BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(1)(2021), 1-13
DOI: 10.7251/BIMVI2101001R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ANNIHILATOR IDEALS IN 0-DISTRIBUTIVE ALMOST LATTICES

G. Nanaji Rao and R. Venkata Aravinda Raju

Abstract. The concept of annihilator ideal is introduced in a $0-$distributive Almost Lattice (AL) L and gave certain examples of annihilator ideals. We proved that the set of all annihilator ideals of L forms a complete Boolean algebra. The concept of an annihilator preserving homomorphism is introduced in L and proved certain properties of annihilator preserving homomorphisms. We proved that the images and the inverse images of an annihilator ideals are again annihilator ideals under annihilator preserving homomorphisms. Finally, a sufficient condition for an AL homomorphism to become annihilator preserving homomorphism is derived.

1. Introduction

In 1973, William H. Cornish [**2**] introduced the concept of annihilator ideals in distributive lattices and studied many properties of annihilator ideals. Mark Mandelker [**3**] studied the properties of relative annihilators in lattices and characterized the distributive lattice in terms of relative annihilators. The concept of almost lattice (AL) was introduced by Nanaji Rao, G. and Habtamu Tiruneh Alemu [**4**] as a common abstraction of almost all lattice theoretic generalizations of Boolean algebra like distributive lattices, almost distributive lattices and established necessary and sufficient conditions for an AL L to become lattice. Also, Nanaji Rao, G. and Habtamu Tiruneh Alemu [**5**] introduced the concept of ideals in an AL and proved that the set $\mathcal{I}(L)$ of all ideals in an AL L forms a complete lattice and proved the $P\mathcal{I}(L)$ of all principal ideals of L is a sublattice of the lattice $\mathcal{I}(L)$. The concept of pseudo-complemented almost lattices was introduced by Nanaji Rao,

2010 *Mathematics Subject Classification.* 06D99, 06D15.
Key words and phrases. Almost lattice, ideal, annihilator ideal, Boolean algebra, complete Boolean algebra, annihilator preserving homomorphism.
Communicated by Daniel A. Romano.

G. and R.Venkata Aravinda Raju [**6**] and proved some basic properties of pseudo-complementation on an AL L. Also, they proved that pseudo-complementation on an AL L is equationally definable and proved that there is a one-to-one correspondence between the set of all pseudo-complementations on L and the set of all maximal elements in L. Later, Nanaji Rao, G. and R.Venkata Aravinda Raju [**7**] introduced the concept of annihilator of a nonempty subset of an AL L and proved certain basic properties of annihilators in L. Also, they introduced the concept of 0-distributive almost lattice and obtained necessary and sufficient conditions for an AL with 0 to become 0-distributive AL in terms of annihilators, ideals and pseudo-complementations.

In this paper the concept of annihilator ideals in an AL is introduced and gave several examples of annihilator ideals and proved some basic properties of the annihilator ideals. We proved that the set $\mathcal{A}(L)$ of all annihilator ideals of an AL L with 0 form complete Boolean algebra. Next, the concept of annihilator preserving homomorphism is introduced and gave certain examples of annihilator preserving homomorphisms. A sufficient condition for an AL homomorphism to become annihilator preserving homomorphism is derived. Finally, we proved that the image and the inverse image of an annihilator ideal is again annihilator ideal under annihilator preserving homomorphisms.

2. Preliminaries

In this section we collect few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1. ([**8**]) Let $(P, \leqslant)$ be a poset. Then P is said to be lattice ordered set if for every pair $x,\ y \in P$, $l.u.b\{x,\ y\}$ and $g.l.b\{x,\ y\}$ exists.

Definition 2.2. ([**8**]) An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following axioms. For any $x, y, z \in L$,

(1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative Law)
(2) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative Law)
(3) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$. (Absorption Law)

It can be easily seen that in any lattice $(L, \vee, \wedge)$, $x \vee x = x$ and $x \wedge x = x$ (Idempotent Law).

Theorem 2.1 ([**8**]). *Let $(L, \leqslant)$ be a lattice ordered set. If we define $x \wedge y$ is the g.l.b of $\{x\ ,\ y\}$ and $x \vee y$ is the l.u.b of $\{x\ ,\ y\}$ $(x, y \in L)$, then $(L, \vee, \wedge)$ is a lattice.*

Theorem 2.2 ([**8**]). *Let $(L, \vee, \wedge)$ be a lattice. If we define a relation $\leqslant$ on L, by $x \leqslant y$ if and only if $x = x \wedge y$, or equivalently $x \vee y = y$. Then $(L, \leqslant)$ is a lattice ordered set.*

Important Note. Theorems 2.1. and 2.2. together imply that the concepts of lattice and lattice ordered set are the same. We refer to it as a lattice in future.

Definition 2.3. ([**8**]) Let $(L, \vee, \wedge)$ be a lattice. Then L is said to be a bounded lattice if L is bounded as a poset.

DEFINITION 2.4. [8] A bounded lattice L with bounds 0 and 1 is said to be complemented if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

THEOREM 2.3 ([**8**]). In any lattice $(L, \vee, \wedge)$, for any $x,\ y,\ z \in L$, the following are equivalent:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(2) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(4) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

DEFINITION 2.5. ([**8**]) A lattice $(L, \vee, \wedge)$ is called a distributive lattice if it satisfies any one of the four conditions in the theorem 2.3.

