (x \circ 0)) \circ 0$. Therefore there exists $c \in x * (x \circ 0)$ such that $c \ll 0$. By (i); we have $c = 0$. Hence $0 \in x * (x \circ 0)$. Similarly, it can be prove that $0 \in x \circ (x * 0)$.

(iii) Using (ii), we have $0 \in x * (x \circ 0)$ and $0 \in x \circ (x * 0)$. Therefore, we get $x \ll x \circ 0$, $x \ll x * 0$.

(iv) By $(HPI3)$ and $(HPI1)$; we find $0 \circ (x \circ y) \subseteq (y \circ y) \circ (x \circ y) \ll y \circ x$. Hence we get $0 \circ (x \circ y) \ll (y \circ x)$. By the similar way, we can see that; $0 * (x * y) \ll (y * x)$.

(v) For all $x \in H$, by $(HPI3)$; we get $x \ll x$. Therefore we find $A \ll A$.

(vi) Let $A \subseteq B$ and $a \in A$. $b \in B$ such that $b = a$ and using $(HPI3)$, we find $a \ll b$. Hence it can be prove that $A \ll B$.

(vii) Let $a \in A$ and $A \ll \{0\}$. Then $a \ll 0$ and using (i), we have $a = 0$. Hence we find $A = \{0\}$.

(viii) By $(HPI2)$; we have $0 \in (x \circ 0) * y = (x * y) \circ 0$. Therefore we get $0 \in d \circ 0$. Hence there exists $d \in x \circ y$ such that $c \ll 0$. By (i); we find $d = 0 \in x \circ y$. The similar way, it can be prove that $x \ll y$ for $x * 0 \ll \{y\}$.

(ix) Let $y \ll z$. Then using $(HPI1)$, we can find $(x \circ z) \circ 0 \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y$ and $(x * z) * 0 \subseteq (x * z) * (y * z) \ll x * y$. Hence we have $(x \circ z) \circ 0 \ll x \circ y$ and $(x * z) * 0 \ll x * y$. For all $a \in x \circ z$ there exists $b \in x \circ y$ such that $a \circ 0 \ll \{b\}$ and for all $c \in x * z$ such that $d \in x * y$ there exists $c * 0 \ll \{d\}$. Moreover, using $(viii)$, we have $a \ll b$, $c \ll d$ and so we find $x \circ z \ll x \circ y$ and $x * z \ll x * y$.

(x) Let $x \circ y = 0$ and $x * y = 0$. Then by $(HPI1)$; we get $(x \circ z) \circ (y \circ z) \ll (x \circ y) = \{0\}$ and using (vii), we have $(x \circ z) \circ (y \circ z) = \{0\}$. Similarly, we have $(x * z) * (y * z) \ll (x * y) = \{0\}$ so that by (vii); we can write $(x * z) * (y * z) = \{0\}$.

(xi) By (i); we can write $a \circ 0 = \{0\}$ for $a \in A$ implies $a = 0$.Then $A \circ \{0\}$ for $a \in A$ implies $A = \{0\}$. The similar way, $b * 0 = \{0\}$ for $b \in A$ implies $b = 0$. Therefore $A * \{0\}$ for $b \in A$ implies $A = \{0\}$. □

Definition 3.2. Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra and S be a nonempty subset of H and S be a hyper pseudo BCI-algebra with $'' * '', '' \circ ''$ two hyper operations on H. Then S is a hyper pseudo subalgebra of H.

Proposition 3.2. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra and S, a nonempty subset of H. If $x \circ y \subseteq S$ and $x * y \subseteq S$ then $0 \in S$ for all $x, y \in S$.*

Proof. Let $x \circ y \subseteq S$ and $x * y \subseteq S$ for all $x, y \in S$. Therefore we have $a \in S$ and $b \in S$. Moreover we can write $a \ll a$, $b \ll b$ and then we get $0 \in a \circ a \subseteq S$ and we can find $0 \in b * b \subseteq S$. Hence we can prove that $0 \in S$. □

Theorem 3.2. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra and S, a nonempty subset of H. Then S is a hyper pseudo subalgebra of H if and only if it satisfies $x \circ y \subseteq S$ and $x * y \subseteq S$ for all $x, y \in S$.*

Proof. $(\Rightarrow)$ It is easily see that $x \circ y \subseteq S$ and $x * y \subseteq S$ for all $x, y \in S$.

