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NOTE ON MEET-DISTRIBUTIVE LATTICE MATRICES

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Abstract. In this paper, we extended the notion of meet-distributive lattice matrices. The properties of meet-distributive lattice matrices are studied and various characterizations of them are given.

1. Introduction

The notion of lattice matrices appeared firstly in the work Lattice Matrices [**5**] by Y. Giveon in 1964. A matrix is called a lattice matrix if its entries belong to a distributive lattice. All Boolean matrices and fuzzy matrices are lattice matrices. Lattice matrices in various special cases become useful tools in various domains like the theory of switching nets, automata theory, and the theory of finite graphs [**5**]. The basic properties of pseudocomplemented lattices and the representation theorem are discussed by G. Birkhoff [**1**] and T. S. Blyth [**2**] in Lattice theory. We know that multiplication of Boolean (lattice) matrices is associative and distributive with respect to join. But, in general it is not true, that this multiplication is distributive with respect to meet. In 1964 T. S. Blyth [**3**] introduce the concept of meet-distributive Boolean matrices.

In this paper, as an analogue to T. S. Blyth [**3**], we generalize the concept of meet-distributive Boolean matrices to meet-distributive lattice matrices and we characterize those matrices.

In classical linear algebra, a QR factorization is a decomposition of a matrix A into a product $A = QR$ of an orthogonal matrix Q and an upper triangular matrix R.

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In this paper, we have proven the meet-distributive lattice matrix can be decomposed into a product $A = QD$ of an invertible (orthogonal) matrix Q and a diagonal matrix D.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

DEFINITION 2.1. ([**1**]) A partially ordered set $(L, \leqslant)$ is a lattice if for all $a, b \in L$, the least upper bound of a, b and the greatest lower bound of a, b exist in L. For any $a, b \in L$, the least upper bound and the greatest lower bound are denoted by $a \vee b$ and $a \wedge b$ (or ab), respectively. An element $a \in L$ is called greatest element of L if $\alpha \leqslant a$, for all $\alpha \in L$. An element $b \in L$ is called least element of L if $b \leqslant \alpha$, for all $\alpha \in L$. We use 1 and 0 to denote the greatest element and the least element of L, respectively.

DEFINITION 2.2. ([**1**]) A maximal element of a subset S of some partially ordered set (poset) is an element of S that is not smaller than any other element in S.

DEFINITION 2.3. ([**1**]) A lattice L is a distributive lattice, if for any $a, b, c \in L$,

(1) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ hold.

For any two elements a, b of a pseudocomplemented distributive lattice, then the following properties are due to Grätzer ([**6**]):

(1) $a^* = \max\{x \in L \mid a \wedge x = 0\}$;
(2) $0^{**} = 0$;
(3) $a \wedge a^* = 0$;
(4) $a \leqslant b$ implies $b^* \leqslant a^*$;
(5) $a \leqslant a^{**}$;
(6) $ab = 0$ if and only if $a \leqslant b^*$ if and only if $b \leqslant a^*$;
(7) $a \leqslant b$ implies $a \wedge b^*{=}0$;
(8) $a^{***} = a^*$;
(9) $(a \vee b)^* = a^* \wedge b^*$;
(10) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

Throughout this paper, unless otherwise stated, we assume that L is a bounded pseudocomplemented distributive lattice with $a^* \vee a^{**}{=}\ 1$, for all $a \in L$.

THEOREM 2.1 ([**2**]). *If L is a pseudocomplemented distributive lattice with $a^* \vee a^{**}{=}\ 1$, for all $a \in L$. Then the following equivalent statements hold for all $x, y \in L$*

(1) *$S(L)$ is a sublattice of L, where the set $S(L) = \{x^{**} \mid x \in L\}$;*
(2) *$(x \wedge y)^* = x^* \vee y^*$;*
(3) *$(x \vee y)^{**} = x^{**} \vee y^{**}$.*

Let $M_n(L)$ be the set of $n \times n$ matrices over a bounded distributive lattice L, the elements of $M_n(L)$ denoted by capital letters and suppose $A \in M_n(L)$, then the $(i,j)^{th}$ entry of A is denoted by a_{ij}. Y. Giveon [5] calls them lattice matrices.

The following definitions are due to Y. Giveon [**5**] for the lattice matrices

$A = [a_{ij}], B = [b_{ij}], C = [c_{ij}] \in M_n(L)$, where $a_{ij}, b_{ij}, c_{ij} \in L$, $1 \leqslant i, j \leqslant n$

$A \leqslant B$ if and only if $a_{ij} \leqslant b_{ij}$;

$A + B = C$ if and only if $c_{ij} = a_{ij} \vee b_{ij}$;

$A \wedge B = C$ if and only if $c_{ij} = a_{ij} \wedge b_{ij} = a_{ij}b_{ij}$;

$A \cdot B = AB = C$ if and only if $c_{ij} = \vee_{k=1}^{n}(a_{ik} \wedge b_{kj})$;

$A^T = C$ if and only if $c_{ij} = a_{ji}$;

$A^0 = I$, where I is the identity matrix;

$A(BC) = (AB)C, AI = IA = A, AO = OA = O$.