DEFINITION 2.6. ([**8**]) A complemented distributive lattice is called a Boolean algebra.

DEFINITION 2.7. ([**8**]) A lattice $(L, \vee, \wedge)$ is called a complete lattice if every nonempty subset of L has both $l.u.b$ and $g.l.b$.

THEOREM 2.4 ([**8**]). *If P is a partial ordered set bounded above each of whose non-void subsets R has an infimum, then each non-void subset P will have a supremum, too, and by the definitions $\bigcap R = infR$, $\bigcup = supR$, then P becomes a complete lattice.*

THEOREM 2.5 ([**1**]). *Let L be a lattice. Then for any $x, y, z \in L$, the following conditions are equivalent:*

(1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(3) $(x \vee y) \wedge z \leqslant x \vee (y \wedge z)$

DEFINITION 2.8. ([**4**]) An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an AL with 0 if, for any $a, b, c \in L$, it satisfies the following conditions:

(1) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
(2) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
(4) $(a \vee b) \vee c = a \vee (b \vee c)$
(5) $a \wedge (a \vee b) = a$
(6) $a \vee (a \wedge b) = a$
(7) $(a \wedge b) \vee b = b$
(8) $0 \wedge a = 0$

It can be easily seen that $a \wedge b = a$ if and only if, $a \vee b = b$ in an AL.

DEFINITION 2.9. ([**4**]) Let L be an AL and $a, b \in L$. Then we define a is less than or equal to b and write $a \leqslant b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$.

THEOREM 2.6 ([**4**]). *The relation $\leqslant$ is a partial ordering on an AL L and hence $(L, \leqslant)$ is a poset.*

DEFINITION 2.10. ([**4**]) Let L be any nonempty set. Define, for any $x, y \in L$, $x \vee y = x = y \wedge x$. Then, clearly L is an AL and is called discrete AL.

Definition 2.11. ([**4**]) An AL L is said to be directed above if for any $a, b \in L$ there exists $c \in L$ such that $a, b \leqslant c$.

Theorem 2.7 ([**4**]). *Let L be an AL. Then the following are equivalent:*

(1) *L is directed above.*
(2) *$\wedge$ is commutative.*
(3) *$\vee$ is commutative.*
(4) *L is a lattice.*

Definition 2.12. ([**1**]) Let L and L' be two ALs with zero elements 0 and $0'$ respectively. Then a mapping $f : L \to L'$ is called a homomorphism if it satisfies the following conditions.

(1) $f(a \vee b) = f(a) \vee f(b)$
(2) $f(a \wedge b) = f(a) \wedge f(b)$
(3) $f(0) = 0'$.

Definition 2.13. ([**5**]) Let L be an AL. Then a nonempty subset I of L is said to be an ideal of L if it satisfies the following conditions.

(1) If $x, y \in I$, then there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.
(2) If $x \in I$ and $a \in L$, then $x \wedge a \in I$.

Lemma 2.1 ([**5**]). *Let L be an AL and I be an ideal of L. Then the following are equivalent:*

(1) *$x, y \in I$ implies $x \vee y \in I$.*
(2) *$x, y \in I$ implies there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.*

Corollary 2.1 ([**5**]). *Let L be an AL and $a \in L$. Then $(a] = \{a \wedge x |\ x \in L\}$ is an ideal of L and is called principal ideal generated by a.*

Corollary 2.2 ([**5**]). *Let L be an AL and $a,\ b \in L$. Then $a \in (b]$ if and only if $a = b \wedge a$.*

Corollary 2.3 ([**5**]). *Let L be an AL and $a,\ b \in L$. Then $(a \wedge b] = (b \wedge a]$.*

Theorem 2.8 ([**5**]). *Let L be an AL. Then the set $\mathcal{I}(L)$ of all ideals of L form a lattice under set inclusion in which the glb and lub for any $I, J \in \mathcal{I}(L)$ are respectively $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : (a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J\}$.*

Theorem 2.9 ([**5**]). *Let L be an AL. Then the set $P\mathcal{I}(L)$ of all principal ideals of L is a sublattice of the lattice $\mathcal{I}(L)$ of all ideals of L.*

Definition 2.14. ([**5**]) Let L be an AL. Then for any ideal I in L, define $I^e = \{(a] : a \in I\}$.

Lemma 2.2 ([**5**]). *Let L be an AL and $I,\ J$ be ideals of L. Then we have the following.*

(1) *I^e is an ideal of the lattice $\mathcal{PI}(L)$*
(2) *$I \subseteq J \Leftrightarrow I^e \subseteq J^e$*
(3) *$(I \vee J)^e = I^e \vee J^e$*

(4) $(I \cap J)^e = I^e \cap J^e$
(5) *I is prime* $\Leftrightarrow$ *I^e is prime.*

Definition 2.15. ([**7**]) Let L be an AL with 0. Then L is said to be 0-distributive if for any $a, b, c \in L$, $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$.

