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α-IDEALS IN 0-DISTRIBUTIVE ALMOST LATTICES

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Abstract. We introduce the concept of an α- ideal in a $0-$distributive almost lattice. We give some examples of the α-ideals. We prove that the set of all α-ideals in a 0-distributive almost lattice L forms a complete distributive lattice. We obtain necessary and sufficient conditions for an ideal in a 0-distributive AL to be an α-ideal. Finally, we establish a necessary and sufficient condition for the contraction of an α-ideal to be an α-ideal.

1. Introduction

In 1973, W. H. Cornish [**2**] introduced the concept of an α-ideal in a distributive lattice with 0 and studied many properties of these ideals. He characterised α-ideals in terms of annulets. The concept of almost lattice (AL) was introduced by G. Nanaji Rao and Habtamu Tiruneh Alemu [**3**] as a common abstraction of almost all lattice theoretic generalizations of Boolean algebra like distributive lattices, almost distributive lattices and established necessary and sufficient conditions for an AL to become a lattice. Also, G. Nanaji Rao and Habtamu Tiruneh Alemu [**4**] introduced the concept of ideals in an AL and proved that the set $\mathcal{I}(L)$ of all ideals in an AL L forms a lattice and proved the set of all principal ideals of L, denoted by $P\mathcal{I}(L)$ is a sublattice of the lattice $\mathcal{I}(L)$. Later, G. Nanaji Rao and R. Venkata Aravinda Raju [**7**] introduced the concept of annihilators of a nonempty subsets of almost lattices and proved some of their basic properties. Also, they introduced the concept of 0-distributive almost lattice and obtained necessary and sufficient conditions for an AL with 0 to become 0-distributive AL in terms of annihilators, ideals and pseudo-complementations.

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In this paper, we introduce the concept of α-ideal in a 0-distributive AL L and give certain examples of α-ideals. Also, we obtain that if I is an ideal of a 0-distributive AL L then the set $\bar{I} = \{x \in L : (a]^* \subseteq (x]^* \text{ for some } a \in I\}$ is the smallest α-ideal containing I. We observe that the lattice $\mathcal{I}(L)$ of all ideals in an AL L need not be distributive.We prove that the set $\mathcal{I}_\alpha(L)$ of all α-ideals of L forms a complete distributive lattice but, in general, not a sublattice of the lattice $\mathcal{I}(L)$. We prove a set of identities for an ideal in a 0-distributive AL to become an α-ideal. We derive necessary and sufficient conditions for an ideal in a 0-distributive AL to become an α-ideal. Also, we characterize $*$-0-distributive almost lattice ($*$-0-DAL) in terms of α-ideals. Finally, obtain a necessary and sufficient condition for the contraction of an α-ideal to be an α-ideal.

2. Preliminaries

In this section we collect few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1. Let $(P, \leqslant)$ be a poset. Then P is said to be lattice ordered set if for every pair $x,\ y \in P$, $l.u.b\{x,\ y\}$ and $g.l.b\{x,\ y\}$ exists.

Definition 2.2. An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following axioms. For any $x, y, z \in L$,

(1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative Law)
(2) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative Law)
(3) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$. (Absorption Law)

It can be easily seen that in any lattice L,
$x \vee x = x$ and $x \wedge x = x$ (Idempotent Law).

Theorem 2.1. *Let $(L, \leqslant)$ be a lattice ordered set. If we define $x \wedge y$ is the g.l.b of $\{x\ ,\ y\}$ and $x \vee y$ is the l.u.b of $\{x\ ,\ y\}$ $(x, y \in L)$, then $(L, \vee, \wedge)$ is a lattice.*

Theorem 2.2. Let $(L, \vee, \wedge)$ be a lattice. If we define a relation $\leqslant$ on L, by $x \leqslant y$ if and only if $x = x \wedge y$, or equivalently $x \vee y = y$. Then $(L, \leqslant)$ is a lattice ordered set.

Important Note: Theorems 2.1. and 2.2. together imply that the concepts of lattice and lattice ordered set are the same. We refer to it as a lattice in future.

Definition 2.3. Let L be a lattice. Then L is said to be a bounded lattice if L is bounded as a poset.

Definition 2.4. A bounded lattice L with bounds 0 and 1 is said to be complemented if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

Theorem 2.3. In any lattice $(L, \vee, \wedge)$ the following are equivalent:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(2) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
(3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
(4) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.5. A lattice L is called a distributive lattice if it satisfies any one of the four conditions in the Theorem 2.3.

Definition 2.6. A complemented distributive lattice is called a Boolean algebra.

Definition 2.7. A lattice L is called a complete lattice if every nonempty subset of L has both $l.u.b$ and $g.l.b$.

Theorem 2.4. *If P is a partial ordered set bounded above each of whose non-void subset R has an infimum, then each non-void subset of P will have a supremum, too, and by the definitions $\bigcap R = inf\, R$, $\bigcup R = sup\, R$, then P becomes a complete lattice.*

Theorem 2.5. *Let L be a lattice. Then for any $x, y, z \in L$, the following conditions are equivalent:*

(1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(3) $(x \vee y) \wedge z \leqslant x \vee (y \wedge z)$.

Definition 2.8. An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an AL with 0 if, for any $a, b, c \in L$, it satisfies the following conditions:

(1) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$,
(2) $(a \vee b) \wedge c = (b \vee a) \wedge c$,
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
(4) $(a \vee b) \vee c = a \vee (b \vee c)$,
(5) $a \wedge (a \vee b) = a$,
(6) $a \vee (a \wedge b) = a$,
(7) $(a \wedge b) \vee b = b$,
(8) $0 \wedge a = 0$.

It can be easily seen that $a \wedge b = a$ if and only if, $a \vee b = b$ in an AL.

Definition 2.9. Let L be an AL and $a, b \in L$. An element a is less than or equal to b and write $a \leqslant b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$.