Theorem 2.2 ([**5**]). *If L is distributive lattice with 0 and 1. $A \in M_n(L)$ is invertible if and only if each row and each column of A is an orthogonal decomposition of 1 in L.*

Definition 2.4. ([**4**]) A lattice vector space V over L(or lattice vector space) is a system $(V, L, +, \cdot)$, where V is a non-empty set, L is a distributive lattice with 1 and 0, $+$ is a binary operation on V called addition and $\cdot$ is a map from $L \times V$ to V called scalar multiplication such that the following properties hold: For every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $a, b \in L$ satisfy ([**4**]):

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$;
(2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$;
(3) there is an element $\mathbf{0}$ in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$, for every $\mathbf{x}$ in V;
(4) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{y} = \mathbf{0}$;
(5) $a \cdot (\mathbf{x} + \mathbf{y}) = a \cdot \mathbf{x} + a \cdot \mathbf{y}$;
(6) $(a \vee b) \cdot \mathbf{x} = a \cdot \mathbf{x} + b \cdot \mathbf{x}$;
(7) $(ab)\mathbf{x} = a \cdot (b \cdot \mathbf{x})$;
(8) $1 \cdot \mathbf{x} = \mathbf{x}$;
(9) $0 \cdot \mathbf{x} = \mathbf{0}$.

3. Meet-distributive lattice matrices

In this section, we extend the concept of meet-distributive matrices over Boolean algebra [**3**] to meet-distributive matrices over lattice matrices, we find such matrices for which the multiplication is distributive with respect to meet and we characterize those matrices.

Definition 3.1. A square matrix $A \in M_n(L)$ is said to be left meet-distributive if it satisfy $A(X \wedge Y) = AX \wedge AY$, for all $X, Y \in M_n(L)$ and is said to be right meet-distributive if it satisfy $(X \wedge Y)A = XA \wedge YA$, for all $X, Y \in M_n(L)$. A matrix is said to be meet-distributive if which is both left meet and right meet-distributive.

Example 3.1. Consider the lattice $L = \{0, a, b, c, d, 1\}$ where the Hasse diagram of L is shown in Figure 1

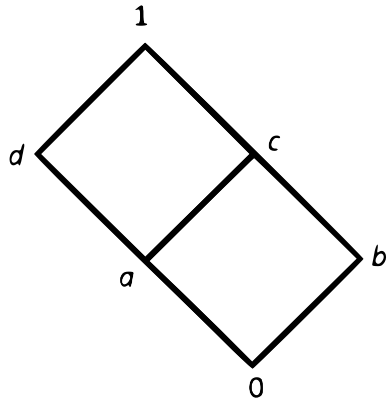

FIGURE 1

Let $A = \begin{pmatrix} d & a \\ a & b \end{pmatrix}$, $X = \begin{pmatrix} a & d \\ c & b \end{pmatrix}$, $Y = \begin{pmatrix} a & c \\ 1 & d \end{pmatrix}$. So,

$$A(X \wedge Y) = \begin{pmatrix} d & a \\ a & b \end{pmatrix} \begin{pmatrix} a & b \\ c & b \end{pmatrix} = \begin{pmatrix} a & b \\ c & b \end{pmatrix}$$

and

$$AX \wedge AY = \begin{pmatrix} a & d \\ c & b \end{pmatrix} \wedge \begin{pmatrix} a & b \\ c & c \end{pmatrix} = \begin{pmatrix} a & b \\ c & b \end{pmatrix}.$$

Clearly A is left meet-distributive matrix.

$$(X \wedge Y)A = \begin{pmatrix} a & b \\ c & b \end{pmatrix} \begin{pmatrix} d & a \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & c \\ b & c \end{pmatrix}$$

and

$$XA \wedge YA = \begin{pmatrix} 0 & c \\ b & c \end{pmatrix} \wedge \begin{pmatrix} a & c \\ d & c \end{pmatrix} = \begin{pmatrix} o & c \\ b & c \end{pmatrix}.$$

Clearly A is right meet-distributive matrix. Therefore, A is meet-distributive matrix.

Now we characterize the left meet-distributive and right meet-distributive lattice matrices.

THEOREM 3.1. *If L is a bounded distributive and a square lattice matrix $P = [p_{ij}]$ over L is left meet-distributive if and only if, for all i, $p_{ij} \wedge p_{ik}$=0, $j \neq k$.*

PROOF. Let $P = [p_{ij}] \in M_n(L)$. Suppose that $P(A \wedge B) = PA \wedge PB$, for all $A = [a_{ij}], B = [b_{ij}] \in M_n(L)$. In particular, we choose $A = I^* = [(\delta^*)_{ij}]$ and $B = I = [\delta_{ij}]$, where δ_{ij}=1 if $i = j$ and 0 if $i \neq j$. Consider, $P(A \wedge B) = P(I^* \wedge I) = AO = O$.