Corollary 2.4 ([**7**]). *Let L be an AL with 0. Then for any ideals I, J of L. we have the following:*

(1) $I \cap I^* = \{0\}$
(2) $I^* = \bigcap\limits_{a \in I} (a]^*$
(3) $(I \cap J)^* = (J \cap I)^*$
(4) $I \subseteq J \Rightarrow J^* \subseteq I^*$
(5) $I^* \cap J^* \subseteq (I \cap J)^*$
(6) $I \subseteq I^{**}$
(7) $I^{***} = I^*$
(8) $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq I^{**}$
(9) $(I \cup J)^* = I^* \cap J^* = (J \cup I)^*$

Corollary 2.5 ([**7**]). *Let L be an AL with 0. Then for any ideals I, J of L, we have the following.*

(1) $(I \cap J)^{**} = I^{**} \cap J^{**}$
(2) $I \cap J = (0] \Leftrightarrow I \subseteq J^* \Leftrightarrow J \subseteq I^*$

Corollary 2.6 ([**7**]). *Let L be an AL with 0. If $\{I_i : i \in \Delta\}$ is a family of ideals of L, then* $(\bigcap\limits_{i \in \Delta} I_i)^{**} = \bigcap\limits_{i \in \Delta} (I_i)^{**}$.

3. Annihilator ideals

In this section, we introduce the concept of annihilator ideal in an AL L, give certain examples of annihilator ideals. Also, we establish a necessary and sufficient condition for an ideal in L to become annihilator ideal. Finally, we prove that the set $\mathcal{A}(L)$ of all annihilator ideals in L forms a complete Boolean algebra. First, we begin this section with the following definition.

Definition 3.1. Let L be an AL with 0 and I be an ideal of L. Then I is called an annihilator ideal of L if $I = A^*$ for some non-empty subset A of L.

It can be easily seen that if I is an annihilator ideal in an AL L then $I = I^{**}$. Also, seen that for any ideal I in L, I^* is an annihilator ideal. Note that, the set of all annihilator ideals in an AL L with 0 is denoted by $\mathcal{A}(L)$. In the following we give certain examples of annihilator ideals.

Example 3.1. Let X be a discrete AL with 0 and with atleast two elements, other than 0. Then $(X^n, \vee, \wedge, 0')$ is an AL with $0' = (0, 0, ..., 0)$, where $\vee$, $\wedge$ are defined coordinate-wise. Now, put $I = \{(0, a_1, a_2, ..., a_{n-1}) : a_i \in X\}$. Then clearly I is an ideal of L and also, $I = I^{**}$. Hence I is an annihilator ideal of L.

EXAMPLE 3.2. Let $(L, +, ., 0)$ be a commutative regular ring with unity. For any $a \in L$, let a^0 be the unique idempotent element in L such that $aL = a^0L$. For any $x, y \in L$, define $x \wedge y = x^0y$ and $x \vee y = x + (1 - x^0)y$. Then clearly $(L, \vee, \wedge, 0)$ is an AL with 0. Now, consider $I = (x^0]$ and $J = (1 - x^0]$. Then clearly I, J are an annihilator ideals in L.

EXAMPLE 3.3. Let $L = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on L as follows:

∨	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

and

∧	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then clearly $(L, \vee, \wedge, 0)$ is an AL with 0. Now, put $I = \{0, a\}$. Then clearly I is an ideal in L. Now, $I^* = \{0, c\}$ and also $I^{**} = \{0, a\} = I$. Thus I is an annihilator ideal in L.

Recall that for any ideal I of an AL L, $I^e = \{(a] : a \in I\}$ is an ideal of the lattice $\mathcal{PI}(L)$. In the following we derive a necessary and sufficient condition for an ideal in an AL to become annihilator ideal.

LEMMA 3.1. *Let L be an AL with 0. Then an ideal I of L is an annihilator ideal in L if and only if I^e is an annihilator ideal in $\mathcal{PI}(L)$.*

PROOF. Suppose I is an annihilator ideal in L. Then we have $I = I^{**}$. Now, we shall prove that $I^e = (I^e)^{**}$. Clearly, $I^e \subseteq (I^e)^{**}$. Let $(a] \in (I^e)^{**}$ and $b \in I^*$. Then for any $c \in I$, $(b] \cap (c] = (b \wedge c] = (0]$. Hence $(b] \in (I^e)^*$. Since $(a] \in (I^e)^{**}$, we get $(a] \cap (b] = (0]$. This implies $(a \wedge b] = (0]$. It follows $a \wedge b = 0$. Thus $a \in I^{**} = I$. Hence $a \in I$. Therefore $(a] \in I^e$. Thus $(I^e)^{**} \subseteq I^e$. Therefore $I^e = (I^e)^{**}$. Hence I^e is an annihilator ideal in $\mathcal{PI}(L)$.

Conversely, suppose I^e is an annihilator ideal in $\mathcal{PI}(L)$. Then $(I^e)^{**} = I^e$. Now, we prove that $I = I^{**}$. We have $I \subseteq I^{**}$. Let $a \in I^{**}$ and $(b] \in (I^e)^*$. Now, for any $c \in I$, $(c] \in I^e$. Hence $(b] \cap (c] = (0]$. This implies $(b \wedge c] = (0]$. Hence $b \wedge c = 0$. Thus $b \in I^*$. Now, let $a \in I^{**}$. Then $a \wedge b = 0$. It follows that $(a \wedge b] = (0]$. This implies $(a] \cap (b] = (0]$. Thus $(a] \in (I^e)^{**} = I^e$. Hence $(a] \in I^e$. Therefore $(a] = (t]$ for some $t \in I$. Now, $a \in (a] = (t]$ and hence $a = t \wedge a \in I$. Thus $a \in I$. Therefore $I^{**} \subseteq I$. Thus $I^{**} = I$ and hence I is an annihilator ideal in L. □

