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COMPARATIVE FILTERS IN QUASI-ORDERED RESIDUATED SYSTEM

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Abstract. The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda as a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $(A, \cdot)$ is a commutative semigroup with the identity 1 as the top element in this ordered monoid under a quasi-order R. In 2020, the author introduced and analyzed the concepts of filters in this type of algebraic structures. In addition to the previous, the author continued to investigate some of the types of filters in quasi-ordered residuated systems such as, for example, implicative and associated filters. In this article, as a continuation of previous author's research, the author introduced and analyzed the concepts of comparative filters in quasi-ordered residuated systems.

1. Introduction

Let $(A, \cdot, 1)$ be a commutative semigroup with the identity 1. Suppose that on the carrier A there exists another operation $\rightarrow$ and one relation R that with multiplication in A have a link $(x{\cdot}y, z) \in R \iff (x, y \rightarrow z) \in R$ for each $x, y, z \in A$. A relational system designed in this way, when R is a quasi-ordered relation on A, is in the focus of this paper.

The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda [**2**]. Previously, this concept was discussed in [**1**]. This paper continues the investigations of quasi-ordered residuated systems and of their filters which were started in the author article [**3**]. In particular, the concept of comparative filters in a quasi-ordered residuated system is introduced and analyzed. This type of filter is compared to the concept of filter and the concept of implicative filters (introduced in [**4**]) in this algebraic system.

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2. Preliminaries

2.1. Concept of quasi-ordered residuated systems. In article [**2**], S. Bonzio and I. Chajda introduced and analyzed the concept of 'residual relational systems'.

Definition 2.1. ([**2**], Definition 2.1) A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:
(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)((x, 1) \in R)$;
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.
We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties for residuated relational systems are subsumed in the following

Theorem 2.1 ([**2**], Proposition 2.1). *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ be a residuated relational system. Then*
(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$,
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$,
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$,
(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$,
(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $' \preccurlyeq '$ on a set A is a binary relation which is reflexive and transitive (Some authors use the term pre-order relation).

Definition 2.2. ([**2**], Definition 3.1) A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

Example 2.1. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
s	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

and

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	a	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1, 1), (a, 1), (b, 1), (c, 1), (d, 1), (b, b), (a, a), (c, a), (d, a), (d, b), (d, c)\}.$$

Example 2.2. For a commutative monoid A, let $\mathfrak{P}(A)$ denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then

$\langle \mathfrak{P}(A), \cdot, \rightarrow, \{1\}, \subseteq\rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\}).$$

EXAMPLE 2.3. Let $\mathbb{R}$ be a field of real numbers. Define a binary operations '$\cdot$' and '$\rightarrow$' on $A = [0, 1] \subset \mathbb{R}$ by

$$(\forall x, y \in [0, 1])(x \cdot y := max\{0, x + y - 1\})$$

and

$$(\forall x, y \in [0, 1])(x \rightarrow y := min\{1, 1 - x + y\}).$$

Then, A is a commutative monoid with the identity 1 and $\langle A, \cdot, \rightarrow, <, 1\rangle$ is a quasi-ordered residuated system.

The following proposition shows the basic properties of quasi-ordered residuated systems.

PROPOSITION 2.1 ([**2**], Proposition 3.1). *Let A be a quasi-ordered residuated system. Then*

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$;

(10) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;

(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

Estimating that this topic is interesting ([**1**], [**2**], [**3**]), it is certain that there is interest in the development of the concept of some substructures such as some types of filters [**4**], [**5**] in these systems.

2.2. Concepts of filters. In the article [**3**], in order to determine the concept of filters in quasi-ordered residuated systems, the relationships between the following conditions are analyzed:

(F0) $1 \in F$;

(F1) $(\forall u, v \in A)((u \cdot v \in F \Longrightarrow (u \in F \wedge v \in F))$;

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \Longrightarrow v \in F)$; and

(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \Longrightarrow v \in F)$.

It is shown ([**3**], Proposition 3.2) that $(F2) \Longrightarrow (F1)$. In addition, it is shown ([**3**], Proposition 3.4) that for every nonempty subset of F of system $\mathfrak{A}$ is valid $(F2) \Longrightarrow (F0)$.