On the other hand

$$\begin{aligned}[PA \wedge PB]_{ik} &= [\vee_j (p_{ij} a_{jk})] \wedge [\vee_l (p_{il} b_{lk})] \\ &= [\vee_j (p_{ij} (\delta)^*_{jk})] \wedge [\vee_l (p_{il} \delta_{lk})] \\ &= (\vee_{j \neq k} p_{ij}) \wedge p_{ik}.\end{aligned}$$

Therefore, the equality gives $(\vee_{j\neq k} p_{ij}) \wedge p_{ik} = \vee_{j\neq k}(p_{ij} \wedge p_{ik}) = 0$. Which implies that $p_{ij} \wedge p_{ik} = 0$, $j \neq k$.

Conversely, suppose that $p_{ij} \wedge p_{ik} = 0$, $j \neq k$, then

$$\begin{aligned}[PA \wedge PB]_{ik} &= [\vee_j (p_{ij} a_{jk})] \wedge [\vee_l (p_{il} b_{lk})] \\ &= \vee_{j,l} (p_{ij} \wedge a_{jk} \wedge p_{il} \wedge b_{lk}) \\ &= \vee_j (p_{ij} \wedge a_{jk} \wedge b_{jk} \\ &= [P(A \wedge B)]_{ik}.\end{aligned}$$

Therefore, $P(A \wedge B) = PA \wedge PB$. □

Similarly, we can obtain the following result:

THEOREM 3.2. *If L is a bounded distributive and a square lattice matrix $P = [p_{ij}]$ over L is right meet-distributive if and only if, for all j, $p_{ij} \wedge p_{kj} = 0$, $i \neq k$.*

EXAMPLE 3.2. Consider the lattice $L = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$ where the Hasse diagram of L is shown in figure 2.

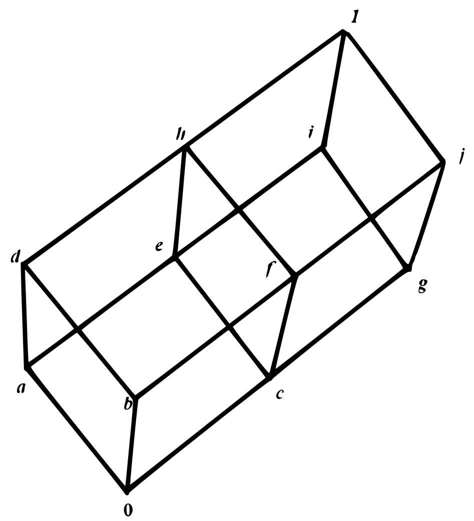

FIGURE 2

Let $P = \begin{pmatrix} g & a & b \\ 0 & j & a \\ d & 0 & c \end{pmatrix}$, here we have, for all j, $p_{ij} \wedge p_{kj} = 0$, $i \neq k$ and for all i, $p_{ik} \wedge p_{ij} = 0$, $j \neq k$. $X = \begin{pmatrix} b & c & d \\ e & f & g \\ a & i & h \end{pmatrix}$ and $Y = \begin{pmatrix} d & f & c \\ b & e & a \\ j & i & h \end{pmatrix}$. Consider,

$$P(X \wedge Y) = \begin{pmatrix} g & a & b \\ 0 & j & a \\ d & 0 & c \end{pmatrix} \begin{pmatrix} b & c & 0 \\ 0 & c & 0 \\ 0 & i & h \end{pmatrix} = \begin{pmatrix} 0 & c & b \\ 0 & e & a \\ b & c & c \end{pmatrix}$$

and now

$$PX \wedge PY = \begin{pmatrix} a & c & b \\ e & h & i \\ b & c & h \end{pmatrix} \wedge \begin{pmatrix} b & e & h \\ b & e & a \\ h & f & c \end{pmatrix} = \begin{pmatrix} 0 & c & b \\ 0 & e & a \\ b & c & c \end{pmatrix}.$$

Therefore, $P(X \wedge Y) = PX \wedge PY$.

We now denote the set of all left meet-distributive matrices of $M_n(L)$ as $M_n^l(L)$ and the set of all right meet-distributive matrices of $M_n(L)$ as $M_n^r(L)$.

Corollary 3.1. *For any $P, Q \in M_n^l(L)$(or $M_n^r(L)$), $A \in M_n(L)$. Then we have*
(a) *$PQ \in M_n^l(L)$(or $M_n^r(L)$).*
(b) *$P \wedge A \in M_n^l(L)$(or $M_n^r(L)$).*

From Corollary 3.1, it follows that both $M_n^l(L)$ and $M_n^r(L)$ forms a subsemigroup and meet-subsemilattice of $M_n(L)$.