Theorem 2.6. *The relation $\leqslant$ is a partial ordering on an AL L and hence $(L, \leqslant)$ is a poset.*

Definition 2.10. Let L be any nonempty set. If we define $x \vee y = x = y \wedge x$ for any $x, y \in L$, then clearly L is an AL and it is called a discrete AL.

Definition 2.11. Let L and L' be two ALs with zero elements 0 and $0'$ respectively. Then a mapping $f : L \to L'$ is called a homomorphism if it satisfies the following conditions.

(1) $f(a \vee b) = f(a) \vee f(b)$,
(2) $f(a \wedge b) = f(a) \wedge f(b)$,
(3) $f(0) = 0'$.

Definition 2.12. Let L be an AL. Then a nonempty subset I of L is said to be an ideal of L if it satisfies the following conditions:

(1) If $x, y \in I$, then there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$,
(2) If $x \in I$ and $a \in L$, then $x \wedge a \in I$.

LEMMA 2.1. *Let L be an AL and I be an ideal of L. Then the following are equivalent:*

(1) *$x, y \in I$ implies $x \vee y \in I$.*
(2) *$x, y \in I$ implies there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.*

COROLLARY 2.1. *Let L be an AL and $a \in L$. Then $(a] = \{a \wedge x | \ x \in L\}$ is an ideal of L and it is called principal ideal generated by a.*

DEFINITION 2.13. Let L be an AL. Then a nonempty subset F of L is said to be a filter if it satisfies the following:

(1) $x, \ y \in F$, implies $x \wedge y \in F$.
(2) $x \in F$ *and* $a \in L$, implies $a \vee x \in F$.

COROLLARY 2.2. *Let L be an AL and $a, \ b \in L$. Then $a \in (b]$ if and only if $a = b \wedge a$.*

COROLLARY 2.3. *Let L be an AL and $a, \ b \in L$. Then $(a \wedge b] = (b \wedge a]$.*

THEOREM 2.7. *Let L be an AL. Then the set $\mathcal{I}(L)$ of all ideals of L forms a lattice under set inclusion in which the glb and lub for any $I, J \in \mathcal{I}(L)$ are respectively $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : (a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J\}$.*

THEOREM 2.8. *Let L be an AL. Then the set $P\mathcal{I}(L)$ of all principal ideals of L is a sublattice of the lattice $\mathcal{I}(L)$ of all ideals of L.*

DEFINITION 2.14. Let L be an AL. Then for any ideal I in L, define $I^e = \{(a] : a \in I\}$.

LEMMA 2.2. *Let L be an AL and $I, \ J$ be ideals of L. Then we have the following.*

(1) *I^e is an ideal of the lattice $\mathcal{PI}(L)$,*
(2) *$I \subseteq J \Leftrightarrow I^e \subseteq J^e$,*
(3) *$(I \vee J)^e = I^e \vee J^e$,*
(4) *$(I \cap J)^e = I^e \cap J^e$,*
(5) *I is prime $\Leftrightarrow I^e$ is prime.*

DEFINITION 2.15. Let L be an AL with 0. Then for any nonempty subset A of L, define $A^* = \{x \in L : x \wedge a = 0$ for all $a \in A\}$. Here, A^* is called an annihilator of A in L.

LEMMA 2.3. *Let L and L' be two ALs with zero elements 0 and $0'$ respectively and $f : L \to L'$ a homomorphism. Then we have the following:*

(1) *If f is onto, then for any ideal I of L, $f(I)$ is an ideal of L'.*
(2) *If J is an ideal of L', then $f^{-1}(J)$ is an ideal of L.*
(3) *$ker(f)$ is an ideal of L.*

Corollary 2.4. *Let L be an AL with* 0. *Then for any ideals I, J of L. we have the following:*

(1) $I \cap I^* = \{0\}$,
(2) $I^* = \bigcap_{a \in I} (a]^*$,
(3) $(I \cap J)^* = (J \cap I)^*$,
(4) $I \subseteq J \Rightarrow J^* \subseteq I^*$,
(5) $I^* \cap J^* \subseteq (I \cap J)^*$,
(6) $I \subseteq I^{**}$,
(7) $I^{***} = I^*$,
(8) $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq I^{**}$,
(9) $(I \cup J)^* = I^* \cap J^* = (J \cup I)^*$.

Corollary 2.5. *Let L be an AL with* 0. *Then for any ideals I, J of L, we have the following:*

(1) $(I \cap J)^{**} = I^{**} \cap J^{**}$,
(2) $I \cap J = (0] \Leftrightarrow I \subseteq J^* \Leftrightarrow J \subseteq I^*$.

Corollary 2.6. *Let L be an AL with* 0. *If $\{I_i : i \in \Delta\}$ is a family of ideals of L, then $(\bigcap_{i \in \Delta} I_i)^{**} = \bigcap_{i \in \Delta} (I_i)^{**}$.*

Theorem 2.9. *Let L be an AL with* 0. *Then for any $x, y \in L$, we have the following:*

(1) $(x] \cap [x]^* = (0]$,
(2) $[x]^* \cap [x]^{**} = (0]$,
(3) $(x]^* = [x]^*$,
(4) $(x]^* \cap [x]^{**} = (0]$,
(5) $x \leqslant y \Rightarrow [y]^* \subseteq [x]^*$,
(6) $[x \wedge y]^* = [y \wedge x]^*$,
(7) $[x \vee y]^* = [y \vee x]^*$,
(8) $(x] \subseteq [x]^{**}$,
(9) $[x]^{***} = [x]^*$,
(10) $[x]^* \subseteq [y]^* \Leftrightarrow [y]^{**} \subseteq [x]^{**}$,
(11) $[x \wedge y]^{**} = [x]^{**} \cap [y]^{**}$.

Corollary 2.7. *Let L be an AL with* 0. *Then for any $0 \neq x \in L$, there exists a maximal filter F of L such that $x \in F$.*

Definition 2.16. Let L and L' be two ALs with 0 and $0'$ respectively. Then a homomorphism $f : L \to L'$ is called annihilator preserving if it satisfies $f(A^*) = (f(A))^*$, for any $\{0\} \subset A \subset L$.