Based on our previous analysis of the interrelationship between conditions (F1), F(2) and (F3) in a quasi-ordered residual system, we introduced the concept of filters in the following definition.

DEFINITION 2.3. ([**3**], Definition 3.1) For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* of $\mathfrak{A}$ if it satisfies conditions (F2) and (F3).

DEFINITION 2.4. ([**4**], Definition 3.1) For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that the *implicative filter* in $\mathfrak{A}$ if (F2) and the following condition

(IF) $(\forall u, v, z \in A)((u \rightarrow (v \rightarrow z) \in F \wedge u \rightarrow v \in F) \Longrightarrow u \rightarrow z \in F)$

are valid.

It is known that every implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ is a filter in $\mathfrak{A}$ ([**4**], Theorem 3.1) but that the reverse does not have to be.

3. The concept of comparative filters

In this section we introduce the concept of comparative filter in quasi-ordered residuated system. We then relate it to the concept of filters (Theorem 3.1 and Example 3.2) and the concept of implicative filters (Theorem 3.3, Theorem 3.4 and Example 3.3) in this algebraic system.

Definition 3.1. For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that a *comparative filter* in $\mathfrak{A}$ if (F2) and the following condition

(FC) $(\forall u, v, z \in A)((u \to ((v \to z) \to v) \in F \wedge u \in F) \implies v \in F)$

are valid.

Example 3.1. Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.1. Then the set $F := \{1, a, b\}$ is a comparative filter in $\mathfrak{A}$.

Since any comparative filter F of $\mathfrak{A}$ satisfies the condition (F2), F also satisfies the condition (F0): $1 \in F$.

In what follows, we need the following lemma

Lemma 3.1 ([**4**], Lemma 3.1). *Let a subset F of a quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition* (F2). *Then the following holds*

(12) $(\forall u \in A)(u \in F \iff 1 \to u \in F)$.

It can be seen immediately that the following proposition is valid

Proposition 3.1. *For any comparative filter F in a quasi-ordered residuated system $\mathfrak{A}$ holds*

(13) $(\forall v, z \in A)((v \to z) \to v \in F \implies v \in F)$.

Proof. If we put $u = 1$ u (FC), we immediately get (13) with respect to (12) and (F0). ☐

Lemma 3.2 ([**4**], Lemma 3.4). *Let a subset F of a quasi-ordered residuated system $\mathfrak{A}$ satisfy the condition* (F2). *Then the following holds*

(14) $(\forall u, v, z \in A)(u \to (v \to z) \in F \iff v \to (u \to z) \in F)$.

Remark 3.1. Considering equivalence (14), condition (FC) is equivalent to condition

(FC (14)) $(\forall u, v, z \in A)(((v \to z) \to (u \to v) \in F \wedge u \in F) \implies v \in F)$.

Let us show that every comparative filter in $\mathfrak{A}$ is a filter of $\mathfrak{A}$. Before that, let us give a technically important lemma.

Lemma 3.3. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*

(15) $(\forall u, v, z \in A)(u \to v \preccurlyeq (v \to z) \to (u \to z))$.

Proof. The proof of this lemma follows directly from the claim (g) of Proposition 3.1 in [**2**]. ☐

Remark 3.2. The condition (15) can be written in the form

$$(\forall u, v, z \in A)(v \to z \preccurlyeq (u \to v) \to (u \to z)),$$

too.

Theorem 3.1. *Every comparative filter of a quasi-ordered residuated system* $\mathfrak{A}$ *is a filter of* $\mathfrak{A}$.

Proof. Let F be a comparative filter of $\mathfrak{A}$ and let $u, v \in A$ be such that $u \in F$ and $u \to v \in F$. Since by (15), we have

$$u \to v \preccurlyeq (v \to z) \to (u \to z)$$

for any $z \in A$, it follows $(v \to z) \to (u \to z) \in F$ from (15) and $u \to v \in F$ with application (F2). Thus $v \in F$ since F is a comparative filter of $\mathfrak{A}$. Therefore, F is a filter of $\mathfrak{A}$. □

The converse of Theorem 3.1 is not true in general as seen in the following example.