Lemma 3.1. *have*
(a) *If $P, Q \in M_n^l(L)$, then $P \vee Q \in M_n^l(L)$ if and only if, for all i,*

$$p_{ij} \wedge q_{ik} = 0, \;\; j \neq k.$$

(b) *If $P, Q \in M_n^r(L)$, then $P \vee Q \in M_n^r(L)$ if and only if, for all j,*

$$p_{ij} \wedge q_{kj} = 0, \;\; i \neq k.$$

Theorem 3.3. *If L is a pseudocomplemented lattice with 0, 1. Let $P = [p_{ij}] \in M_n^l(L)$ is left meet-distributive, the matrix $T = [t_{ij}] \in M_n^l(L)$ defined by $t_{ij} = p_{ij}^{**}$, $j \neq i$ and $t_{ii} = \wedge_{k \neq i} p_{ik}^*$. Then T is maximal element of set left meet-distributive matrices, containing P.*

Proof. Let $P = [p_{ij}] \in M_n^l(L)$, then we have $p_{ij} \wedge (\vee_{k \neq j} p_{ik}) = 0$, so which implies that

$$p_{ij} \leqslant (\vee_{k \neq j} p_{ik})^* = \wedge_{k \neq j} p_{ik}^* \text{ or } \vee_{k \neq j} p_{ik} = \wedge_{k \neq j} p_{ik} \leqslant p_{ij}^*.$$

Now we construct the matrix $T = [t_{ij}]$, where $t_{ij} = p_{ij}^{**}$, $j \neq i$ and $t_{ii} = \wedge_{k \neq i} p_{ik}^*$. Consider for $i \neq k$, $t_{ii} t_{ik} = \wedge_{k \neq i} p_{ik}^* p_{ik}^{**} = 0$. For $i \neq j \neq k$, $t_{ij} t_{ik} = p_{ij}^{**} p_{ik}^{**} = (p_{ij} p_{ik})^{**} = (0)^{**} = 0$. Therefore, $T = [t_{ij}] \in M_n^l(L)$.

To prove T is maximal in $M_n^l(L)$, consider any $X = [x_{ij}] \in M_n^l(L)$ such that $T \leqslant X$ and we prove $T = X$ (or $X \leqslant T$). Now consider, for $i = j$,

$$x_{ii} \leqslant \wedge_{k\neq i} x_{ik}^* \leqslant \wedge_{k\neq i} t_{ik}^* \leqslant \wedge_{k\neq i} p_{ik}^{***} = \wedge_{k\neq i} p_{ik}^* = t_{ii}.$$

For $i \neq j$,

$$\begin{aligned} x_{ij} \leqslant \wedge_{k\neq j} x_{ik}^* &= x_{ii}^* \wedge [\wedge_{i\neq j\neq k} x_{ik}^*] \\ &\leqslant t_{ii}^* \wedge [\wedge_{i\neq j\neq k} t_{ik}^*] \\ &= [\wedge_{i\neq k} p_{ik}^*]^* \wedge [\wedge_{i\neq j\neq k} p_{ik}^*] \\ &= [\vee_{k\neq i} p_{ik}^{**}] \wedge [\wedge_{i\neq j\neq k} p_{ik}^*] \\ &= [p_{ij}^{**} \vee [\vee_{i\neq j\neq k} p_{ik}^{**}]] \wedge [\wedge_{i\neq j\neq k} p_{ik}^*] \\ &= p_{ij}^{**} \wedge [\wedge_{i\neq j\neq k} p_{ik}^*] \\ &\leqslant p_{ij}^{**} \\ &= t_{ij}. \end{aligned}$$

So, $x_{ij} \leqslant t_{ij}$, for all i, j and since $t_{ij} \leqslant x_{ij}$, for all i, j which gives $x_{ij} = t_{ij}$, for all i, j. Therefore, $T = X$. Consequently T is maximal in $M_n^l(L)$.

Similarly, if $P = [p_{ij}] \in M_n^r(L)$, the matrix $T = [t_{ij}]$ defined by $t_{ij} = p_{ij}^{**}$, $j \neq i$ and $t_{ii} = \wedge_{k\neq i} p_{ki}^*$. Then T is maximal element of $M_n^r(L)$ containing P. □

Corollary 3.2. *If L is a pseudocomplemented lattice with 0, 1. Then $T = [t_{ij}] \in M_n^l(L)$ is maximal in $M_n^l(L)$ whenever $\vee_j t_{ij} = 1$.*

Proof. Let $T = [t_{ij}] \in M_n^l(L)$ is maximal in $M_n^l(L)$, then there exist $P = [p_{ij}] \in M_n^l(L)$ such that $P \leqslant T$ and $t_{ij} = p_{ij}^{**}$, $j \neq i$ and $t_{ii} = \wedge_{k\neq i} p_{ik}^*$. Consider

$$\begin{aligned} \vee_j t_{ij} &= t_{ii} \vee \vee_{j\neq i} t_{ij} \\ &= [\wedge_{k\neq i} p_{ik}^*] \vee [\vee_{j\neq i} p_{ij}^{**}] \\ &= [\vee_{i\neq j} p_{ij}]^* \vee [\vee_{j\neq i} p_{ij}]^{**} \\ &= 1. \end{aligned}$$

Suppose $T = [t_{ij}] \in M_n^l(L)$ and $\vee_j t_{ij} = 1$. Since $T = [t_{ij}] \in M_n^l(L)$, then $t_{ii} \wedge (\vee_{j\neq i} t_{ij}) = 0$. Which implies that $t_{ii} \leqslant (\vee_{j\neq i} t_{ij})^*$.