Definition 2.17. Let L be an AL with 0. Then L is said to be 0-distributive if for any $a, b, c \in L$, $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$.

Lemma 2.4. *Let L be a* 0*-distributive AL. Then for any $x, y, z \in L$,*

$$[(x \vee y) \wedge z]^* = [(x \wedge z) \vee (y \wedge z)]^*.$$

3. α-Ideals in 0-Distributive Almost Lattices

In this section, we introduce the concept of an α-ideal in a 0-distributive AL L and give certain examples of α-ideals. For any ideal I of an AL L with 0, we prove that $\bar{I} = \{x \in L : (a]^* \subseteq (x]^* \text{ for some } a \in I\}$ is the smallest α-ideal containing I and also we prove some basic properties of $\bar{I}$. We prove that the set $\mathcal{I}_\alpha(L)$ of all α-ideals of L forms a complete distributive lattice. We derive a set of identities for an ideal in a 0-distributive AL to become an α-ideal. Also, we obtain a necessary and sufficient condition for an ideal in a 0-distributive AL to become an α-ideal. Finally, in this section we characterise the $*$-0-DAL in terms of α-ideals. First, we begin this section with the following definition.

Definition 3.1. Let L be a 0-distributive AL and I be an ideal of L. Then I is called an α-ideal of L, if $(x]^{**} \subseteq I$ for all $x \in I$.

Note that we denote the set of all α-ideals in an AL L by $\mathcal{I}_\alpha(L)$. In the following, we give certain examples of an α-ideals.

Example 3.1. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Write $L = A \times B = \{(0,0), (0,b_1), (0,b_2), (a,0), (a,b_1), (a,b_2)\}$. Then clearly $(L, \vee, \wedge, 0')$ is an AL with zero under point-wise operations, where $0' = (0,0)$. Now, put $I = \{(0,0), (0,b_1), (0,b_2)\}$. Then clearly I is an ideal of L. Also, clearly $((0,0)]^{**} = \{(0,0)\}$ and $((0,b_1)]^{**} = ((0,b_2)]^{**} = \{(0,0), (0,b_1), (0,b_2)\} \subseteq I$. Thus I is an α-ideal of L.

Example 3.2. Let $L = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

and

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then clearly $(L, \vee, \wedge, 0)$ is a 0-distributive AL. Now, put $I = \{0, a\}$. Then clearly I is an ideal in L. Also, clearly $(0]^{**} = \{0\}$ and $(a]^{**} = \{0, a\} \subseteq I$. Thus I is an α-ideal of L.

Next, we prove some basic properties of α-ideals in 0-distributive ALs.

Theorem 3.1. *Let L be a 0-distributive AL and let S be a nonempty subset of L which is closed under the operation $\wedge$. Then the set $I = \{x \in L : x \wedge y = 0$ for some $y \in S\}$ is an α-ideal of L.*

Proof. Clearly $0 \in I$ and hence I is nonempty. Let $a, b \in I$. Then $a \wedge r = 0$ and $b \wedge s = 0$ for some $r, s \in S$. This implies $a \wedge (r \wedge s) = 0$ and $b \wedge (r \wedge s) = 0$. It follows that $(a \vee b) \wedge (r \wedge s) = 0$, we get $a \vee b \in I$. Again, let $x \in I$ and $r \in L$. Then $x \wedge s = 0$ for some $s \in S$. Now, consider $(x \wedge r) \wedge s = (r \wedge x) \wedge s = r \wedge (x \wedge s) = r \wedge 0 = 0$. Therefore $x \wedge r \in I$. Hence I is an ideal of L. Clearly, $I = \bigcup_{x \in S} (x]^*$. Let $a \in I$. Then $a \in (t]^*$ for some $t \in S$. This implies $(a] \subseteq (t]^*$. Thus $(a]^{**} \subseteq (t]^{***} = (t]^*$. Hence $(a]^{**} \subseteq \bigcup_{x \in S} (x]^* = I$. Therefore I is an α-ideal of L. □

Corollary 3.1. *Let L be a 0-distributive AL and let F be a filter of L. Then the set $I = \{x \in L : x \wedge y = 0 \text{ for some } y \in F\}$ is an α-ideal of L.*

Theorem 3.2. *The set union of any chain of α-ideals of a 0-distributive AL L is itself an α-ideal in L.*

Proof. Suppose $\{I_i\}_{i\in\Delta}$ be an arbitrary family of α-ideals of L. Now, put $I = \bigcup_{i\in\Delta} I_i$. Then clearly I is an ideal of L. Now, we shall prove I is an α-ideal of L. Suppose $x \in I = \bigcup_{i\in\Delta} I_i$. Then $x \in I_i$ for some $i \in \Delta$. This implies $(x]^{**} \subseteq I_i$, since each I_i is an α-ideal. It follows that $(x]^{**} \subseteq \bigcup_{i\in\Delta} I_i = I$. Therefore I is an α-ideal of L. □

We now define an extension of an ideal I of a 0-distributive AL, which leads to a useful characterization of α-ideals.