Example 3.2. Let $\mathfrak{A}$ be a quasu-ordered residuated system as in Example 2.1. Then $F := \{1, b\}$ is a filter of $\mathfrak{A}$, but it is not a comparative filter of $\mathfrak{A}$, since for $u = b$, $v = a$ and $z = d$, we have $b \to ((a \to d) \to a) = 1 \in F$ and $b \in F$, but $a \notin F$.

Let us show that the condition (13) is sufficient for a filter of a quasi-ordered residuated system $\mathfrak{A}$ to be a comparative filter of $\mathfrak{A}$.

Theorem 3.2. *Let F be a filter of a quasi-ordered residuated system $\mathfrak{A}$. Then F is a comparative filter of $\mathfrak{A}$ if and only if the condition* (13) *is valid.*

Proof. The necessity of the condition (13) is shown in Proposition 3.1.

Let F be a filter of $\mathfrak{A}$ that satisfies (13). Let $u, v, z \in A$ be such that $u \to ((v \to z) \to v) \in F$ and $u \in F$. Then $(v \to z) \to v \in F$ by (F3). Thus $v \in F$ by (13). Therefore F is a comparative filter of $\mathfrak{A}$. □

In what follows, we assume that a quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition

(16) $(\forall u, v \in A)(u \to v = ((u \to v) \to v) \to v)$.

Proposition 3.2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system that satisfies the condition (16). Then any comparative filter F in $\mathfrak{A}$ satisfies the the condition*

(17) $(\forall u, v \in A)((u \to v) \to v \in F \implies (v \to u) \to u \in F)$.

Proof. Suppose F is a comparative filter of $\mathfrak{A}$. Let $u, v \in A$ be such that $(u \to v) \to v \in F$.

As $u \preccurlyeq (v \to u) \to u$ is valid according to (11), hence it follows $((v \to u) \to u) \to v \preccurlyeq u \to v$ in accordance with (10) by the action on the right with v on the previous inequality.

On the other hand, according to (10), we have

$$(u \to v) \to v \preccurlyeq (v \to u) \to ((u \to v) \to u).$$

Hence $(v \to u) \to ((u \to v) \to u) \in F$ with respect (F2) and hypothesis $(u \to v) \to v \in F$. Then $(u \to v) \to ((v \to u) \to u) \in F$ by (14). Thus, we have $(((v \to u) \to u) \to v) \to ((v \to u) \to u) \in F$ by (16). Finally, we have

$$1 \to ((((v \to u) \to u) \to v) \to ((v \to u) \to u)) \in F$$

by (12). Since $1 \in F$ and F is a comparative filter in $\mathfrak{A}$, it follows immediately $(v \to u) \to u \in F$. □

THEOREM 3.3. *Let F be an implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ satisfying (17). Then F is a comparative filter of $\mathfrak{A}$.*

PROOF. Suppose F is an implicative filter of $\mathfrak{A}$ satisfying the condition (17). Then F is a filter of $\mathfrak{A}$ according to Theorem 3.1. u [**4**]. To prove that F is a comparative filter in $\mathfrak{A}$, it suffices to prove, according Theorem 3.2, that F satisfies condition (13). Let $u, v \in A$ be such that $(u \to v) \to u \in F$. Applying the condition (15) on $(u \to v) \to u$, we have

$$(u \to v) \to u \preccurlyeq (u \to v) \to ((u \to v) \to v).$$

Thus, by (F2), we have $(u \to v) \to ((u \to v) \to v) \in F$. Since F is an implicative filter in $\mathfrak{A}$, it satisfies condition (F4) in Proposition 3.1 in [**4**]. So, the following holds $(u \to v) \to v \in F$. Hence $(v \to u) \to u \in F$ according to (17).