Consider

$$\begin{aligned} 1 &= t_{ii} \vee (\vee_{j\neq i} t_{ij}) \\ &\leqslant (\vee_{j\neq i} t_{ij})^* \vee (\vee_{j\neq i} t_{ij}) \\ &\leqslant (\vee_{j\neq i} t_{ij})^* \vee (\vee_{j\neq i} t_{ij})^{**} \\ &= 1 \end{aligned}$$

we obtain from above, $t_{ij} = t_{ij}^{**}$, $j \neq i$ and $t_{ii} = \wedge_{j\neq i} t_{ij}^*$. Therefore, $T = [t_{ij}]$ is maximal in $M_n^l(L)$. □

Similarly, we can prove $T = [t_{ij}] \in M_n^r(L)$ is maximal in $M_n^r(L)$ whenever $\vee_i t_{ij} = 1$.

Remark 3.1. If L is a pseudocomplemented distributive lattice with 0, 1. Let $T = [t_{ij}] \in M_n(L)$ is maximal element of both $M_n^l(L)$ and $M_n^r(L)$, then $t_{ij} = p_{ij}^{**}$, $j \neq i$ and $t_{ii} = \wedge_{i \neq k} p_{ik}^* = \wedge_{k \neq i} p_{ki}^*$.

Corollary 3.3. *$A \in M_n(L)$ is maximal element of both $M_n^l(L)$ and $M_n^r(L)$ if and only if it is invertible.*

Example 3.3. Consider the lattice $L = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$ where the Hasse diagram of L is shown in figure 2. Let P= $[p_{ij}] = \begin{pmatrix} d & 0 & c \\ g & a & 0 \\ 0 & c & a \end{pmatrix}$, where, for all j, $p_{ij} \wedge p_{kj}$=0, $i \neq k$ and for all i, $p_{ik} \wedge p_{ij}$=0, $j \neq k$. Then the maximal element in both $M_n^l(L)$ and $M_n^r(L)$ will be $T = \begin{pmatrix} d & 0 & g \\ g & d & 0 \\ 0 & g & d \end{pmatrix}$ and here, for all i,

$$t_{ij} t_{ik} = 0, j \neq k \ \ \vee_i \, t_{ij} = 1;$$

for all j,

$$t_{ij} t_{kj} = 0, i \neq k; \ \vee_j t_{ij} = 1.$$

Therefore, T is invertible.

Now we characterise meet-distributive lattice matrices.

Theorem 3.4. *If L is a pseudocomplemented distributive lattice with 0, 1. Then $A \in M_n(L)$ is meet-distributive if and only if there exists an invertible matrix $P \in M_n(L)$ such that $A \leqslant P$.*

Proof. Suppose $A \leqslant P$, where P is invertible, then by Corollary 3.1, we have $A = A \wedge P \in M_n^l(L)$ and $A = A \wedge P \in M_n^r(L)$. Therefore, A is meet-distributive.

Conversely, suppose A is meet-distributive, then we wish to show that A is contained in some invertible matrix C. Now we build up systematically a sequence of matrices

$$A \leqslant N_1^n \ \leqslant N_2^n \ \cdots \ \leqslant N_n^n \ (= C \,\text{say})$$

in which each N_i^n is meet-distributive and N_n^n $(= C)$ is invertible. Since A is meet-distributive, we have for all i, $a_{ij} a_{ik} = 0$, $j \neq k$ and for all j, $a_{ij} a_{kj} = 0$, $i \neq k$ which gives that $a_{ij} \leqslant \wedge_{k \neq j} a_{ik}^* \wedge \wedge_{k \neq i} a_{kj}^*$.

Now we observe that the previous former relations remain unaltered if, for a given a_{ij} we replace this a_{ij} by the right-hand side of later one. We use this fact repeatedly in building up the sequence $A \leqslant N_1^n \leqslant N_2^n \cdots \leqslant N_n^n$ $(= C)$ in the following way. We begin by replacing the leading element of A, then proceed along the first row and then down the first column. At this stage, we will have the matrix N_1^n of the sequence which is meet-distributive, contains A and is such that its first row and column orthogonal decomposition 1 in L.