Definition 3.2. Let L be 0-distributive AL. Then for any ideal I of L, define $\bar{I} = \{x \in L : (a]^* \subseteq (x]^* \text{ for some } a \in I\}$

Theorem 3.3. *Let L be a 0-distributive AL. Then for any ideal I of L, $\bar{I}$ is an ideal of L.*

Proof. By the definition of $\bar{I}$, it is clear that $I \subseteq \bar{I}$. Therefore $\bar{I}$ is a nonempty subset of L. Let $x, y \in \bar{I}$. Then $(a]^* \subseteq (x]^*$ and $(b]^* \subseteq (y]^*$ for some $a, b \in I$. Hence $(a]^* \cap (b]^* \subseteq (x]^* \cap (y]^*$. This implies $(a \vee b]^* \subseteq (x \vee y]^*$ and $a \vee b \in I$. Hence $x \vee y \in \bar{I}$. Again, let $x \in \bar{I}$ and $r \in L$. Then $(a]^* \subseteq (x]^*$ for some $a \in I$. This implies $(x]^{**} \subseteq (a]^{**}$. Since $x \in (x]^{**} \subseteq (a]^{**}$, $x \in (a]^{**}$ and $r \in (r]^{**}$. It follows that $x \wedge r \in (a]^{**} \cap (r]^{**} = (a \wedge r]^{**}$. This implies $(x \wedge r] \subseteq (a \wedge r]^{**}$. It follows that $(a \wedge r]^* \subseteq (x \wedge r]^*$ and we have $a \wedge r \in I$. Thus $x \wedge r \in \bar{I}$. Therefore $\bar{I}$ is an ideal of L. □

Theorem 3.4. *Let L be a 0-distributive AL. Then for any ideals I, J of L, we have the following.*

(1) $I \subseteq \bar{I}$,
(2) $I \subseteq J \Rightarrow \bar{I} \subseteq \bar{J}$,
(3) $\overline{I \cap J} = \bar{I} \cap \bar{J}$,
(4) $\bar{I} \vee \bar{J} \subseteq \overline{I \vee J}$,
(5) $\overline{(\bar{I})} = \bar{I}$,
(6) $\overline{(I^*)} = I^*$.

Proof. Proof (1) is clear.

(2) Suppose $I \subseteq J$ and suppose $x \in \bar{I}$. Then $(a]^* \subseteq (x]^*$ for some $a \in I$. Hence $x \in \bar{J}$, since $a \in I \subseteq J$. Therefore $\bar{I} \subseteq \bar{J}$

(3) Clearly, $\overline{(I \cap J)} \subseteq \bar{I} \cap \bar{J}$. Conversely, suppose $x \in \bar{I} \cap \bar{J}$. Then $x \in \bar{I}$ and $x \in \bar{J}$. This implies $(a]^* \subseteq (x]^*$ and $(b]^* \subseteq (x]^*$, where $a \in I$ and $b \in J$. It follows that $(x]^{**} \subseteq (a]^{**}$ and $(x]^{**} \subseteq (b]^{**}$. Hence $(x]^{**} \subseteq (a]^{**} \cap (b]^{**} = (a \wedge b]^{**}$. This implies $(a \wedge b]^* \subseteq (x]^*$ and $a \wedge b \in I \cap J$. Thus $x \in \overline{I \cap J}$. Hence $\bar{I} \cap \bar{J} \subseteq \overline{I \vee J}$. Therefore $\overline{I \cap J} = \bar{I} \cap \bar{J}$.

Proof (4) is clear.

(5) We have $I \subseteq \bar{I}$. Therefore by condition (2), we get $\bar{I} \subseteq \overline{(\bar{I})}$. Conversely, suppose $x \in \overline{(\bar{I})}$. Then $(a]^* \subseteq (x]^*$, where $a \in \bar{I}$. Again, since $a \in \bar{I}$, $(b]^* \subseteq (a]^*$ for some $b \in I$. It follows that $(b]^* \subseteq (x]^*$ and $b \in I$. Thus $x \in \bar{I}$. Hence $\overline{(\bar{I})} \subseteq \bar{I}$. Therefore $\overline{(\bar{I})} = \bar{I}$.

(6) Clearly $I^* \subseteq \overline{(I^*)}$. Suppose $x \in \overline{(I^*)}$. Then $(a]^* \subseteq (x]^*$ for some $a \in I^*$. This implies $(x]^{**} \subseteq (a]^{**}$. Since $a \in I^*$, we get $(a] \subseteq I^*$. It follows that $(a]^{**} \subseteq I^*$. Hence $x \in (x]^{**} \subseteq (a]^{**} \subseteq I^*$. Thus $x \in I^*$. Hence $\overline{(I^*)} \subseteq I^*$. Therefore $\overline{(I^*)} = I^*$. □

Corollary 3.2. *Let L be a 0-distributive AL and $\{I_i\}_{i\in\Delta}$ be a family of α-ideals in L. Then* $\overline{(\bigcap_{i\in\Delta} I_i)} = \bigcap_{i\in\Delta} \bar{I}_i = \bigcap_{i\in\Delta} I_i$.

In the following we characterise an ideal $\bar{I}$ in 0-distributive AL.

Theorem 3.5. *Let L be a 0-distributive AL and I be an ideal of L. Then $\bar{I}$ is the smallest α-ideal containing I.*

Proof. Clearly, $\bar{I}$ is an ideal of L containing I. Suppose $x \in \bar{I}$. Then $(a]^* \subseteq (x]^*$ for some $a \in I$. Let $t \in (x]^{**}$. Then we have $(x]^* \subseteq (t]^*$. Thus $(a]^* \subseteq (x]^* \subseteq (t]^*$. It follows that $t \in \bar{I}$, since $a \in I$. Thus $(x]^{**} \subseteq \bar{I}$. Therefore $\bar{I}$ is an α-ideal containing I. Suppose K is an α-ideal in L such that $I \subseteq K$. Now, let $x \in \bar{I}$. Then $(a]^* \subseteq (x]^*$ for some $a \in I$, since K is an α-ideal containing I, $(a]^{**} \subseteq K$. Now, since $t \in (x]^{**} \subseteq (a]^{**} \subseteq K$, $t \in K$. Hence $\bar{I} \subseteq K$. Therefore $\bar{I}$ is the smallest α-ideal containing I. □

Next, we prove that the set $\mathcal{I}_\alpha(L)$ of all α-ideals in a 0-distributive AL is a complete distributive lattice. For, this first we need the following lemma whose proof is straight forward.