$(\Leftarrow)$ Using Proposition 3.2, we have

$$x \circ y \subseteq S \text{ and } x * y \subseteq S \text{ for all } x, y \in S \text{ implies that } 0 \in S.$$

For all $x, y, z \in S$, we get

$$x \circ z \subseteq S, x * z \subseteq S \quad y \circ z \subseteq S, y * z \subseteq S \quad x \circ y \subseteq S, x * y \subseteq S.$$

Hence, we have

$$(x \circ z) \circ (y \circ z) = \bigcup_{a \in x \circ z, b \in y \circ z} a \circ b \subseteq S, \quad (x * z) * (y * z) = \bigcup_{c \in x * z, d \in y * z} c * d \subseteq S.$$

Therefore $(HPI1)$ holds. The similar way, $(HPI2)$, $(HPI3)$, $(HPI4)$, $(HPI5)$ are true for S. Hence S is a hyper pseudo subalgebra of H. □

EXAMPLE 3.2. In Example 3.1. for $S_1 = \{0,1\} \subseteq H$, S_1 is a hyper pseudo subalgebra of H. On the other hand, $S_2 = \{0,2\} \subseteq H$, S_2 is not a hyper pseudo subalgebra of H because we have $2 \in S$, $2 * 2 = \{0,1,2\} \nsubseteq S_2$.

THEOREM 3.3. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra and $S(H) := \{x \in H | 0 \circ x = 0 * x = \{0\}\}$ be a nonempty subset of H. Then $S(H)$ is a hyper pseudo subalgebra of H.*

PROOF. Let $x, y \in S(H)$ and $a \in x \circ y$, $b \in x * y$. Then using $(HPI1)$, we have

$$0 \circ (x \circ y) = (0 \circ y) \circ (x \circ y) \ll 0 \circ x = 0,$$

$$0 * (x * y) = (0 * y) * (x * y) \ll 0 * x = 0.$$

Hence by Proposition 3.1 (vii), we find

$$0 \circ (x \circ y) = \{0\}, \quad \text{and} \quad 0 * (x * y) = \{0\}.$$

Therefore we have $x \circ y \subseteq S(H)$ and $x * y \subseteq S(H)$. Hence $S(H)$ is a hyper pseudo subalgebra of H. □

PROPOSITION 3.3. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra and S_k, a nonempty set be defined such that*

$$S_k := \{x \in H | 0 \wedge x = x \circ (x * 0) = \{0\}\}.$$

If S_k is a nonempty subset of H then it satisfies $x \circ y \subseteq S_k$.

PROOF. Let $x, y \in S_k$ and $x \circ (x * 0) = \{0\}$. By $(HPI2)$, $(HPI1)$; we get $(x \circ y) \circ ((x \circ y) * 0) = (x \circ y) \circ ((x * 0) \circ y) \ll x \circ (x * 0) = \{0\}$. Hence we have $0 \wedge (x \circ y) = \{0\}$. Therefore we get $x \circ y \subseteq S_k$. □

THEOREM 3.4. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra and S_I be a nonempty subset of H and it be defined by*

$$S_I := \{x \in H | x \circ x = x * x = \{0\}\}.$$

Then S_I is a hyper pseudo subalgebra of H.