We begin, therefore, with the matrix M_1^1 defined from A by

$$[M_1^{(1)}]_{ij} = \begin{cases} \wedge_{k>1} a_{1k}^* \wedge \wedge_{k>1} a_{k1}^*, & i = 1, j = 1 \\ a_{ij}, & i \neq j \end{cases}$$

We now proceed along the first row, defining recursively the sequence

$$A \leqslant M_1^1 \leqslant M_1^2 \cdots \leqslant M_1^n$$

in the following way:

$$[M_1^{(r)}]_{ij} = \begin{cases} [M_1^{(j)}]_{1j}, & i=1, j<r \\ \wedge_{k<r}[M_1^{(k)}]^*_{1k} \wedge_{k>r} a^*_{1k} \wedge \wedge_{k>1} a^*_{kr}, & j=r, i=1 \\ a_{ij}, & otherwise \end{cases}.$$

Denoting for convenience M_1^n by N_1^1 we now proceed down the first column, thus defining the sequence $N_1^1 \leqslant N_1^2 \cdots \leqslant N_1^n$ in the following recursive way:

$$[N_1^{(r)}]_{ij} = \begin{cases} [N_1^{(i)}]_{i1}, & j=1, i<r \\ \wedge_{k<r}[N_1^{(k)}]^*_{k1} \wedge_{k>1} a^*_{rk} \wedge \wedge_{k>r} a^*_{k1}, & i=r, j=1 \\ a_{ij}, & otherwise \end{cases}.$$

At this stage, we have the matrix N_1^n of the sequence $A \leqslant N_1^n \leqslant N_2^n \cdots \leqslant N_n^n$ (= C), and by its construction it satisfies the meet-distributive conditions and in which each element of these matrices is a pseudocomplemented of some element.

Consider now the first row of N_1^n ($= B = b^*_{ij}$ say). Taking the union of this and using repeatedly the formula $a^* \vee (a^{**} \wedge b^*) = a^* \vee b^*$ and the distributive law, we have, $\vee_j[N_1^n]_{1j} = \vee_j[b^*_{1j}] = (\wedge_j[b_{1j}])^* = (0)^* = 1$.

Similarly, we can obtain that $\vee_i[N_1^n]_{i1} = 1$.

We may now re-start the process of substitution and build up the matrix N_2^n from N_1^n exactly as we built up N_1^n from A, though in this case we leave the first row and column alone and deal with the second row and second column. We then build up N_3^n from N_2^n by concentrating on the third row and column of $N_2^n$and so on.

The entire sequence of matrices is given by

$A \leqslant M_1^1 \leqslant M_1^2 \cdots \leqslant M_1^n = N_1^1 \leqslant N_1^2 \cdots \leqslant N_1^n = M_2^1 \leqslant M_2^2 \cdots \leqslant M_2^n = N_2^2 \leqslant N_2^3 \cdots \leqslant N_2^n = M_3^2 \leqslant M_3^3 \cdots \leqslant M_3^n = N_3^3 \leqslant N_3^4 \cdots \leqslant N_3^n = \cdots M_{n-1}^{n-2} \leqslant M_{n-1}^{n-1} \leqslant M_{n-1}^n = N_{n-1}^{n-1} \leqslant N_{n-1}^n = M_n^{n-1} \leqslant M_n^n = N_n^n$ (= C).

In this way, we construct the sequence $A \leqslant N_1^n \leqslant N_2^n \cdots \leqslant N_n^n (= C)$ and eventually arrive at the matrix $N_n^n (= C)$ which is meet-distributive, contains A and satisfies the condition $\vee_i[N_n^n]_{ij} = 1 = \vee_j[N_n^n]_{ij}$. Consequently, N_n^n (= C) is invertible. □

For simulation,consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \leqslant \begin{pmatrix} a^*_{12}a^*_{21} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\leqslant \begin{pmatrix} b_{11} & b^*_{11}a^*_{22} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix} \leqslant \begin{pmatrix} b_{11} & b_{12} \\ b^*_{11}a^*_{22} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & a_{22} \end{pmatrix}$$

$$\leqslant \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b^*_{12}b^*_{21} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = B.$$

Consider

$$\begin{aligned} b_{11} \vee b_{12} &= a_{12}^* a_{21}^* \vee (a_{12}^* a_{21}^*)^* (a_{22}^*) \\ &= (a_{12} \vee a_{21})^* \vee (a_{12} \vee a_{21})^{**} (a_{22}^*) \\ &= (a_{12}^* a_{21}^*) \vee (a_{22}^*) \\ &= (a_{12} a_{22})^* \vee (a_{21} a_{22})^* \\ &= 1 \end{aligned}$$

$$\begin{aligned} b_{11} \vee b_{21} &= a_{12}^* a_{21}^* \vee (a_{12}^* a_{21}^*)^* (a_{22}^*) \\ &= (a_{12} \vee a_{21})^* \vee (a_{12} \vee a_{21})^{**} (a_{22}^*) \\ &= (a_{12}^* a_{21}^*) \vee (a_{22}^*) \\ &= (a_{12} a_{22})^* \vee (a_{21} a_{22})^* \\ &= 1 \end{aligned}$$