Lemma 3.1. *Let L be a 0-distributive AL and I be an ideal of L. Then the following are equivalent:*

(1) *I is an α-ideal,*
(2) *$I = \bar{I}$.*

Recall that the set $\mathcal{I}(L)$ of all ideals in an AL L is lattice. But, in the following we give an example of an AL in which $\mathcal{I}(L)$ is not a distributive lattice.

Example 3.3. Let $L = \{0, a, b, c, d\}$ and define $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	d	d	d
b	b	d	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

and

$\wedge$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
d	0	a	b	c	d

Then clearly $(L, \vee, \wedge, 0)$ is an AL. Now, Put $I = \{0, a\}$, $J = \{0, b\}$ and $K = \{0, c\}$. Then clearly I, J and K are ideals of L. Also, clearly $J \cap K = \{0\}$ and hence $I \vee (J \cap K) = \{0, a\}$. On the other hand $I \vee J = \{0, a, b, c, 1\} = L$ and $I \vee K = \{0, a, b, c, 1\} = L$. Therefore $(I \vee J) \cap (I \vee K) = L$. Thus $I \vee (J \cap K) \neq (I \vee J) \cap (I \vee K)$. Therefore the lattice $\mathcal{I}(L)$ is not a distributive lattice.

Again, in the following we give an example of an AL L in which the set $\mathcal{I}_\alpha(L)$ of all α-ideal in L is not a sublattice of the lattice $\mathcal{I}(L)$.

Example 3.4. Let $L = \{0, a, b, c, d\}$ and define $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	c	c	d
b	b	c	b	c	d
c	c	c	c	c	d
d	d	d	d	d	d

and

$\wedge$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
d	0	a	b	c	d

Then clearly $(L, \vee, \wedge, 0)$ is a 0-distributive AL. Now, put $I = \{0, a\}$ and $J = \{0, b\}$. Then clearly I and J are α-ideals of L. But, we have $I \vee J = \{0, a, b, c\}$, which is not an α-ideal, since $(c]^{**} = L \nsubseteq I \vee J$. Thus $I \vee J$ is not an α-ideal in L. Therefore $\mathcal{I}_\alpha(L)$ is not a sublattice of $\mathcal{I}(L)$.

However, in the following we prove that if L is a 0-distributive AL then the set $\mathcal{I}_\alpha(L)$ of α-ideals in L is a distributive lattice on its own. For this, first we need the following lemma whose proof is straight forward.

Lemma 3.2. *Let L be an AL with 0. Then for any $I, J \in \mathcal{I}(L)$, $I^{**} \vee J^{**} \subseteq (I \vee J)^{**}$*

But, the converse of the lemma 3.2 is not true. For, in example 3.4, we have $I^{**} = I$, $J^{**} = J$ and hence $I^{**} \vee J^{**} = \{0, a, b, c\}$. On the other hand, we have $(I \vee J)^{*} = \{0\}$ and hence $(I \vee J)^{**} = L$. Therefore $I^{**} \vee J^{**} \neq (I \vee J)^{**}$.

Now, we prove the following theorem.

Theorem 3.6. *Let L be a 0-distributive AL in which $I^{**} \vee J^{**} = (I \vee J)^{**}$ for every $I, J \in \mathcal{I}_\alpha(L)$. Then the set $\mathcal{I}_\alpha(L)$ is a distributive lattice with respect to set inclusion, where for any $I, J \in \mathcal{I}_\alpha(L)$, g.l.b.$(I, J) = I \cap J$ and l.u.b.$(I, J) = I \bar{\vee} J = \overline{I \vee J}$.*

Proof. Clearly, $\mathcal{I}_\alpha(L)$ is a poset with respect to set inclusion. Let $I, J \in \mathcal{I}_\alpha(L)$. Then clearly, $I \cap J$ is an α-ideal and hence $I \cap J \in \mathcal{I}_\alpha(L)$. Also, clearly $I \cap J$ is the *g.l.b.* of I and J. Now, we shall prove $I \bar{\vee} J$ is the *l.u.b.* of I and J in $\mathcal{I}_\alpha(L)$. Now, by Theorem 3.5, we get $\overline{I \vee J}$ is the smallest α-ideal containing $I \vee J$ and hence $I \bar{\vee} J = \overline{I \vee J} \in \mathcal{I}_\alpha(L)$. Since $I, J \subseteq I \vee J$, $\bar{I}, \bar{J} \subseteq \overline{I \vee J} = I \bar{\vee} J$. It follows that $I \bar{\vee} J$ is an upper bound of I and J. Suppose $K \in \mathcal{I}_\alpha(L)$ such that K is an upper bound of I and J. Then $I \subseteq K$ and $J \subseteq K$. This implies $I \vee J \subseteq K$. Hence $\overline{I \vee J} \subseteq \bar{K} = K$. Thus $I \bar{\vee} J$ is the *l.u.b.* of I and J. Therefore $\mathcal{I}_\alpha(L)$ is a lattice.