PROOF. Let $x, y \in S(I)$ and $a \in x \circ y$, $b \in x * y$. Then we find

$$(x \circ y) \circ (x \circ y) \ll x \circ x = \{0\}, \quad \text{and} \quad (x * y) \circ (x * y) \ll x * x = \{0\}.$$

By Proposition 3.1 (vii), we can write $(x \circ y) \circ (x \circ y) = \{0\}$ and $(x * y) \circ (x * y) = \{0\}$. Therefore we get $a \circ a = \{0\}$ and $a, b \in S_I$ so that we find $b * b = \{0\}$. Then $x \circ y \subseteq S(I)$ and $x * y \subseteq S(I)$. Hence, using Theorem 3.2, we can write S_I is a hyper pseudo subalgebra of H. □

THEOREM 3.5. *Let $(H, \circ, *, 0)$ be a hyper pseudo BCI-algebra and S_I, a nonempty subset of H be defined by as follows:*

$$S_I := \{x \in H | x \circ x = x * x = \{0\}\}.$$

Then $(S_I, \circ, *)$ is a pseudo BCI-algebra.

PROOF. Let $x, y \in S(I)$ and $a, b \in x \circ y$, $c, d \in x * y$. Then we have

$$a \circ b \subseteq (x \circ y) \circ (x \circ y) \ll x \circ x = \{0\},$$

$$c * d \subseteq (x * y) \circ (x * y) \ll x * x = \{0\}.$$

Therefore we find $a \circ b = \{0\}$, $c * d = \{0\}$ and so we have $a \ll b$ and $c \ll d$. The similar way, it can be proved that $b \ll a$ and $d \ll c$. Using $(HPI4)$, we can find $a = b$, $c = d$. Hence $x \circ y$ and $x * y$ are singleton. Using Theorem 3.1, we find that S_I is a pseudo BCI-algebra. □

References

[1] R. A. Borzooei, A. Rezazadeh and R. Ameri. On hyper pseudo BCK-algebras. *Iran. J. Math. Sci. Inform.*, **9**(1)(2014), 13–29.

[2] W. A. Dudek and Y. B. Jun. Pseudo-BCI algebras. *East Asian Math. J.*, **24**(2)(2008), 187–190.

[3] G. Dymek. P-semisimple pseudo-BCI-algebras. *J. Mult.-Val. Log. Soft Comput.*, **19**(5)(2012), 461–474.

[4] K. Iseki and S. Tanaka. An introduction to the theory of BCK-algebras. *Math. Japan*, **23**(1)(1978), 1–26.

[5] K. Iseki. On BCI-Algebras. *Math. Semin. Notes, Kobe Univ.*, **8**(1)(1980), 125–130.

[6] Y. B. Jun and X. L. Xin. On derivations of BCI-algebras. *Information Sciences*, **159**(3-4)(2004), 167–176.

[7] Y. B. Jun, H. S. Kim and J. Neggers. On pseudo-BCI ideals of pseudo BCI-algebras. *Mat. Vesn.*, **58**(1-2)(2006), 39–46.

[8] K. J. Lee and C. H. Park. Some ideals of pseudo-BCI algebras. *J. Appl. Math. Inform.*, **27**(1-2)(2009), 217–231.

[9] F. Marty. Surene generalization de la notion de group. In eigth Congress Math. Scandinaves, Stockholm, (1934), 45–49.

[10] X. L. Xin. Hyper BCI-algebras. *Discuss. Math., Gen. Algebra Appl.*, **26**(1)(2006), 5–19.

[11] D. S. Uzay and A. Firat. On multipler of hyper BCI-algebras. Gazi University Journal of Science, **32**(2)(2019), 660–665.

[12] D. S. Uzay and A. Firat. On derivation of pseudo BCI-algebras. *Asian-Eur. J. Math.*, **12**(6)(2019), ID: 2040011.

Received by editors 22.01.2020; Revised version 14.06.2020; Available online 06.07.2020.

Department of Mathematics, Ege University, Izmir, Turkey,
E-mail address: didemsurgevil@hotmail.com

Department of Mathematics, Ege University, Izmir, Turkey,
E-mail address: alev.firat@ege.edu.tr