$$\begin{aligned} b_{12} \vee b_{22} &= (a_{12}^* a_{21}^*)^* (a_{22}^*) \vee [(a_{12}^* a_{21}^*)^* (a_{22}^*)]^* \\ &= [(a_{12} \vee a_{21})^{**}](a_{22}^*) \vee (a_{12} \vee a_{21})^* \vee (a_{22}^*) \\ &= [(a_{12} \vee a_{21})^{**}] \vee [(a_{12} \vee a_{21})^*] \vee (a_{22}^*) \\ &= 1 \vee (a_{22}^*) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} b_{21} \vee b_{22} &= (a_{12}^* a_{21}^*)^* (a_{22}^*) \vee [(a_{12}^* a_{21}^*)^* (a_{22}^*)]^* \\ &= [(a_{12} \vee a_{21})^{**}](a_{22}^*) \vee (a_{12} \vee a_{21})^* \vee (a_{22}^*) \\ &= [(a_{12} \vee a_{21})^{**}] \vee [(a_{12} \vee a_{21})^*] \vee (a_{22}^*) \\ &= 1 \vee (a_{22}^*) \\ &= 1 \end{aligned}$$

with $b_{11}b_{12} = b_{21}b_{22} = 0 = b_{11}b_{21} = b_{12}b_{22}$.

Therefore, B is invertible and $A \leqslant B$.

Example 3.4. Consider the lattice

$$L = \{0, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, 1\}$$

where the Hasse diagram of L is shown figure 3:

Consider the matrix $A = \begin{pmatrix} g & a & c \\ c & m & a \\ d & c & g \end{pmatrix}$, here we have, for all i, $a_{ij} \wedge a_{kj}$=0, $i \neq k$ and for all j, $a_{ik} \wedge a_{ij}$=0, $j \neq k$. Then

$$A = \begin{pmatrix} g & a & c \\ c & m & a \\ d & c & g \end{pmatrix} \leqslant \begin{pmatrix} a^*c^*d^* & a & c \\ c & m & a \\ d & c & g \end{pmatrix} = \begin{pmatrix} m & a & c \\ c & m & a \\ d & c & g \end{pmatrix}$$

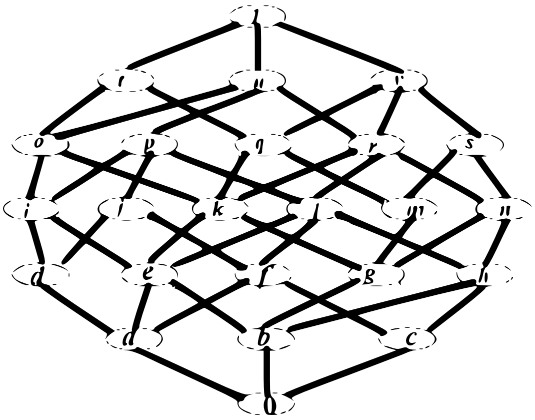

FIGURE 3

$$
\begin{aligned}
&\leqslant \begin{pmatrix} m & m^*c^* & c \\ c & m & a \\ d & c & g \end{pmatrix} = \begin{pmatrix} m & d & c \\ c & m & a \\ d & c & g \end{pmatrix} \leqslant \begin{pmatrix} m & d & m^*d^*a^*g^* \\ c & m & a \\ d & c & g \end{pmatrix} \\
&= \begin{pmatrix} m & d & c \\ c & m & a \\ d & c & g \end{pmatrix} \leqslant \begin{pmatrix} m & d & c \\ m^*d^*a^* & m & a \\ d & c & g \end{pmatrix} = \begin{pmatrix} m & d & c \\ c & m & a \\ d & c & g \end{pmatrix} \\
&\leqslant \begin{pmatrix} m & d & c \\ c & m & a \\ m^*c^*g^* & c & g \end{pmatrix} = \begin{pmatrix} m & d & c \\ c & m & a \\ d & c & g \end{pmatrix} \leqslant \begin{pmatrix} m & d & c \\ c & d^*c^*a^* & a \\ d & c & g \end{pmatrix} \\
&= \begin{pmatrix} m & d & c \\ c & m & a \\ d & c & g \end{pmatrix} \leqslant \begin{pmatrix} m & d & c \\ c & m & c^*m^*g^* \\ d & c & g \end{pmatrix} = \begin{pmatrix} m & d & c \\ c & m & d \\ d & c & g \end{pmatrix} \\
&\leqslant \begin{pmatrix} m & d & c \\ c & m & d \\ d & d^*m^*g^* & g \end{pmatrix} = \begin{pmatrix} m & d & c \\ c & m & d \\ d & c & g \end{pmatrix} \leqslant \begin{pmatrix} m & d & c \\ c & m & d \\ d & c & d^*c^* \end{pmatrix} \\
&= \begin{pmatrix} m & d & c \\ c & m & d \\ d & c & m \end{pmatrix} = P.
\end{aligned}
$$

Clearly P is invertible and $A \leqslant P$.