Now, we shall prove $\mathcal{I}_\alpha(L)$ is a distributive lattice. Let $I, J, K \in \mathcal{I}_\alpha(L)$. Since $\mathcal{I}_\alpha(L)$ is a lattice, we have $I\bar{\vee}(J \cap K) \subseteq (I\bar{\vee}J) \cap (I\bar{\vee}K)$. Conversely, suppose $x \in (I\bar{\vee}J) \cap (I\bar{\vee}K)$. Then $x \in \overline{(I \vee J)} \cap \overline{(I \vee K)} = \overline{(I \vee J) \cap (I \vee K)}$. It follows that $[a]^* \subseteq [x]^*$ for some $a \in (I \vee J) \cap (I \vee K)$. Hence $a \in I \vee J$ and $a \in I \vee K$. This implies $(i_1 \vee j) \wedge a = a$ for some $i_1 \in I$, $j \in J$ and $(i_2 \vee k) \wedge a = a$ for some $i_2 \in I$, $k \in K$. Now, since $[a]^* \subseteq [x]^*$, $[x]^{**} \subseteq [a]^{**}$. Now, we have $[(i_1 \vee j) \wedge a]^{**} = [a]^{**}$. It follows that $[i_1 \vee j]^{**} \cap [a]^{**} = [a]^{**}$. Thus $[a]^{**} \subseteq [i_1 \vee j]^{**}$. Similarly, we get $[a]^{**} \subseteq [i_2 \vee k]^{**}$. Therefore $[a]^{**} \subseteq [i_1 \vee j]^{**} \cap [i_2 \vee k]^{**}$. This implies $[x]^{**} \subseteq [i_1 \vee j]^{**} \cap [i_2 \vee k]^{**} = [(i_1 \vee j) \wedge (i_2 \vee k)]^{**} = [((i_1 \vee j) \wedge i_2) \vee ((i_1 \vee j) \wedge k)]^{**} = (((i_1 \vee j) \wedge i_2) \vee ((i_1 \vee j) \wedge k)]^{**} = (((i_1 \vee j) \wedge i_2] \vee ((i_1 \vee j) \wedge k])^{**} = ((i_1 \vee j) \wedge i_2]^{**} \vee ((i_1 \vee j) \wedge k]^{**} = ((i_1 \wedge i_2) \vee (j \wedge i_2)]^{**} \vee ((i_1 \wedge k) \vee (j \wedge k)]^{**} = (((i_1 \wedge i_2) \vee (j \wedge i_2)] \vee ((i_1 \wedge k) \vee (j \wedge k)])^{**} = (((i_1 \wedge i_2) \vee (j \wedge i_2) \vee (i_1 \wedge k)) \vee (j \wedge k)])^{**} = [t \vee (j \wedge k)]^{**}$, where $t = (i_1 \wedge i_2) \vee (j \wedge i_2) \vee (i_1 \wedge k) \in I$. Thus $[x]^{**} \subseteq [t \vee (j \wedge k)]^{**}$. Hence $[t \vee (j \wedge k)]^* \subseteq [x]^*$, where $t \vee (j \wedge k) \in I \vee (J \cap K)$. Thus $x \in \overline{I \vee (J \cap K)} = I\bar{\vee}(J \cap K)$. Therefore $(I\bar{\vee}J) \cap (I\bar{\vee}K) \subseteq I\bar{\vee}(J \cap K)$. Hence $I\bar{\vee}(J \cap K) = (I\bar{\vee}J) \cap (I\bar{\vee}K)$. Therefore $\mathcal{I}_\alpha(L)$ is a distributive lattice. □

In view of theorem 3.6 and corollary 3.2, we have the following.

THEOREM 3.7. *Let L be a 0-distributive AL. Then the set $\mathcal{I}_\alpha(L)$ of all α-ideals of L forms a complete distributive lattice, ordered by set inclusion.*

Recall that the intersection of all minimal prime ideals in a 0-distributive AL is $\{0\}$. In the following we derive set of identities of an ideal in a 0-distributive AL to become an α-ideal. For, this first we need the following lemma.

LEMMA 3.3. *Let L be a 0-distributive AL. Then for any $x \in L$, $(x]^* = \bigcap_{P \in M_x} P$.*

PROOF. Suppose P is a minimal prime ideal of L such that $x \notin P$. Suppose $t \in (x]^*$. Then $t \wedge x = 0 \in P$. Hence, we get $t \in P$. Therefore $(x]^* \subseteq \bigcap_{P \in M_x} P$.

Conversely, suppose $t \notin (x]^*$. Then $t \wedge x \neq 0$. Then there exists a minimal prime ideal (say) P such that $t \wedge x \notin P$. This implies $t \notin P$ and $x \notin P$. Hence $t \notin \bigcap_{P \in M_x} P$. Thus $\bigcap_{P \in M_x} P \subseteq (x]^*$. Therefore $(x]^* = \bigcap_{P \in M_x} P$ □

Now, we prove the following theorem.

THEOREM 3.8. *Let L be a 0-distributive AL. Then for any ideal I of L, the following are equivalent:*

(1) *I is an α-ideal,*
(2) *$I = \bar{I}$,*
(3) *For any $x, y \in L$, $[x]^* = [y]^*$ and $x \in I$ imply $y \in I$,*
(4) *$I = \bigcup_{x \in I} [x]^{**}$,*
(5) *For any $x, y \in L$, $h(x) = h(y)$ and $x \in I$ imply $y \in I$.*

PROOF. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): Assume (2). Let $x, y \in L$ such that $[x]^* = [y]^*$ and $x \in I$. Then by condition (2), we get $x \in \bar{I}$. Therefore $[a]^* \subseteq [x]^*$ for some $a \in I$. Hence $[a]^* \subseteq [y]^*$ and $a \in I$. Therefore $y \in \bar{I} = I$. Thus $y \in I$.

(3) $\Rightarrow$ (4): Assume (3). Clearly $I \subseteq \bigcup_{x \in I} [x]^{**}$. Conversely, suppose $y \in \bigcup_{x \in I} [x]^{**}$. Then $y \in [x]^{**}$ for some $x \in I$. It follows that $[y]^{**} \subseteq [x]^{**}$. Thus $[y]^{**} = [x]^{**} \cap [y]^{**} = [x \wedge y]^{**}$ and $x \in I$. Therefore by condition (3), we get $y \in I$. Hence $\bigcup_{x \in I} [x]^{**} \subseteq I$. Therefore $I = \bigcup_{x \in I} [x]^{**}$.

(4) $\Rightarrow$ (1) is clear, since by the definition of α-ideal. (3) $\Leftrightarrow$ (5) also clear, since $h(x) = h(y) \Leftrightarrow [x]^* = [y]^*$. □

Recall that if L is a 0-distributive AL then for any $(a] \in \mathcal{PI}(L)$, $\{(a]\}^* = \{(x] \in \mathcal{PI}(L) : (a] \cap (x] = (0]\}$ is an ideal of the lattice $\mathcal{PI}(L)$. In the following we derive a necessary and sufficient condition for an ideal in a 0-distributive AL to become an α-ideal. For this, first we need the following.