THEOREM 3.1. *Let L be a 0-distributive AL. Then for any ideal I of L, $(I^e)^*$ is an annihilator ideal in $\mathcal{PI}(L)$.*

PROOF. Suppose I is an ideal of L. Then we have I^e is an ideal of $\mathcal{PI}(L)$. Now, we prove that $(I^e)^*$ is an ideal of $\mathcal{PI}(L)$. Since $(0] \cap (a] = (0 \wedge a] = (0]$ for all $(a] \in I^e$, $(0] \in (I^e)^*$. Hence $(I^e)^*$ is non empty subset of $\mathcal{PI}(L)$. Let $(x], (y] \in (I^e)^*$. Then $(x] \cap (a] = (0]$ and $(y] \cap (a] = (0]$ for all $(a] \in I^e$. This implies $(x \wedge a] = (0]$ and $(y \wedge a] = (0]$ for all $(a] \in I^e$. It follows that $x \wedge a = 0$ and $y \wedge a = 0$ for all $a \in I$. Since L is 0-distributive AL, we get $(x \vee y) \wedge a = 0$ for all

$a \in I$. This implies $((x \vee y) \wedge a] = (0]$ for all $a \in I$. It follows that $(x \vee y] \cap (a] = ((x] \vee (y]) \cap (a] = (0]$ for all $(a] \in I^e$. Therefore $(x] \vee (y] \in (I^e)^*$. Let $(x] \in (I^e)^*$ and $(r] \in \mathcal{PI}(L)$. Then $(x] \cap (a] = (0]$ for all $(a] \in I^e$. Now, for any $(t] \in I^e$, consider $((x] \cap (r]) \cap (t] = ((x] \cap (t]) \cap (r] = (x \wedge t] \cap (r] = (0] \cap (r] = (0 \wedge r] = (0]$. Thus $(x] \cap (r] \in (I^e)^*$. Therefore $(I^e)^*$ is an ideal. Clearly, $(I^e)^*$ is an annihilator ideal in the lattice $\mathcal{PI}(L)$. □

Corollary 3.1. *Let L be a 0-distributive AL. Then for any annihilator ideal I in L, $(I^e)^*$ is an annihilator ideal in $\mathcal{PI}(L)$.*

Recall that the set $\mathcal{I}(L)$ of all ideals of an AL L is a lattice. But, the set $\mathcal{A}(L)$ of all annihilator ideals in L is not a sublattice of $\mathcal{I}(L)$. For, consider the following example.

Example 3.4. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is as follows:

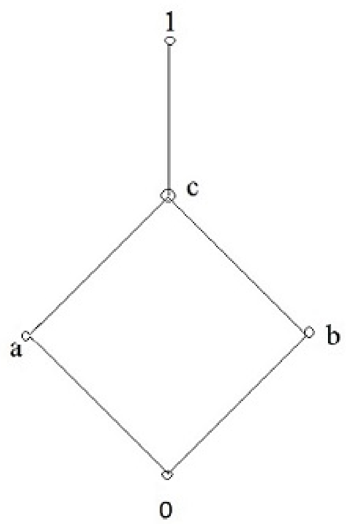

Figure 1

Now, put $I = \{0, a\}$ and $J = \{0, b\}$. Then clearly I and J are ideals of L. Now, $I^* = \{0, b\} = J$ and $J^* = \{0, a\} = I$. It follows that I and J are annihilator ideals in L. Now, $I \vee J = \{0, a, b, c\}$. Therefore $(I \vee J)^* = \{0\}$. Hence $(I \vee J)^{**} = \{0\}^* = L \neq I \vee J$. Therefore $I \vee J$ is not an annihilator ideal in L.

However, we prove that the set $\mathcal{A}(L)$ of all annihilator ideals in an AL L is a complete Boolean algebra. For, this first we need the following.

Lemma 3.2. *Let L be 0-distributive AL. Then for any $I, J \in \mathcal{A}(L)$, $I \cap J = (I^* \vee J^*)^*$.*

Proof. Suppose $I, J \in \mathcal{A}(L)$. Since $I^* \subseteq I^* \vee J^*$, we get $(I^* \vee J^*)^* \subseteq I^{**} = I$. Similarly, we get $(I^* \vee J^*)^* \subseteq J$. Hence $(I^* \vee J^*)^* \subseteq I \cap J$.

Conversely, suppose $x \in I \cap J$ and $y \in I^* \vee J^*$. Then $y = (a \vee b) \wedge y$ for some $a \in I^*$ and $b \in J^*$. It follows that $x \wedge a = 0$ and $x \wedge b = 0$. Now, consider $x \wedge y = x \wedge ((a \vee b) \wedge y) = (x \wedge (a \vee b)) \wedge y = 0 \wedge y = 0$. Therefore $x \in (I^* \vee J^*)^*$. Hence $I \cap J \subseteq (I^* \vee J^*)^*$. Thus $I \cap J = (I^* \vee J^*)^*$. □

Theorem 3.2. *Let L be a 0-distributive AL with 0. Then the set $\mathcal{A}(L)$ of all annihilator ideals of L forms a complete Boolean algebra.*