DEFINITION 3.2. Let V be a lattice vector space. An inner product on V is a function $<,>: V \times V \to L$ which satisfy:

(1) $<\mathbf{x},\mathbf{x}> \geqslant 0$, for all $\mathbf{x} \in V$ and $<\mathbf{x},\mathbf{x}> = 0$ iff $\mathbf{x} = 0_V$;
(2) $<\mathbf{x}+\mathbf{y},\mathbf{z}> = <\mathbf{x},\mathbf{z}> \vee <\mathbf{y},\mathbf{z}>$, for all $\mathbf{x},\mathbf{y},\mathbf{z} \in V$;
(3) $<\alpha\mathbf{x},\mathbf{y}> = \alpha <\mathbf{x},\mathbf{y}>$, for all $\mathbf{x},\mathbf{y} \in V$ and $\alpha \in L$;
(4) $<\mathbf{y},\mathbf{x}> = <\mathbf{x},\mathbf{y}>$, for all $\mathbf{x},\mathbf{y} \in V$.

DEFINITION 3.3. A vector space V together with an inner product $<,>$ is called an inner product space which is denoted by $(V, <,>)$.

EXAMPLE 3.5. Let $V = V_n(L) = \{\mathbf{x} \in V | \mathbf{x} = (x_1, x_2, \ldots x_n)\}$(set of all n-tuples). By defining "+" on V as $(x_1, x_2, \ldots x_n) + (y_1, y_2, \ldots y_n) = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$ and scalar multiplication $\cdot$ on V as $a(x_1, x_2, \ldots x_n) = (ax_1,\ ax_2,\ \ldots,\ ax_n)$, $V_n(L)$ forms a vector space over L. For any $\mathbf{x}, \mathbf{y} \in V$, by defining $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^T \mathbf{y}$ or $\mathbf{y}^T \mathbf{x}$, $(V_n(L), <,>\ )$ forms an inner product space.

Now we will see another characterization of meet-distributive lattice matrices.

THEOREM 3.5. *If L is a pseudocomplemented distributive lattice with 0, 1 and a matrix in $M_n(L)$ is meet-distributive lattice matrix if and only if it is a product of an orthogonal and a diagonal matrices.*

PROOF. Let $A = [a]_{ij} \in M_n(L)$ be a meet-distributive lattice matrix, then by Theorem 3.4 there exist an invertible (orthogonal) matrix $Q = [q]_{ij} \in M_n(L)$ such that $A \leqslant Q$. Let the columns of A be $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ and the columns of Q be $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n$. Now we construct a matrix $D = [d]_{ij}$ as $d_{ij} =< \mathbf{q}_i, \mathbf{a}_j >$, for all i, j. Then we can obtain that, $< \mathbf{q}_i, \mathbf{a}_j > = 0$, for $i \neq j$. Therefore, $A = QD$ and $Q^T A = D$.

Converse obvious, that is, the product of an orthogonal and a diagonal lattice matrices is a meet-distributive lattice matrix. □

EXAMPLE 3.6. Consider the lattice $L = \{0, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, 1\}$ where the Hasse diagram of L is shown in Figure 3.

Consider the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] = \begin{pmatrix} g & a & c \\ c & m & a \\ d & c & g \end{pmatrix}$, here we have, for all i, $a_{ij} \wedge a_{kj}$=0, $i \neq k$ and for all j, $a_{ik} \wedge a_{ij}$=0, $j \neq k$. Then by Theorem 3.4, there exists a invertible (orthogonal) matrix $Q = [\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3] = \begin{pmatrix} m & d & c \\ c & m & d \\ d & c & m \end{pmatrix}$ and construct $D = \begin{pmatrix} < \mathbf{q}_1, \mathbf{a}_1 > & 0 & 0 \\ 0 & < \mathbf{q}_2, \mathbf{a}_2 > & 0 \\ 0 & 0 & < \mathbf{q}_3, \mathbf{a}_3 > \end{pmatrix}$. So,

$$QD = \begin{pmatrix} m & d & c \\ c & m & d \\ d & c & m \end{pmatrix} \begin{pmatrix} < \mathbf{q}_1, \mathbf{a}_1 > & 0 & 0 \\ 0 & < \mathbf{q}_2, \mathbf{a}_2 > & 0 \\ 0 & 0 & < \mathbf{q}_3, \mathbf{a}_3 > \end{pmatrix}$$

$$= \begin{pmatrix} m & d & c \\ c & m & d \\ d & c & m \end{pmatrix} \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & r \end{pmatrix} = \begin{pmatrix} g & a & c \\ c & m & a \\ d & c & g \end{pmatrix} = A.$$

Conclusion. In this paper, we extended and studied the notion of meet-distributive lattice matrices. The properties of meet-distributive lattice matrices and various characterizations of them are given. Also, as an analogue to T. S.

Blyth [3], we generalize the concept of meet-distributive Boolean matrices to meet-distributive lattice matrices and characterize those matrices. It is proved that, the meet-distributive lattice matrix can be decomposed into a product $A = QD$ of an invertible (orthogonal) matrix Q and a diagonal matrix D.

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