Lemma 3.4. *Let L be a 0-distributive AL. Then for any $a, b \in L$, we have the following.*

(1) $x \in (a]^* \Leftrightarrow (x] \in \{(a]\}^*$,
(2) $(a]^* = (b]^* \Leftrightarrow \{(a]\}^* = \{(b]\}^*$.

Proof. We have $x \in (a]^* \Leftrightarrow x \wedge a = 0 \Leftrightarrow (x \wedge a] = (0] \Leftrightarrow (x] \cap (a] = (0] \Leftrightarrow (x] \in \{(a]\}^*$. Thus $x \in (a]^* \Leftrightarrow (x] \in \{(a]\}^*$.

Suppose $(a]^* = (b]^*$. Then $(x] \in \{(a]\}^* \Leftrightarrow (x] \cap (a] = (0] \Leftrightarrow (x \wedge a] = (0] \Leftrightarrow x \wedge a = 0 \Leftrightarrow x \wedge b = 0 \Leftrightarrow (x \wedge b] = (0] \Leftrightarrow (x] \cap (b] = (0] \Leftrightarrow (x] \in \{(b]\}^*$. Therefore $\{(a]\}^* = \{(b]\}^*$. Conversely, suppose $\{(a]\}^* = \{(b]\}^*$. Then $x \in (a]^* \Leftrightarrow x \wedge a = 0 \Leftrightarrow (x \wedge a] = (0] \Leftrightarrow (x] \cap (a] = (0] \Leftrightarrow (x] \in \{(a]\}^* \Leftrightarrow (x] \in \{(b]\}^* \Leftrightarrow (x] \cap (b] = (0] \Leftrightarrow (x \wedge b] = (0] \Leftrightarrow x \wedge b = 0 \Leftrightarrow x \in (b]^*$. Therefore $(a]^* = (b]^*$ □

Now, we prove the following theorem.

Theorem 3.9. *Let L be a 0-distributive AL and I be an ideal of L. Then I is an α-ideal in L if and only if I^e is an α-ideal in $\mathcal{PI}(L)$.*

Proof. Suppose I is an α-ideal in L. Clearly, I^e is an ideal in $\mathcal{PI}(L)$. Let $(a], (b] \in \mathcal{PI}(L)$ such that $\{(a]\}^* = \{(b]\}^*$ and $(a] \in I^e$. This implies $(a] = (t]$ for some $t \in I$. It follows that $a \in I$. Since $\{(a]\}^* = \{(b]\}^*$, by lemma 3.4, $(a]^* = (b]^*$. Again, since I is an α-ideal of L, we get $b \in I$. Hence $(b] \in I^e$. Therefore I^e is an α-ideal in $\mathcal{PI}(L)$.

Conversely, suppose I^e is an α-ideal in $\mathcal{PI}(L)$. Let $a, b \in L$ such that $(a]^* = (b]^*$ and $a \in I$. Then $\{(a]\}^* = \{(b]\}^*$ and $(a] \in I^e$. Since I^e is an α-ideal in $\mathcal{PI}(L)$, we get $(b] \in I^e$. It follows that $b \in I$. Therefore I is an α-ideal in L. □

Finally, in this section we derive sufficient condition for a 0-distributive AL to become a $*$-0-DAL in terms of α-ideals.

Theorem 3.10. *Let L be a 0-distributive AL. If every α-ideal in L is a principal ideal, then L is a $*$-0-DAL.*

Proof. Suppose $x \in L$. Then for any $a \in (x]^*$, we get $(a]^{**} \subseteq (x]^*$. Therefore $(x]^*$ is an α-ideal of L. Thus by the hypothesis, we get $(x]^* = (a]$ for some $a \in L$. Thus $(x]^{**} = (a]^*$. Therefore L is a $*$-0-DAL. □

4. Annihilator Preserving Epimorphisms

In this section, we prove that the image of an α-ideal under an annihilator preserving epimorphism is again an α-ideal. We derive a necessary and sufficient condition for the contraction of an α-ideal is an α-ideal. For this, first we need the following lemma.

Lemma 4.1. *Let L and L' be two ALs with 0 and $0'$ respectively and let $f : L \to L'$ be a homomorphism. Then for any $a \in L$, $f((a]) \subseteq (f(a)]$. Moreover, if f is onto, then $f((a]) = (f(a)]$.*

Proof. Let $a \in L$ and let $t \in f((a])$. Then $t = f(x)$ for some $x \in (a]$. It follows that $t = f(x) = f(a \wedge x) = f(a) \wedge f(x)$. Hence $t \in (f(a)]$. Therefore $f((a]) \subseteq (f(a)]$. Now, suppose f is onto. Let $t \in (f(a)]$. Then $t = f(a) \wedge s$ for some $s \in L'$. Since f is onto, there exists $x \in L$ such that $f(x) = s$. Therefore $t = f(a) \wedge s = f(a) \wedge f(x) = f(a \wedge x)$. Now, we have $a \wedge x \in (a]$ and hence $f(a \wedge x) \in (f(a)]$. Therefore $t \in (f(a)]$. Thus $(f(a)] \subseteq f((a])$. Thus $f((a]) = (f(a)]$. □

Theorem 4.1. *Let L and L' be two 0-distributive ALs and let $f : L \to L'$ be an annihilator preserving epimorphism. If I is an α-ideal in L, then $f(I)$ is an α-ideal in L'.*

Proof. Suppose I is an α-ideal in L. Then by lemma 2.3, we get $f(I)$ is an ideal in L'. Now, let $x \in f(I)$. Then $x = f(a)$, for some $a \in I$. Now, since I is an α-ideal and $a \in I$, $(a]^{**} \subseteq I$. Therefore $f((a]^{**}) \subseteq f(I)$. It follows that $(f((a]))^{**} \subseteq f(I)$. Hence $(f(a)]^{**} \subseteq f(I)$. Thus $(x]^{**} \subseteq f(I)$. Therefore $f(I)$ is an α-ideal in L'. □

Definition 4.1. Let L and L' be two ALs and let $f : L \to L'$ is a homomorphism. Then for any ideal I of L', $f^{-1}(I)$ is called the contraction of I.