Proof. Let L be an AL with 0. Then clearly, $\mathcal{A}(L)$ is non empty, since $(0], L \in \mathcal{A}(L)$. Also, clearly $\mathcal{A}(L)$ is a poset with respect to the set inclusion. Let $I, J \in \mathcal{A}(L)$. Then $I = I^{**}$ and $J = J^{**}$. Now, consider $(I \cap J)^{**} = I^{**} \cap J^{**} = I \cap J$. Therefore $I \cap J \in \mathcal{A}(L)$ and clearly $I \cap J$ is the g.l.b of I, J in $\mathcal{A}(L)$. Define $I \underline{\vee} J = (I^* \cap J^*)^*$. Now, consider $(I \underline{\vee} J)^{**} = ((I^* \cap J^*)^*)^{**} = (I^* \cap J^*)^{***} = (I^* \cap J^*)^* = I \underline{\vee} J$. Thus $(I \underline{\vee} J)^{**} = I \underline{\vee} J$. Hence $I \underline{\vee} J$ is an annihilator ideal. Therefore $I \underline{\vee} J \in \mathcal{A}(L)$. Let $x \in I$ and $y \in I^* \cap J^*$. Then $x \in I$ and $y \in I^*$. It follows that $x \wedge y \in I \cap I^* = (0]$. Thus $x \wedge y = 0$. Hence $x \in (I^* \cap J^*)^*$. Therefore $I \subseteq (I^* \cap J^*)^* = I \underline{\vee} J$. Similarly, we get $J \subseteq I \underline{\vee} J$. Therefore $I \underline{\vee} J$ is an upper bound of I, J in $\mathcal{A}(L)$.

Suppose $H \in \mathcal{A}(L)$ such that H is an upper bound of I, J. Then $I \subseteq H$ and $J \subseteq H$. This implies $H^* \subseteq I^*$ and $H^* \subseteq J^*$. Hence $H^* \subseteq I^* \cap J^*$. Therefore $(I^* \cap J^*)^* \subseteq H^{**} = H$. Thus $I \underline{\vee} J \subseteq H$. Therefore $I \underline{\vee} J$ is the l.u.b of I, J. Hence $(\mathcal{A}(L), \wedge, \underline{\vee})$ is a lattice. Clearly, $(0]$ and L are the least and greatest elements in $\mathcal{A}(L)$ respectively. Hence $\mathcal{A}(L)$ is a bounded lattice. Let $I \in \mathcal{A}(L)$. Then, we have $I^* \in \mathcal{A}(L)$ and $I \cap I^* = (0]$. Now, consider $I \underline{\vee} I^* = (I^* \cap I^{**})^* = (0]^* = L$. Thus every element in $\mathcal{A}(L)$ has a complement in $\mathcal{A}(L)$. Therefore $\mathcal{A}(L)$ is a complemented lattice.

Now, we shall prove $\mathcal{A}(L)$ is a distributive lattice. Let $I, J, K \in \mathcal{A}(L)$. Since $\mathcal{A}(L)$ is a lattice, $I \underline{\vee} (J \cap K) \subseteq (I \underline{\vee} J) \cap (I \underline{\vee} K)$. First we shall prove that $(I \underline{\vee} J) \cap K \subseteq I \underline{\vee} (J \cap K)$. We have $I \cap (K \cap (I^* \cap (J \cap K)^*)) = (0]$. This implies $K \cap (I^* \cap (J \cap K)^*) \subseteq I^*$. Also, we have $J \cap (K \cap (I^* \cap (J \cap K)^*)) = (0]$. Therefore $K \cap (I^* \cap (J \cap K)^*) \subseteq J^*$. Hence $K \cap (I^* \cap (J \cap K)^*) \subseteq I^* \cap J^*$. Thus $(K \cap (I^* \cap (J \cap K)^*)) \cap (I^* \cap J^*)^* \subseteq (I^* \cap J^*) \cap (I^* \cap J^*)^* = (0]$. Therefore $(K \cap (I^* \cap (J \cap K)^*)) \cap (I^* \cap J^*)^* = (0]$. This implies $(I^* \cap (J \cap K)^*) \cap (K \cap (I^* \cap J^*)^*) = (0]$. Thus $K \cap (I^* \cap J^*)^* \subseteq (I^* \cap (J \cap K)^*)^*$. It follows that $(I^* \cap J^*)^* \cap K \subseteq (I^* \cap (J \cap K)^*)^*$. Hence $(I \underline{\vee} J) \cap K \subseteq I \underline{\vee} (J \cap K)$. Now, $(I \underline{\vee} J) \cap (I \underline{\vee} K) \subseteq I \underline{\vee} (J \cap (I \underline{\vee} K)) = I \underline{\vee} ((I \underline{\vee} K) \cap J) \subseteq I \underline{\vee} (I \underline{\vee} (K \cap J)) = (I \underline{\vee} I) \underline{\vee} (K \cap J) = I \underline{\vee} (J \cap K)$. Thus $(\mathcal{A}(L), \cap, \underline{\vee}, ^*, (0], L)$ is a Boolean algebra. Also, by theorem 2.4. and by corollary 2.6., we get $\mathcal{A}(L)$ is a complete Boolean algebra. □

4. Annihilator Preserving homomorphisms

In this section, we introduce the concepts of an annihilator preserving homomorphism and dense AL and give examples of annihilator preserving homomorphisms in terms of dense ALs. Next, we prove certain basic properties of annihilator preserving homomorphisms. Finally, we derive sufficient condition for an AL homomorphism to become annihilator preserving homomorphism. For, this first we need the following.