On the other hand, by $v \preccurlyeq u \to v$ according (11), using (10) we have $(u \to v) \to u \preccurlyeq v \to u$. Then $v \to u \in F$ by (F2). Finally, from $v \to y \in F$ and $(v \to u) \to u \in F$ it follows $u \in F$ by (F3). This completes the proof. □

THEOREM 3.4. *Any comparative filter of a quasi-ordered residuated system $\mathfrak{A}$ is an implicative filter of $\mathfrak{A}$.*

PROOF. Let F be a comparative filter of $\mathfrak{A}$. Then F is a filter of A by Theorem 3.1. To prove that F is an implicative filter let us take $u, v, z \in A$ such that $u \to (v \to z) \in F$ and $u \to v \in F$. Then $v \to (u \to z) \in F$ by (14). From here and from

$$v \to (u \to z) \preccurlyeq (u \to v) \to (u \to (u \to z))$$

it follows $(u \to v) \to (u \to (u \to z))$. Thus $u \to (u \to z) \in F$ by (F3). On the other hand, note that $u \to (((u \to z) \to z) \to z) = u \to (u \to v) \in F$. Now, we have $((u \to z) \to z) \to (u \to z) \in F$. Hence $1 \to (((u \to z) \to z) \to (u \to z)) \in F$. Since F is a comparative filter, we have $x \to z \in F$ by (FC). This completes the proof. □

EXAMPLE 3.3. Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 3.2. Then the subset $F := \{1, b\}$ is an implicative filter but it is not a comparative filter.

We end this section with the following theorem.

THEOREM 3.5. *The family $\mathfrak{F}_c(A)$ of all comparative filters of a quasi-ordered residuated system $\mathfrak{A}$ forms a complete lattice.*

PROOF. Let $\{F_k\}_{k\in\Lambda}$ be a family of comparative filters of $\mathfrak{A}$ where Λ is index set. It is clear that $1 \in \bigcap_{k\in\Lambda} F_k$. Let $u, v, z \in A$ be such that $(u \to z) \to (u \to z) \in \bigcap_{k\in\Lambda} F_k$ and $u \in \bigcap_{k\in\Lambda} F_k$. Then $(v \to z) \to (u \to z) \in F_k$ and $u \in F_k$ for any $k \in \Lambda$. Thus $v \in F_k$ for all $k \in \Lambda$. Hence $v \in \bigcap_{k\in\Lambda} F_k$. So, the intersection $\bigcap_{k\in\Lambda} F_k$ satisfies the condition (FC). Therefore $\bigcap_{k\in\Lambda} F_k$ is a comparative filter of $\mathfrak{A}$.

Let $\mathfrak{X}$ be the family of all comparative filters containing the union $\bigcup_{k\in\Lambda} F_k$. Then $\cap\mathfrak{X}$ is a comparative filter of $\mathfrak{A}$ according to the first part of this proof.

If we put $\sqcap_{k\in\Lambda} F_k = \bigcap_{k\in\Lambda} F_k$ and $\sqcup_{k\in\Lambda} F_k = \cap\mathfrak{X}$, then $(\mathfrak{F}_c(A), \sqcap, \sqcup)$ is a complete lattice. □

COROLLARY 3.1. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any subset T of A, there is the unique minimum comparative filter in $\mathfrak{A}$ that contains T.*

PROOF. The proof of this Corillary follows directly from the second part of the proof of the previous theorem. □

COROLLARY 3.2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any element x of A, there is the unique minimum comparative filter in $\mathfrak{A}$ that contains x.*

PROOF. The proof of this Corollary follows from the previous Corollary if we take $T = \{x\}$. □

4. Conclusion and further work

The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda in [**2**]. This author introduced and analyzed the substructure of filters in this algebraic construction ([**3**]) and then the class of implicative filters ([**4**]) and the the class of associated filters ([**5**]) In this system, the concept of comparative filter is such system is introduced and analyzed. In addition, a link between comparative and implicative filters in this algebraic structure has been established.

The possibility of further research into this area that opens with this article is, among other things, besides finding new types of filters and to compare the concept of a comparative filter with the concept of associated filters. For example, it is possible to investigate the condition

$$(\forall u, v, z \in A)((u \to (v \to z) \in F \ \wedge\ u \in F) \implies ((z \to v) \to v) \to z \in F)$$

which could, combined with condition (F2), it determine some new filter class in quasi-ordered residuated systems.

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