It can be easily seen that the contraction of an ideal is an ideal. But, the contraction of an α-ideal need not be an α-ideal. For, consider the following example.

Example 4.1. Let $L = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on L as follows.

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

and

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly $(L, \vee, \wedge, 0)$ is an AL with 0. Again, let $A = \{0, a'\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Write

$$L' = A \times B = \{(0,0), (0,b_1), (0,b_2), (a',0), (a',b_1), (a',b_2)\}.$$

Then clearly, L' is an AL with zero element $0' = (0,0)$, under point-wise operations. Now, define a mapping $f : L \to L'$ as follows. $f(0) = 0'$, $f(c) = (a', 0)$, $f(a) = (a', b_1)$, $f(b) = (a', b_2)$. Then clearly f is a homomorphism from L onto L'. Now, put $J = \{(0,0), (a', 0)\}$. Then, clearly J is an α-ideal of L', but, $f^{-1}(J) = \{0, c\}$ is not an α-ideal in L, since $(c]^{**} = (0]^* = L \not\subseteq f^{-1}(J)$.

It can be easily seen that for every a in a 0-distributive AL, $[a]^*$ is an α-ideal. In the following we give necessary and sufficient condition for a contraction of an α-ideal is an α-ideal.

THEOREM 4.2. *Let L and L' be two* 0*-distributive ALs and let $f : L \to L'$ be a homomorphism. Then contraction of every α-ideal is an α-ideal if and only if contraction of $[a]^*$ is an α-ideal for every $a \in L'$.*

PROOF. Suppose contraction of an α-ideal is an α-ideal and suppose $a \in L'$. Then we have $[a]^*$ is an α-ideal. Therefore by assumption, $f^{-1}([a]^*)$ is an α-ideal. Thus contraction of $[a]^*$ is an α-ideal.

Conversely, assume the condition. Suppose J is an α-ideal of L'. Now, we shall prove that $f^{-1}(J)$ is an α-ideal of L. Let $x, y \in L$ such that $[x]^* = [y]^*$ and $x \in f^{-1}(J)$. First we prove that $[f(x)]^* = [f(y)]^*$. Let $t \in [f(x)]^*$. Then $t \wedge f(x) = 0$. This implies $f(x) \in [t]^*$. Hence $x \in f^{-1}([t]^*)$. Since $f^{-1}([t]^*)$ is an α-ideal(by assumption), $y \in f^{-1}([t]^*)$. Therefore $f(y) \in [t]^*$. This implies $f(y) \wedge t = 0$. Hence $t \in [f(y)]^*$. Thus $[f(x)]^* \subseteq [f(y)]^*$. Similarly, we can prove that $[f(y)]^* \subseteq [f(x)]^*$. Therefore $[f(x)]^* = [f(y)]^*$ and we have $f(x) \in J$. It follows that $f(y) \in J$, since J is an α-ideal. Therefore $y \in f^{-1}(J)$. Thus $f^{-1}(J)$ is an α-ideal of L. □

THEOREM 4.3. *Let L and L' be two* 0*-distributive ALs. If $f : L \to L'$ is an annihilator preserving epimorphism, then contraction of every α-ideal is an α-ideal.*

PROOF. Suppose L and L' be two 0-distributive ALs and $f : L \to L'$ is an annihilator preserving epimorphism. Suppose J is an α-ideal of L'. Now, we shall prove $f^{-1}(J)$ is an α-ideal of L. Clearly, $f^{-1}(J)$ is an ideal of L. Now, let $x, y \in L$ such that $[x]^* = [y]^*$ and $x \in f^{-1}(J)$. Now, since $[x]^* = [y]^*$, $f([x]^*) = f([y]^*)$. This implies $f((x]^*) = f((y]^*)$. It follows that $(f((x]))^* = (f((y]))^*$. Therefore by lemma 4.1, we get $(f(x)]^* = (f(y)]^*$. Hence we get $[f(x)]^* = [f(y)]^*$. Now, since $x \in f^{-1}(J)$, $f(x) \in J$. Therefore $f(y) \in J$, since J is an α-ideal. Thus $y \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is an α-ideal. □

It can be easily seen that for any nonempty subset A of a 0-distributive AL L, A^* is an α-ideal. Now, we have the following.

COROLLARY 4.1. *Let L and L' be two* 0*-distributive ALs and let $f : L \to L'$ be an annihilator preserving epimorphism. Then for any nonempty subset A of L', $f^{-1}(A^*)$ is an α-ideal of L containing $(f^{-1}(A))^*$.*

PROOF. Suppose A is a nonempty subset of L'. Then we have A^* is an α-ideal. Therefore by Theorem 4.3, we get $f^{-1}(A^*)$ is an α-ideal. Now, we shall prove $(f^{-1}(A))^* \subseteq f^{-1}(A^*)$. Let $x \in (f^{-1}(A))^*$. Then $x \wedge t = 0$ for all $t \in f^{-1}(A)$.

This implies $f(x \wedge t) = f(0)$ for all $t \in f^{-1}(A)$. It follows that $f(x) \wedge f(t) = 0'$ for all $f(t) \in A$. Thus $f(x) \in A^*$. Hence $x \in f^{-1}(A^*)$. Therefore $(f^{-1}(A))^* \subseteq f^{-1}(A^*)$. □

COROLLARY 4.2. *Let L and L' be two 0-distributive ALs and let $f : L \to L'$ be an annihilator preserving epimorphism. Then $ker(f)$ is an α-ideal of L.*

PROOF. We have $\{0'\}$ is an α-ideal of L' and also, we have $ker(f) = f^{-1}\{0'\}$. It follows that $ker(f)$ is an α-ideal of L. □

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