LEMMA 4.1. *Let L and L' be two ALs with zero elements 0 and $0'$ respectively and $f : L \to L'$ a homomorphism. Then we have the following.*

(1) *If f is onto, then for any ideal I of L, $f(I)$ is an ideal of L'.*
(2) *If J is an ideal of L', then $f^{-1}(J)$ is an ideal of L.*
(3) *$ker(f)$ is an ideal of L.*

PROOF. (1) Suppose I is an ideal of L. Then we have $f(I) = \{f(x) : x \in I\}$. Clearly $f(I)$ is non empty, since $f(0) \in f(I)$. Let $f(a), f(b) \in f(I)$ where $a, b \in I$. Then $a, b \in I$ and hence $a \vee b \in I$. Therefore $f(a \vee b) \in f(I)$. It follows $f(a) \vee f(b) \in f(I)$. Again, let $f(a) \in f(I)$ and $r \in L'$. Since f is onto and $r \in L'$, there exist $t \in L$ such that $f(t) = r$. Now, since $a \in I$ and $t \in L$, $a \wedge t \in I$. Therefore $f(a \wedge t) \in f(I)$. It follows $f(a) \wedge f(t) \in f(I)$. Hence $f(a) \wedge r \in f(I)$. Therefore $f(I)$ is an ideal in L'.

(2) Suppose J is an ideal in L'. Then $f^{-1}(J) = \{x \in L : f(x) \in J\}$. Since $f(0) = 0' \in J$, we get $0 \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is non empty subset of L. Let $a, b \in f^{-1}(J)$. Then $f(a), f(b) \in J$. Since J is an ideal in L', $f(a) \vee f(b) \in J$. This implies $f(a \vee b) \in J$. Hence $a \vee b \in f^{-1}(J)$. Again, let $a \in f^{-1}(J)$ and $x \in L$. Then $f(a) \in J$ and $f(x) \in f(L) \subseteq L'$. Since J is an ideal in L', $f(a) \wedge f(x) \in J$. Therefore $f(a \wedge x) \in J$. Hence $a \wedge x \in f^{-1}(J)$. Thus $f^{-1}(J)$ is an ideal in L.

(3) We have $\{0'\}$ is an ideal of L'. Therefore by condition (2), we get $f^{-1}(\{0'\})$ is an ideal of L. But, $f^{-1}(\{0'\}) = \{x \in L : f(x) \in \{0'\}\} = \{x \in L : f(x) = 0'\} = ker(f)$. Thus $ker(f)$ is an ideal of L. □

LEMMA 4.2. *Let $f : L \to L'$ be a homomorphism. Then for any non empty subset A of L, $f(A^*) \subseteq (f(A))^*$.*

PROOF. Let $a \in f(A^*)$ and $y \in f(A)$. Then there exists $b \in A^*$ and $x \in A$ such that $a = f(b)$ and $y = f(x)$. Now, since $b \in A^*$ and $x \in A$, $b \wedge x = 0$. Therefore $a \wedge y = f(b) \wedge f(x) = f(b \wedge x) = f(0) = 0'$. Thus $a \in (f(A))^*$. Therefore $f(A^*) \subseteq (f(A))^*$. □

But, the converse of the lemma 4.2. is not true. For, consider the following example.

EXAMPLE 4.1. Let $L = \{0, a, b, c\}$ be a discrete AL. Define a mapping $f : L \to L$ by $f(x) = 0$ for all $x \in L$. Then clearly f is a homomorphism. Now, put $A = \{a, b\}$. Then clearly $A^* = \{0\}$ and $f(A) = \{0\}$. Hence $f(A^*) = \{0\}$ and $(f(A))^* = L$. Thus $(f(A))^* \nsubseteq f(A^*)$.

In view of above observation, we introduce the concept of annihilator preserving homomorphism.

Definition 4.1. Let L and L' be two ALs with 0 and $0'$ respectively. Then a homomorphism $f : L \to L'$ is called annihilator preserving if it satisfies $f(A^*) = (f(A))^*$, for any $\{0\} \subset A \subset L$.

In the following we give an example of annihilator preserving homomorphism.

Example 4.2. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Write $L = A \times B = \{(0,0),(0,b_1),(0,b_2),(a,0),(a,b_1),(a,b_2)\}$. Then $(L, \vee, \wedge, (0,0))$ is an AL under point-wise operations with the zero element $(0,0)$. Also, put $L' = \{0', a', b', c'\}$ and define the operations $\vee'$ and $\wedge'$ on L' as follows.

$\vee'$	$0'$	a'	b'	c'
$0'$	$0'$	a'	b'	c'
a'	a'	a'	c'	c'
b'	b'	c'	b'	c'
c'	c'	c'	c'	c'

and

$\wedge'$	$0'$	a'	b'	c'
$0'$	$0'$	$0'$	$0'$	$0'$
a'	$0'$	a'	$0'$	a'
b'	$0'$	$0'$	b'	b'
c'	$0'$	a'	b'	c'

Then clearly $(L', \vee', \wedge', 0')$ is an AL with zero. Now, define a mapping $f : L \to L'$ by $f((0,0)) = 0'$, $f((a,0)) = a'$, $f((0,b_1)) = f((0,b_2)) = b'$, $f((a,b_1)) = f((a,b_2)) = c'$. Then clearly f is a homomorphism from L onto L' and also, clearly f is an annihilator preserving homomorphism.

Next, we introduce the concept of dense AL and give an example of dense AL. Also, we establish an example of annihilator preserving homomorphism interms of dense ALs. For, this first we need the following.

Definition 4.2. An element a of an AL L is called a dense element if $[a]^* = \{0\}$.

It can be easily observed that every maximal element is dense. But, dense element need not be maximal. For, consider the following example.

Example 4.3. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Now, put $L = A \times B = \{(0,0),(0,b_1),(0,b_2),(a,0),(a,b_1),(a,b_2)\}$ and define operations $\vee$ and $\wedge$ on L as follows.

$\vee$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	(a,b_1)	(a,b_1)	(a,b_1)
$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	(a,b_2)	(a,b_2)	(a,b_2)
$(a,0)$	$(a,0)$	(a,b_1)	(a,b_2)	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)
(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)

and

$\wedge$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,b_1)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(a,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
(a,b_1)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_2)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)

Then clearly, $(L,\vee,\wedge,(0,0))$ is an AL with $(0,0)$ as its zero element. Now, let $L'=\{(0,0),(0,b_1),(0,b_2),(a,b_1),(a,b_2)\}$. Then L' is a sub AL of $(L,\vee,\wedge,(0,0))$. In L', (a,b_1), (a,b_2) are only maximal elements. Now, $[(0,b_1)]^*=\{(0,0)\}$. So that $(0,b_1)$ is a dense element, but not a maximal element in L'. Because $(0,b_1)\wedge(a,b_1)=(0,b_1)\neq(a,b_1)$. Similarly $(0,b_2)$ is also a dense element but not a maximal element.

Definition 4.3. Let L be an AL with 0. Then L is said to be dense AL if every nonzero element in L is dense.

It can be easily seen that every discrete AL is a dense AL. More generally, we have the following theorem.

Theorem 4.1. *Let L and L' be two dense ALs. Then every homomorphism from L into L' is an annihilator preserving homomorphism.*

Proof. Suppose L and L' are two dense ALs and $f:L\to L'$ is a homomorphism. Suppose $\{0\}\subset A\subset L$. Now, we shall prove that $f(A^*)=(f(A))^*$. Since L is dense and $A^*=\bigcap\limits_{a\in A}[a]^*$, $A^*=\{0\}$. Therefore $f(A^*)=f(\{0\})=\{0\}$. On the other hand $(f(A))^*=\bigcap\limits_{f(a)\in f(A)}[f(a)]^*=\{0\}$, since L' is dense. Therefore $f(A^*)=(f(A))^*$. Thus f is an annihilator preserving homomorphism. □

Now, we prove some properties of annihilator preserving homomorphisms.

Theorem 4.2. *Let L and L' be two ALs with zero elements 0 and $0'$ respectively and let $f:L\to L'$ be an annihilator preserving homomorphism such that $ker(f)=\{0\}$. Then for any non empty subsets A and B of L, $A^*=B^*$ if and only if $(f(A))^*=(f(B))^*$.*

Proof. Suppose A and B are two non empty subsets of L such that $A^*=B^*$. Then clearly $f(A^*)=f(B^*)$. Since f is annihilator preserving, $(f(A))^*=(f(B))^*$. Conversely, suppose $(f(A))^*=(f(B))^*$.

$$
\begin{aligned}
\text{Now, } t\in A^* &\Rightarrow t\wedge a=0 \text{ for all } a\in A.\\
&\Rightarrow f(t\wedge a)=f(0)\\
&\Rightarrow f(t)\wedge f(a)=0' \text{ for all } a\in A\\
&\Rightarrow f(t)\in(f(A))^*\\
&\Rightarrow f(t)\in(f(B))^*\\
&\Rightarrow f(t)\wedge f(b)=0' \text{ for all } b\in B\\
&\Rightarrow f(t\wedge b)=0' \text{ for all } b\in B
\end{aligned}
$$

$\Rightarrow t \wedge b \in ker(f) = \{0\}$ for all $b \in B$
$\Rightarrow t \wedge b = 0$ for all $b \in B$
$\Rightarrow t \in B^*$.

Therefore $A^* \subseteq B^*$. Similarly, we get $B^* \subseteq A^*$. Therefore $A^* = B^*$. □

THEOREM 4.3. *Let L and L' be two ALs with zero elements 0 and $0'$ respectively and let $f : L \to L'$ be a homomorphism. Then we have the following.*

(1) *If f is annihilator preserving and onto, then $f(I)$ is an annihilator ideal of L' for every annihilator ideal I of L.*
(2) *If f^{-1} preserves annihilators, then $f^{-1}(J)$ is an annihilator ideal of L for every annihilator ideal J of L'.*

PROOF. (1) Suppose I is an annihilator ideal of L. Then by lemma 4.1(1), $f(I)$ is an ideal of L'. Since f is annihilator preserving, $(f(I))^{**} = f(I^{**}) = f(I)$. Therefore $f(I)$ is an annihilator ideal in L'.

(2) Suppose J is an annihilator ideal of L'. Then by lemma 4.1(2), $f^{-1}(J)$ is an ideal of L. Since f^{-1} preserves annihilators, $(f^{-1}(J))^{**} = f^{-1}(J^{**}) = f^{-1}(J)$. Therefore $f^{-1}(J)$ is an annihilator ideal in L. □

COROLLARY 4.1. *Let L and L' be two ALs with zero elements 0 and $0'$ respectively and let $f : L \to L'$ be a homomorphism such that f^{-1} preserves annihilators. Then $ker(f)$ is an annihilator ideal of L.*

PROOF. By lemma 4.1(3), we get $ker(f)$ is an ideal of L. Also, we have $\{0'\}$ is an annihilator ideal in L'. Now, we have $ker(f) = f^{-1}(\{0'\})$. Therefore $(ker(f))^{**} = (f^{-1}(\{0'\}))^{**} = f^{-1}(\{0'\}^{**}) = f^{-1}(\{0'\}) = ker(f)$. Therefore $ker(f)$ is an annihilator ideal. □

It can be easily seen that if $f : L \to L'$ is homomorphism such that f is one-one, then $ker(f) = \{0\}$. But, converse need not be true. For, consider the following example.

EXAMPLE 4.4. Let $L = \{0, a, b\}$ and $L' = \{0', c'\}$ be two discrete ALs. Define a mapping $f : L \to L'$ by $f(0) = 0'$ and $f(a) = f(b) = c'$. Then clearly f is a homomorphism from L into L' such that $ker(f) = \{0\}$. Now, we have $f(a) = f(b)$, but $a \neq b$. Hence f is not one-one.

Finally, we give sufficient condition for a homomorphism to become annihilator preserving.

THEOREM 4.4. *Let L and L' be two ALs with zero elements 0 and $0'$ respectively and let $f : L \to L'$ be a homomorphism. If $ker(f) = \{0\}$ and f is onto, then both f and f^{-1} are annihilator preserving.*

PROOF. Suppose f is onto and $ker(f) = \{0\}$. Let A be a non empty subset of L such that $\{0\} \subset A \subset L$. Then we have $f(A^*) \subseteq (f(A))^*$. Now, let $x \in (f(A))^*$. Since f is onto, there exists $y \in L$ such that $f(y) = x$. This implies $f(y) \wedge f(a) = 0'$ for all $a \in A$. Hence $f(y \wedge a) = 0'$ for all $a \in A$. Therefore $y \wedge a \in ker(f) = \{0\}$ for all $a \in A$. Hence $y \wedge a = 0$ for all $a \in A$. It follows that $y \in A^*$. This implies $x = f(y) \in f(A^*)$. Therefore $(f(A))^* \subseteq f(A^*)$ and hence $f(A^*) = (f(A))^*$. Therefore f is annihilator preserving homomorphism.

We shall prove that f^{-1} preserves annihilators. Let B be a non empty subset of L'. Let $x \in (f^{-1}(B))^*$. Then $x \wedge b = 0$ for all $b \in f^{-1}(B)$. This implies $f(x) \wedge f(b) = f(x \wedge b) = 0'$ for all $f(b) \in B$. It follows that $f(x) \in B^*$. Hence $x \in f^{-1}(B^*)$. Therefore $(f^{-1}(B))^* \subseteq f^{-1}(B^*)$. Conversely, suppose $x \in f^{-1}(B^*)$ and $b \in f^{-1}(B)$. Then $f(x) \in B^*$ and $f(b) \in B$. Hence $f(x \wedge b) = f(x) \wedge f(b) = 0'$. Therefore $x \wedge b \in ker(f) = \{0\}$ and hence $x \wedge b = 0$ for all $b \in f^{-1}(B)$. Hence $x \in (f^{-1}(B))^*$. Thus $f^{-1}(B^*) \subseteq (f^{-1}(B))^*$. Therefore $f^{-1}(B^*) = (f^{-1}(B))^*$. Thus f^{-1} preserves annihilators. □

COROLLARY 4.2. *Let L and L' be two ALs with zero elements 0 and $0'$ respectively and let $f : L \to L'$ be a homomorphism. If f is one-one and onto, then both f and f^{-1} are annihilator preserving.*

References

[1] Birkhoff, G. *Lattice Theory*. American Mathematical Society, Vol. XXV, Providence, 1967.

[2] Cornish, W. H. Annulets and α-ideals in Distributive Lattices. *J. Austral. Math. Soc.*, **15**(1)(1973), 70–77.

[3] Mandelkar, M. Relative annihilators in lattices. *Duke Math. J.*, **37**(2)(1970), 377–389.

[4] Nanaji Rao, G. and Habtamu, T. A. Almost Lattices. *J. Int. Mat. Virtual Inst.*, **9**(1)(2019), 155–171 .

[5] Nanaji Rao, G. and Habtamu T. A. Ideals in Almost Lattices. *Bull. Int. Mat. Virtual Inst.*, **10**(1)(2019), 37–50.

[6] Nanaji Rao, G. and R.Venkata Aravinda Raju. Pseudo-complementation on Almost Lattices. *Annals of Pure and Applied Mathematics*, **19**(1)(2019), 37–51.

[7] Nanaji Rao, G., R.Venkata Aravinda Raju. 0-Distributive Almost Lattices. *Bull. Int. Mat. Virtual Inst.*, **10**(2)(2020), 239–248.

[8] Szasz, G. *Introduction to Lattice Theory*, Academic Press, New York, 1963.

Received by editors 05.11.2019; Revised version 11.06.2020; Available online 22.06.2020.

G. NANAJI RAO. DEPARTMENT OF MATHEMATICS, ANDHRA UNIVERSITY, VISAKHAPATNAM, INDIA

E-mail address: nani6us@yahoo.com

R. VENKATA ARAVINDA RAJU. DEPARTMENT OF MATHEMATICS, ANDHRA UNIVERSITY, VISAKHAPATNAM, INDIA

E-mail address: aravindaraju.1@gmail.com